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COMPACT OPERATORS UNDER ORLICZ FUNCTION[1]

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In this paper we investigate the compact operators under Orlicz function, named noncommutative Orlicz sequence space (denoted by $S_\varphi(\mathcal{H})$), where $\mathcal{H}$ is a complex, separable Hilbert space. We will show that the space generalizes the Schatten classes $S_p(\mathcal{H})$ and the classical Orlicz sequence space respectively. After getting some relations of trace and norm, we will give some operator inequalities, such as Hölder inequality and some other classical operator inequalities. Also we will give the dual space and reflexivity of $S_\varphi(\mathcal{H})$ which generalizes the results of $S_p(\mathcal{H})$. Finally, as an application, we will show that the Toeplitz operator $T_{1-|z|^2}$ on the Bergman space $L_a^2(\mathbb{D})$ belongs to some $S_\varphi(\mathcal{H})$, and the norm satisfies $1 = \sum\limits_{n\geq 0} \varphi\left(\frac{1}{(n+2)\|T_{1-|z|^2}\|_\varphi}\right)$. Especially, if $\varphi(T) = |T|^p,\ p > 1$, the norm is $\|T_{1-|z|^2}\|_p = \left[\sum\limits_{n\geq 0} \frac{1}{(n+2)^p}\right]^{\frac{1}{p}} = (\zeta(p)-1)^{\frac{1}{p}}$, where $\zeta(p)$ is the Riemann function.

Key words : Noncommutative Orlicz sequence spaces; compact operators; Schatten classes; Orlicz function.

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1. Introduction and Preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators in a complex, separable Hilbert space $\mathcal{H}$. If T in $\mathcal{B}(\mathcal{H})$ is a compact operator and T^* is its adjoint $T^*T : \mathcal{H} \to \mathcal{H}$ is positive and compact, also it T^*T has a unique positive square root denoted by $|T| = (T^*T)^{1/2}$. Correspondingly, $\mathcal{H}$ has an orthonormal basis $\{e_k\}_k$, each of which is an eigenvector of $|T|$.

The eigenvalue $\lambda_k(|T|)$ corresponding to each eigenvector e_k is necessarily non-negative, and it is called a singular value of the operator T, which is denoted by $s_k(T)$. Without loss of generality, by reordering the members of the orthonormal basis $\{e_k\}$ if needed, we assume $s_k(T)$ are arranged in decreasing order, that is, $s_{k+1}(T) \leq s_k(T)$ for $k = 1, 2, \cdots$.

If $\sum\limits_{k=1}^{\infty} s_k(T) < \infty$, the operator $T : \mathcal{H} \to \mathcal{H}$ is called a trace class operator. The set of all trace class operators is denoted by $S_1(\mathcal{H})$. It can be shown that $S_1(\mathcal{H})$ is a Banach space in which the trace norm $\|\cdot\|_1$ is given by $\|T\|_1 = \sum\limits_{k=1}^{\infty} s_k(T)$.

On the other side, if $T : \mathcal{H} \to \mathcal{H}$ is an operator in $S_1(\mathcal{H})$ and $\{e_k\}_k(k = 1, 2, \ldots)$ is any orthonormal basis for $\mathcal{H}$, the series $\sum\limits_{k=1}^{\infty} \langle Te_k, e_k\rangle$ is absolutely convergent and the sum is independent of the choice of the orthonormal basis e_k. Thus, we can define the trace $\mathrm{Tr}(T)$ of any linear operator $T : \mathcal{H} \to \mathcal{H}$ in $S_1(\mathcal{H})$ by

$$\mathrm{tr}(T) := \sum_{k=1}^{\infty} \langle Te_k, e_k\rangle.$$

By the spectral decomposition theorem of compact operators, for any $\xi \in \mathcal{H}$ there exists an orthonormal sequence $\{\xi_n\}_n \in \mathcal{H}$ such that,

$$T\xi = \sum_{n\geq 1} s_n(T)\langle \xi_n, \xi\rangle \xi_n,$$

hence one get

$$\mathrm{tr}(T) = \sum_{n\geq 1} s_n(T).$$

For more details of $S_1(\mathcal{H})$ please refer to [9, 14].

The Schatten classes $S_p(\mathcal{H})$ is given as follows: for any $0 < p < \infty$, let T a compact operator of $\mathcal{H}$, we define

$$S_p(\mathcal{H}) = \left\{ T : \ \mathrm{tr}\left(|T|^p\right) = \sum_{n\geq 1} s_n\left(T^p\right) = \sum_{n\geq 1} \left(s_n\left(T\right)\right)^p < \infty \right\},$$

and for any $T \in S_p(\mathcal{H})$, the norm is

$$\|T\|_p = \left(\sum_{n \geq 1} (s_n(T))^p \right)^{\frac{1}{p}}.$$

Especially, $S_2(\mathcal{H})$ is called Hilbert-Schmidt operator space [3], and the norm is

$$\|T\|_2 = \left(\sum_{n \geq 1} (s_n(T))^2 \right)^{\frac{1}{2}}.$$

Literatures on the Schatten class operators and their applications are very rich, please see the very interesting recent papers ([2, 5, 7, 12, 18]) and references cited therein. Especially, Gil', M.I. extended some useful results on determinants of Schatten-von Neumann operators to the operators with the Orlicz type norms and modular function, respectively ([7, 8]). In this paper we investigate the compact operators under Orlicz function, named noncommutative Orlicz sequence space (denoted by $S_\varphi(\mathcal{H})$), where $\mathcal{H}$ is a complex, separable Hilbert space. The many results of the space could be extended the Schatten class $S_p(\mathcal{H})$, such as the Hölder Inequality and other classical operator inequalities. Besides, some applications are given to support our theory, such as Toeplitz operator on the Bergman space.

A convex function $\varphi : [0, \infty) \to [0, \infty]$ is called an Orlicz function if it is nondecreasing and continuous for $\alpha \geq 0$ and satisfying $\varphi(0) = 0$, $\varphi(\alpha) > 0$ and $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$.

Let φ an Orlicz function and l^0 the set of all real sequences. We define the Orlicz sequence space as follows

$$l_\varphi = \left\{ x \in l^0 : I_\varphi(\lambda x(i)) = \sum_{i=1}^{\infty} \varphi(\lambda x(i)) < \infty, \exists \lambda > 0 \right\}.$$

It is well known that l_φ equipped with so called the Luxemburg norm

$$\|x\| = \inf \left\{ \lambda > 0 : I_\varphi \left(\frac{x(i)}{\lambda} \right) \leq 1 \right\}$$

is a Banach space.

Further we say an Orlicz function φ satisfies condition δ_2 near zero, shortly $\varphi \in \delta_2(0)$, if there exist constant $u_0 > 0$ and $k > 2$ such that $\varphi(2u) \leq k\varphi(u)$ for all $|u| \leq u_0$. For the background of Orlicz function and Orlicz space please see [4, 16].

2. NONCOMMUTATIVE ORLICZ SEQUENCE SPACE

By functional calculus, if φ is a continuous function, we have

$$\mathrm{tr}(\varphi(T)) = \sum_{k=1}^{\infty}\langle\varphi(T)e_k, e_k\rangle = \sum_{n\geq 1}\varphi(s_n(T)) = \sum_{n\geq 1}s_n(\varphi(T)).$$

Hence, if φ is an Orlicz function and T a compact operator of $\mathcal{H}$, the definitions of the noncommutative Orlicz sequence spaces and its a useful subspace could be given.

In this section, after the definitions of noncommutative Orlicz sequence spaces and a subspace which first appear in [6] are given, we will show this space could generalizes the $S_p(\mathcal{H})$ and the classical Orlicz sequence spaces respectively. At the end of this section, the Orlicz norm of the rank-one operator $e_n \otimes e_n$ will be given.

Definition 2.1 — If $T \in \mathcal{B}(\mathcal{H})$ is a compact operator, we define the noncommutative Orlicz sequence spaces $S_\varphi(\mathcal{H})$ and the subspace $E_\varphi(\mathcal{H})$ as follows:

$$S_\varphi(\mathcal{H}) = \left\{T : \mathrm{tr}\left(\varphi(\lambda T)\right) = \sum_{n\geq 1}\varphi(\lambda s_n(T)) < \infty, \exists\lambda > 0\right\},$$

$$E_\varphi(\mathcal{H}) = \left\{T : \mathrm{tr}\left(\varphi(\lambda T)\right) = \sum_{n\geq 1}\varphi(\lambda s_n(T)) < \infty, \forall\lambda > 0\right\}.$$

where tr is the usual trace functional of $\mathcal{B}(\mathcal{H})$.

For convenience, we use S_φ, E_φ to denote $S_\varphi(\mathcal{H})$ and $E_\varphi(\mathcal{H})$ respectively, which could be equipped with the Luxemburg norm as

$$\begin{aligned}
\|T\|_\varphi &= \inf\left\{\lambda > 0, \mathrm{tr}\left(\varphi\left(\frac{T}{\lambda}\right)\right) \leq 1\right\} \\
&= \inf\left\{\lambda > 0, \sum_{n\geq 1}\varphi\left(s_n\left(\frac{T}{\lambda}\right)\right) \leq 1\right\} \\
&= \inf\left\{\lambda > 0, \sum_{n\geq 1}\varphi\left(\frac{s_n\left(T\right)}{\lambda}\right) \leq 1\right\} \\
&= \|s(T;n)\|,
\end{aligned}$$

where $s(T;n)$ is the singular value sequence of T and $\|\cdot\|$ is the Luxemburg norm of classical Orlicz space. By the theory of classical Orlicz sequence, it is easy to know $(S_\varphi, \|T\|_\varphi)$ and $(E_\varphi, \|T\|_\varphi)$ are Banach spaces.

Also, one can define another norm which is named as Orlicz norm of S_φ as follows

$$\|T\|_\varphi^o = \sup\{\mathrm{tr}(|TB|) : \mathrm{tr}(\psi(B)) \leq 1\},$$

where $T \in S_\varphi$ and $\psi : [0,\infty) \to [0,\infty]$ is defined by $\psi(u) = \sup\{uv - \varphi(v) : v \geq 0\}$. Here we call ψ the complementary function of φ. Similar with the classical case, one also could give another equivalent definition of the Orlciz norm which is named as Amemiya norm, $\|T\|_\varphi^A = \inf\limits_{k>0} \frac{1}{k}(1 + \mathrm{tr}(\varphi(kT)))$. From [11] it is easy to know, $\|T\|_\varphi^o = \|T\|_\varphi^A$.

By the classical Orlicz theory we also have the following relation of these norms

$$\|T\|_\varphi < \|T\|_\varphi^o \leq 2\|T\|_\varphi.$$

It is obvious that $E_\varphi \subsetneqq S_\varphi$, but if $\varphi \in \delta_2(0)$ one could get the following theorem.

***Theorem* 2.1** — *$S_\varphi = E_\varphi$ if and only if $\varphi \in \delta_2(0)$.*

PROOF : If $\varphi \in \delta_2(0)$. It is obvious that $E_\varphi \subseteq S_\varphi$. Hence we only need to prove $S_\varphi \subseteq E_\varphi$.

For any $T \in S_\varphi$, there exists a $\lambda_0 > 0$ such that

$$\mathrm{tr}\,(\varphi(\lambda_0 T)) = \sum_{n\geq 1} \varphi(\lambda_0 s_n(T)) < \infty,$$

then for any $\lambda > 0$, one could get

$$\mathrm{tr}\,(\varphi(\lambda T)) = \sum_{n\geq 1} \varphi(\lambda s_n(T)) < \infty.$$

In fact, if $\frac{\lambda}{\lambda_0} \leq 1$, by convexity of φ, we can obtain

$$\mathrm{tr}\,(\varphi(\lambda T)) = \sum_{n\geq 1} \varphi\left(\frac{\lambda}{\lambda_0}\lambda_0 s_n(T)\right) \leq \frac{\lambda}{\lambda_0}\sum_{n\geq 1} \varphi\left(\lambda_0 s_n(T)\right) < \infty.$$

If $\frac{\lambda}{\lambda_0} > 1$, define $n_0 = \sup\{n, s_n(T) \leq u_0\}$, where $u_0 \in (0, s_1)$. Since $\varphi \in \delta_2(0)$, we have

$$\begin{aligned}
\mathrm{tr}\,(\varphi(\lambda T)) &= \sum_{n\geq 1} \varphi\left(\lambda s_n(T)\right) \\
&= \sum_{n=1}^{n_0} \varphi\left(\lambda s_n(T)\right) + \sum_{n=n_0+1}^{\infty} \varphi\left(\frac{\lambda}{\lambda_0}\lambda_0 s_n(T)\right) \\
&\leq n_0\varphi(\lambda s_1(T)) + k\sum_{n=n_0+1}^{\infty} \varphi\left(\lambda_0 s_n(|T|)\right) \\
&\leq n_0\varphi(\lambda s_1(T)) + k\mathrm{tr}(\varphi(\lambda_0 T)) \\
&< +\infty.
\end{aligned}$$

Now, if $S_\varphi = E_\varphi$, similar to the Example 1.19 of [4] we have $\varphi \in \delta_2(0)$. $\square$

Remark 1 : (1) In the case $\varphi(T) = |T|^p,\ 1 \le p < \infty$ for any compact operator $T \in \mathcal{B}(\mathcal{H})$, S_φ is nothing but the Schatten classes S_p and the Luxemburg norm is

$$\|T\|_p = \mathrm{tr}(|T|^p) = \sum_{n \ge 0} ((s_n(T))^p)^{\frac{1}{p}}.$$

Especially, if $p = 1$, $S_\varphi = S_1$ and if $p = 2$, S_φ is the Hilbert-Schmidt operator space S_2 which satisfy $\varphi \in \delta_2(0)$.

(2) If $\varphi \in \delta_2(0)$, let $T : l_2(\mathbb{N}) \to l_2(\mathbb{N})$ with $Te_n = a_n e_n$, where e_n is the orthonormal basis of $l_2(\mathbb{N})$, then it is obvious that $T \in S_\varphi$ if and only if $a = \{a_n\} \in l_\varphi$ and $\|T\|_\varphi = \|a\|$.

Let $\mu(n)$ denote the multiplicity of singular value $s_n(T)$. The following theorem gives the Orlicz norm of rank one operator $e_n \otimes e_n$.

***Theorem* 2.2** — *For each positive operator $B \in S_\psi$ with $\mathrm{tr}(\psi(B)) \le 1$, where ψ is invertible on $\mathbb{R}^+$, and $\psi \in \delta_2(0)$. Let $\{e_n\}$ a orthonormal bases, then*

$$\|e_n \otimes e_n\|_\varphi^o = \psi^{-1}\left(\frac{1}{\mu(n)}\right)\mu(n),$$

where $e \otimes h$ is rank one operator (a bounded operator) on $\mathcal{H} \to \mathcal{H}$ with $\xi \mapsto \langle h, \xi\rangle e$.

PROOF : First, by the Jensen Inequality,

$$\psi(s_n(B)) = \psi\left(\frac{1}{\mu(n)}\sum_{\mu(n)} s_n(B)\right) \le \frac{1}{\mu(n)}\mathrm{tr}(\psi(B)) \le \frac{1}{\mu(n)}.$$

Using spectral decomposition of compact operators, we have $B = \sum_{n \ge 1} s_n(B) e_n \otimes e_n$. Therefore,

$$\|e_n \otimes e_n\|_\varphi^o = \sup\left\{\sum_{\mu(n)} s_n(B) : \mathrm{tr}(\psi(B)) \le 1\right\} \le \psi^{-1}\left(\frac{1}{\mu(n)}\right)\mu(n).$$

On the other hand, observing that

$$\mathrm{tr}\left(\psi\left(\psi^{-1}\left(\frac{1}{\mu(n)} e_n \otimes e_n\right)\right)\right) = \frac{1}{\mu(n)}\mathrm{tr}(e_n \otimes e_n) \le 1,$$

we obtain

$$\|e_n \otimes e_n\|_\varphi^o \ge \sum_{n=1}^{\infty} s_n\left(\psi^{-1}\left(\frac{1}{\mu(n)} e_n \otimes h_n\right) e_n \otimes e_n\right) = \psi^{-1}\left(\frac{1}{\mu(n)}\right)\mu(n).$$

Thus,

$$\|e_n \otimes e_n\|_\varphi^o = \psi^{-1}\left(\frac{1}{\mu(n)}\right)\mu(n).\square$$

Corollary 2.1 — If $T \in S_p$ is positive and $\mu(n)$ is the multiplicity of singular value $s_n(T)$. Then

$$\|e_n \otimes e_n\|_p^o = \mu(n)^{1-p}.$$

Corollary 2.2 — For a positive operator T, if $\varphi(T) = \frac{1}{\alpha}|T|^\alpha$ and $\alpha > 1$. Then

$$\|e_n \otimes e_n\|_\varphi^o = \left(\frac{\alpha}{\mu(n)}\right)^{\frac{1}{\alpha}}\mu(n).$$

3. Trace and the Luxemburg Norm

In this section, we will give some relations of trace and the Luxemburg norm, which generalize the corresponding results of S_p.

Lemma 3.1 — If $T \in S_\varphi$ and $B \in S_\psi$, then we have

(1) If $\|T\|_\varphi > 1$, then $\mathrm{tr}(\varphi(T)) \geq \|T\|_\varphi$;

(2) If $T \neq 0$, then $\mathrm{tr}\left(\varphi\left(\frac{T}{\|T\|_\varphi}\right)\right) \leq 1$;

(3) If $\|T\|_\varphi \leq 1$, then $\mathrm{tr}(\varphi(T)) \leq \|T\|_\varphi$;

(4) (Hölder Inequality) $\mathrm{tr}(|TB|) \leq \|T\|_\varphi^o \|B\|_\psi$;

(5) $TB, BT \in S_1$.

PROOF : (1) If $\|T\|_\varphi > 1$, we choose $0 < \varepsilon < 1$ such that $(1-\varepsilon)\|T\|_\varphi > 1$. Then

$$\begin{aligned} 1 &< \mathrm{tr}\left(\varphi\left(\frac{T}{(1-\varepsilon)\|T\|_\varphi}\right)\right) \\ &\leq \frac{1}{(1-\varepsilon)\|T\|_\varphi}\mathrm{tr}(\varphi(T)) + \left(1 - \frac{1}{(1-\varepsilon)\|T\|_\varphi}\right)\mathrm{tr}(\varphi(0)) \\ &= \frac{1}{(1-\varepsilon)\|T\|_\varphi}\mathrm{tr}(\varphi(T)), \end{aligned}$$

which implies

$$(1-\varepsilon)\|T\|_\varphi < \mathrm{tr}(\varphi(T)).$$

Let $\varepsilon \to 0$ we can get the conclusion.

(2) For any $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that $\operatorname{tr}\left(\varphi\left(\frac{T}{\lambda_\varepsilon}\right)\right) \leq 1$, by the definition of Luxemburg norm. If $\lambda_\varepsilon < \varepsilon + \|T\|_\varphi$ one can get that $\operatorname{tr}\left(\varphi\left(\frac{T}{\|T\|_\varphi + \varepsilon}\right)\right) \leq 1$, by the arbitrariness of ε, then we have

$$\operatorname{tr}\left(\varphi\left(\frac{T}{\|T\|_\varphi}\right)\right) \leq 1.$$

(3) If $T \in S_\varphi$ and $\|T\|_\varphi \leq 1$, with $T \neq 0$. By the definition of $\|T\|_\varphi$, there exists a decreasing sequence $\{\lambda_n\}$ which converges to $\|T\|_\varphi$ such that

$$\operatorname{tr}\left(\varphi\left(\frac{T}{\lambda_n}\right)\right) = \sum_{n\geq 1} \varphi\left(\frac{1}{\lambda_n} s_n(T)\right) \leq 1.$$

By Levy's theorem,

$$\operatorname{tr}\left(\varphi\left(\frac{T}{\|T\|_\varphi}\right)\right) = \sum_{n\geq 1} \varphi\left(\frac{1}{\|T\|_\varphi} s_n(T)\right) \leq 1.$$

Thanks to the convexity of φ and $\|T\|_\varphi \leq 1$,

$$\frac{1}{\|T\|_\varphi} \operatorname{tr}(\varphi(T)) \leq \operatorname{tr}\left(\varphi\left(\frac{T}{\|T\|_\varphi}\right)\right) \leq 1,$$

hence,

$$\operatorname{tr}(\varphi(T)) = \sum_{n\geq 1} \varphi\left(s_n(T)\right) \leq \|T\|_\varphi.$$

(4) By (3) and the definition of the Orlicz norm we have $\operatorname{tr}\left(\frac{|TB|}{\|B\|_\psi}\right) \leq \|T\|_\varphi^o$, which could deduce the conclusion.

(5) Using 2.13 of [9] and the Orlicz Yöung inequality, we have

$$\begin{aligned}
\operatorname{tr}(|TB|) &= \sum_{n+m=1}^{\infty} s_{n+m+1}(|TB|) \\
&\leq \sum_{n+m=1}^{\infty} \left(s_{n+1}(|T|) s_{m+1}(|B|)\right) \\
&\leq \sum_{n+m=1}^{\infty} \left(\varphi\left(s_{n+1}(|T|)\right) + \psi\left(s_{m+1}(|B|)\right)\right) \\
&< \infty.
\end{aligned}$$

Hence, we have $TB \in S_1$. Using the same method, one can get $BT \in S_1$.

***Theorem* 3.1** — *If $\varphi \in \delta_2(0)$, for any $T, B \in S_\varphi$ we have*

(1) $\mathrm{tr}\left(\varphi\left(\frac{T}{\|T\|_\varphi}\right)\right) = 1$*;*

(2) $\|T\|_\varphi = 1$ *if and only if* $\mathrm{tr}(\varphi(T)) = 1$*;*

(3) $\mathrm{tr}(\varphi(T_n)) \to 0$ *if and only if* $\|T_n\|_\varphi \to 0$*;*

(4) $\mathrm{tr}(\varphi(T+B)) \leq \frac{k}{2}\left[\mathrm{tr}(\varphi(T)) + \mathrm{tr}(\varphi(B))\right]$*, where* k *is the real number which satisfies* $\varphi(2\alpha) \leq k\varphi(\alpha)$ *for any* $\alpha > 0$*.*

PROOF : (1) Let $T \in S_\varphi$, by Proposition 0 of [1],

$$\begin{aligned}
\mathrm{tr}\left(\varphi\left(\frac{T}{\|T\|_\varphi}\right)\right) &= \sum_{n\geq 1}\varphi\left(\frac{1}{\|T\|_\varphi}s_n(T)\right) \\
&= \sum_{n\geq 1}\varphi\left(\frac{1}{\|s(T;n)\|}s_n(T)\right) \\
&= 1.
\end{aligned}$$

(2) If $\|T\|_\varphi = 1$, then obvious $\mathrm{tr}(\varphi(T)) = 1$. If $\mathrm{tr}(\varphi(T)) = 1$, by Remark 8.15 of [15] we can get the conclusion.

(3) This can be obtained by the Theorem 8.14 of [15] .

(4) We know there exist partially isometric operators U, V such that $|T+B| \leq U|T|U^* + V|B|V^*$, by the convexity of φ,

$$\begin{aligned}
\mathrm{tr}(\varphi(T+B)) &\leq \mathrm{tr}\left(\varphi\left(U|T|U^* + V|B|V^*\right)\right) \\
&= \mathrm{tr}\left(\varphi\left(2\frac{U|T|U^* + V|B|V^*}{2}\right)\right) \\
&\leq \frac{k}{2}\left[\mathrm{tr}\left(\varphi\left(U|T|U^*\right)\right) + \mathrm{tr}\left(\varphi\left(V|B|V^*\right)\right)\right] \\
&= \frac{k}{2}\left[\sum_{n\geq 1}\varphi\left(s_n(U|T|U^*)\right) + \sum_{n\geq 1}\varphi\left(s_n(V|B|V^*)\right)\right] \\
&\leq \frac{k}{2}\left[\sum_{n\geq 1}\varphi\left(s_n(|T|)\right) + \sum_{n\geq 1}\varphi\left(s_n(|B|)\right)\right] \\
&= \frac{k}{2}\left[\mathrm{tr}(\varphi(T)) + \mathrm{tr}(\varphi(B))\right].\square
\end{aligned}$$

Corollary 3.1 — (1) $\mathrm{tr}\left(\frac{T^p}{\|T\|_\varphi^p}\right) = 1$. Especially, if T belongs to S_1 and S_2 respectively, we have $\mathrm{tr}\left(\frac{T}{\|T\|_1^1}\right) = 1$ and $\mathrm{tr}\left(\frac{T^2}{\|T\|_2^2}\right) = 1$.

(2) If $T \in S_p$, then $\varphi(2T) = |2T|^p = 2^p|T|^p$. Let $k = 2^p$ and we can get

$$\operatorname{tr}\left(|T+B|^p\right) \leq 2^{p-1}\left[\operatorname{tr}\left(|T|^p\right) + \operatorname{tr}\left(|B|^p\right)\right]$$

or

$$\sum_{n\geq 1} s_n\left((T+B)^p\right) \leq 2^{p-1}\left[\sum_{n\geq 1} s_n\left(T^p\right) + \sum_{n\geq 1} s_n\left(B^p\right)\right].$$

Especially, if $T \in S_1$, we have

$$\sum_{n\geq 1} s_n\left(T+B\right) \leq \sum_{n\geq 1} s_n\left(T\right) + \sum_{n\geq 1} s_n\left(B\right);$$

If $T \in S_2$, we have

$$\sum_{n\geq 1} s_n\left((T+B)^2\right) \leq 2\left[\sum_{n\geq 1} s_n\left(T^2\right) + \sum_{n\geq 1} s_n\left(B^2\right)\right].$$

(3) (Hölder Inequality of S_p): If $T \in S_p,\ B \in S_q$, where $\frac{1}{p} + \frac{1}{q} = 1$ with $p \geq 1$, we have

$$\operatorname{tr}(|TB|) \leq \|T\|_p \|B\|_q$$

or

$$\begin{aligned}\sum_{n=1}^{\infty} s_n(TB) &\leq \left(\sum_{n=1}^{\infty} s_n^p(T)\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} s_n^q(B)\right)^{\frac{1}{q}} \\ &= \left(\operatorname{tr}\left(|T|^p\right)\right)^{\frac{1}{p}} \left(\operatorname{tr}\left(|T|^q\right)\right)^{\frac{1}{q}}.\end{aligned}$$

PROOF : First, if $T \in S_\varphi$, similar with the Theorem 1.29 of [4] we know, if there exists $k > 0$ such that $\sum_{n=1}^{\infty} \psi(h(s_n(kT))) = 1$, then $\|T\|_\varphi^o = \frac{1}{k}\left[1 + \operatorname{tr}(\varphi(k(|T|)))\right]$, where ψ is the complementary function of φ satisfying $\varphi(T) = \int_0^{|T|} h(t)\mathrm{d}t$.

If $T \in S_p$ which means $\varphi(T) = |T|^p$, then $h(t) = pt^{p-1}$ and $\psi(B) = cB^q$, where $c = \frac{1}{p^{\frac{p}{q}}} \cdot \frac{1}{q}$. If

$\sum\limits_{n=1}^{\infty} \psi(h(s_n(kT))) = 1$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} \psi(pk^{p-1}(s_n^{p-1}(T)) &= \sum_{n=1}^{\infty} cp^q k^{(pq-q)}(s_n^{(pq-q)}(T)) \\
&= \sum_{n=1}^{\infty} \frac{1}{p^{\frac{q}{p}}} \frac{1}{q} p^q k^p (s_n^p(T)) \\
&= \sum_{n=1}^{\infty} \frac{p}{q} k^p (s_n^p(T)) \\
&= \sum_{n=1}^{\infty} (p-1) k^p (s_n^p(T)) \\
&= 1
\end{aligned}$$

hence,

$$k^p = \frac{1}{(p-1) \sum\limits_{n=1}^{\infty} (s_n^p(T))}$$

or

$$k = \frac{1}{(p-1)^{\frac{1}{p}} \|T\|_p}.$$

Now one can get

$$\begin{aligned}
\|T\|_{\varphi}^{o} &= \frac{1}{k} \left[1 + \operatorname{tr}(\varphi(k(|T|)))\right] \\
&= (p-1)^{\frac{1}{p}} \|T\|_p \left[1 + \sum_{n=1}^{\infty} k^p (s_n^p(T))\right] \\
&= (p-1)^{\frac{1}{p}} \|T\|_p \left[1 + \frac{\sum\limits_{n=1}^{\infty} (s_n^p(T))}{(p-1) \sum\limits_{n=1}^{\infty} (s_n^p(T))}\right] \\
&= p(p-1)^{\frac{1}{p}-1} \|T\|_p \\
&= \frac{p}{(p-1)^{\frac{1}{q}}} \|T\|_p.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|y\|_\varphi &= \inf\left\{\lambda \geq 0,\ \operatorname{tr}\left(\psi\left(\frac{B}{\lambda}\right)\right) \leq 1\right\} \\
&= \inf\left\{\lambda \geq 0,\ \sum_{n=1}^{\infty}\left(\psi\left(\frac{s_n(B)}{\lambda}\right)\right) \leq 1\right\} \\
&= \inf\left\{\lambda \geq 0,\ \sum_{n=1}^{\infty} c\left(\frac{s_n^q(B)}{\lambda^q}\right) \leq 1\right\} \\
&= \inf\left\{\lambda \geq 0,\ c^{\frac{1}{q}}\sum_{n=1}^{\infty}(s_n^q(B))^{\frac{1}{q}} \leq \lambda\right\} \\
&= \frac{1}{p^{\frac{1}{p}}}\frac{1}{q^{\frac{1}{q}}}\|B\|_q.
\end{aligned}$$

Finally, by (4) of Lemma 3.1, we get

$$\operatorname{tr}(|TB|) \leq \|T\|_\varphi^o\|B\|_\varphi = \frac{p}{(p-1)^{\frac{1}{q}}}\|T\|_p \cdot \frac{1}{p^{\frac{1}{p}}}\frac{1}{q^{\frac{1}{q}}}\|B\|_q = \|T\|_p\|B\|_q,$$

or

$$\sum_{n=1}^{\infty} s_n(TB) \leq \left(\sum_{n=1}^{\infty} s_n^p(T)\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} s_n^q(B)\right)^{\frac{1}{q}}.$$

In particular, for $p = 2$, we get the Schwarz inequality

$$\operatorname{tr}(|TB|) \leq \|T\|_2\|B\|_2,$$

or

$$\sum_{n=1}^{\infty} s_n(TB) \leq \left(\sum_{n=1}^{\infty} s_n^2(T)\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} s_n^2(B)\right)^{\frac{1}{2}}.\square$$

Remark 2 : This corollary is just the Hölder Inequality of S_p which has been obtained by Ruskai in [17].

(4) If φ is invertible on $\mathbb{R}^+$, and $\varphi \in \delta_2(0)$, then

$$\|e_n \otimes e_n\|_\varphi = \frac{1}{\varphi^{-1}(1)}.$$

PROOF : Since $\|e_n \otimes e_n\| = 1$ and $e_n \otimes e_n$ is a one dimensional compact operator, where $\|\cdot\|$ is the operator norm, then $s_n(e_n \otimes e_n) = 1$ for any $n = 1, 2, \cdots$. By (1) of Theorem 3.1

$$1 = \operatorname{tr}\left(\varphi\left(\frac{e_n \otimes e_n}{\|e_n \otimes e_n\|_\varphi}\right)\right) = \varphi\left(s_n\left(\frac{e_n \otimes e_n}{\|e_n \otimes e_n\|_\varphi}\right)\right) = \varphi\left(\frac{1}{\|e_n \otimes e_n\|_\varphi}\right),$$

therefore, we can get the conclusion. $\square$

Example 3.1 : In [19], in order to study entropy in information geometry the author consider the quantum Orlicz function $\varphi(T) = coshT - 1$ with $T \geq 1$. It is obvious a Orlicz function and $\varphi \in \delta_2(0)$, then we have $\|e_n \otimes e_n\|_\varphi = \frac{1}{\ln(2+\sqrt{3})}$. In fact, since $\varphi(T) = coshT - 1$ we have $\varphi^{-1}(1) = \ln(2+\sqrt{3})$, by (3) of Corollary 3.1 we can get the conclusion.

***Theorem* 3.2** — *If $T \in \mathcal{B}(\mathcal{H})$ is a positive compact operator, then $T \in S_\varphi$ if and only if $\varphi(T) \in S_1$.*

PROOF : Since T is a positive compact operator, then

$$\varphi(T)\eta = \sum_{n=1}^{\infty} \varphi(\lambda_n)\langle \eta, e_n \rangle e_n, \ \ \eta \in \mathcal{H}$$

where $\{0\} \cup \{\varphi(\lambda_n)\}$ is spectral set. By the spectral mapping theorem, $\{\varphi(\lambda_n)\} = \sigma(\varphi(T))\backslash\{0\}$. Since φ is nondecreasing and $\varphi(\lambda_n)$ is the singular value sequence of $\varphi(T)$ we can get the conclusion. $\square$

***Theorem* 3.3** — *For any $T \in S_\varphi$, we have*

(1) $\|T\|_\varphi = \||T|\|_\varphi = \|T^*\|_\varphi$;

(2) *S_φ is a $*$-two sides ideal of $\mathcal{B}(\mathcal{H})$, which means that for any $B, C \in \mathcal{B}(\mathcal{H})$, one has $BTC, CTB \in S_\varphi$ and*

$$\|BTC\|_\varphi \leq \|B\|_\infty \|T\|_\varphi \|C\|_\infty.$$

PROOF : (1) By the definition of $\|\cdot\|_\varphi$ we can get $\|T\|_\varphi = \||T|\|_\varphi$. Let $T = U|T|$ be the polar decomposition of T. Then for any $\lambda > 0$ we have $U\left|\frac{T}{\lambda}\right|U^* = |\frac{T^*}{\lambda}|$, by the induction we get $U\left|\frac{T}{\lambda}\right|^n U^* = |\frac{T^*}{\lambda}|^n$, hence for any polynomial P we have $UP\left(\left|\frac{T}{\lambda}\right|\right)U^* = P\left(\left|\frac{T^*}{\lambda}\right|\right)$. Then for any continuous function φ, one get $U\left(\varphi\left(\left|\frac{T}{\lambda}\right|\right)\right)U^* = \varphi\left(\left|\frac{T^*}{\lambda}\right|\right)$. By the property of tr we get

$$\mathrm{tr}\left(U\varphi\left(\left|\frac{T}{\lambda}\right|\right)U^*\right) = \mathrm{tr}\left(\varphi\left(\left|\frac{T}{\lambda}\right|\right)\right) = \mathrm{tr}\left(\varphi\left(\left|\frac{T^*}{\lambda}\right|\right)\right),$$

which can deduce the conclusion.

(2) If $T \in S_\varphi$, then there exists a $\lambda_0 > 0$ such that

$$\mathrm{tr}(\varphi(\lambda_0 T)) = \sum_{n\geq 1} \varphi\left(\lambda_0 s_n(T)\right) < \infty.$$

By the inequality 2.3 of [9] and the nondecreasing property of φ, for any $B \in \mathcal{B}(\mathcal{H})$ let $\lambda_1 =$

$\frac{\lambda_0}{\|B\|_\infty} > 0$, we have

$$\begin{aligned} \mathrm{tr}(\varphi(\lambda_1 TB)) &= \sum_{n\geq 1} \varphi\left(\lambda_1 s_n(TB)\right) \\ &\leq \sum_{n\geq 1} \varphi\left(\lambda_1 s_n(T)\|B\|_\infty\right) \\ &= \sum_{n\geq 1} \varphi\left(\lambda_0 s_n(T)\right) \\ &< \infty. \end{aligned}$$

On the other hand, combing the inequality 2.3 of [9], the nondecreasing property of φ with (2) of Lemma 3.1, we have

$$\sum_{n\geq 1} \varphi\left(\frac{s_n(TB)}{\|T\|_\varphi \|B\|_\infty}\right) \leq \sum_{n\geq 1} \varphi\left(\frac{s_n(T)\|B\|_\infty}{\|T\|_\varphi \|B\|_\infty}\right) \leq 1.$$

By the definition of the Luxemburg we get

$$\|TB\|_\varphi \leq \|T\|_\varphi \|B\|_\infty.$$

Hence, one gets that S_φ is a right ideal of $\mathcal{B}(\mathcal{H})$. Furthermore using the equality 2.1 of [9] $(s_n(T) = s_n(T^*))$ one gets S_φ is a $*$-right ideal of $\mathcal{B}(\mathcal{H})$ and

$$\|TB\|_\varphi \leq \|T\|_\varphi \|B\|_\infty. \tag{1}$$

Then, by the equality 2.1 of [9] and using the similar methods above we get S_φ is a $*$-left ideal of $\mathcal{B}(\mathcal{H})$ and

$$\|BT\|_\varphi \leq \|B\|_\infty \|T\|_\varphi. \tag{2}$$

At last, for any $B, C \in \mathcal{B}(\mathcal{H})$, since $BT \in S_\varphi$ and $TC \in S_\varphi$ hence $BTC \in S_\varphi$ with

$$\|BTC\|_\varphi \leq \|B\|_\infty \|T\|_\varphi \|C\|_\infty.$$

Using (1) and (2) one can complete the proof. □

***Theorem* 3.4** — *$(E_\varphi)^* = S_\psi^o$ in the sense that for any $(E_\varphi)^*$, there is a unique $B \in S_\psi^o$ such that*

$$L_B(T) = \mathrm{tr}(TB) \ \ (T \in (E_\varphi)^*)$$

and the mapping $L_B \mapsto B$ is isometric from $(E_\varphi)^*$ to S_ψ^o.

PROOF : By [4] of Lemma 3.1, one get that $|L_B(T)| = |\mathrm{tr}(TB)| \leq \|B\|_\psi^o \|T\|_\varphi$ and $\|L_B\| \leq \|B\|_\psi^o$.

We define $\Phi : S_\psi^o \to (E_\varphi)^*$ with $\Phi(B) = L_B$, then Φ is a linear bounded operator. For any $\varepsilon > 0$, find a unit vector $e \in \mathcal{H}$ such that $\|Be\|_\psi^o \geq \|B\|_\psi^o - \varepsilon$. Find another unit vector $h \in \mathcal{H}$ such that $\|Be\|_\psi^o = \langle Be, h\rangle = \mathrm{tr}(B(e \otimes h))$, hence

$$L_B(e \otimes h) = \mathrm{tr}(B(e \otimes h)) = \langle Be, h\rangle = \|B\|_\psi^o - \varepsilon.$$

It is obvious $\|e \otimes h\|_\varphi = \|e\|\|h\| = 1$. Hence, one get $\|L_B\| \geq \|B\|_\psi^o$ which implies that $\|L_B\| = \|B\|_\psi^o$. So, the mapping $\Phi : S_\psi^o \mapsto (E_\varphi)^*$ is isometric.

Now, letting $M \in (E_\varphi)^*$ it is easy to verify $M(e \otimes h)$ is a bounded bilinear functional, because

$$|M(e \otimes h)| \leq \|M\|\|e \otimes h\| = \|M\|\|e\|\|h\|.$$

By Riesz theorem, there exists a $B \in S_\psi$ such that $M(e \otimes h) = \langle Be, h\rangle = \mathrm{tr}(B(e \otimes h))$. Hence for any finite rank operator F, we have $M(F) = \mathrm{tr}(BF)$. Since finite rank operator is dense on E_φ, then $M(T) = \mathrm{tr}(TB) = L_B(T)$ for any $T \in E_\varphi$, which means $M = T_B$ and $M : S_\psi^o \mapsto (E_\varphi)^*$ is an isometric isomorphism. Then we can get the conclusion. $\square$

Remark 3 : Using the similar method we also can get $S_\varphi = (E_\psi^o)^*$.

Combing the Theorems 2.1, 3.4 and Remark 1, one can gets the following corollary,

Corollary 3.2 — If $\varphi \in \delta_2(0)$ and $\psi \in \delta_2(0)$, S_φ is reflexive. Especially, S_p is reflexive for $p > 1$.

4. Application

The Bergman space $L_a^2(\mathbb{D})$ on open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is as following

$$L_a^2(\mathbb{D}) = \left\{ f(z) \text{ is analytic functions on } \mathbb{D} : \|f\|^2 = \frac{1}{\pi}\int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A(z) < \infty \right\},$$

where $\mathrm{d}A(z)$ is area metric. Then $L_a^2(\mathbb{D})$ is a Hilbert space which has an orthonormal basis $e_n(z) = \sqrt{n+1}z^n, n = 0, 1, 2, \cdots$. From the Example 5.17 of [10] we know, the Toeplitz operator $T_{1-|z|^2}$ on Bergman space is a compact self-adjoint operator and $T_{1-|z|^2} = \bigoplus\limits_{n=0}^{\infty} \frac{1}{n+2} e_n \otimes e_n$ is the spectral decomposition. Now we show $T_{1-|z|^2}$ belongs to some S_φ.

In fact, one could find some Orlicz functions φ with $\sum\limits_{n\geq 0} \varphi\left(\frac{1}{n+1}\right) < \infty$, then

$$\varphi\left(T_{1-|z|^2}\right) = \bigoplus_{n=0}^{\infty} \varphi\left(\frac{1}{n+2}\right) e_n \otimes e_n$$

and

$$\operatorname{tr}\left(\varphi\left(T_{1-|z|^2}\right)\right)=\sum_{n\geq 0}\varphi\left(\frac{1}{n+1}\right)<\infty.$$

Hence, $T_{1-|z|^2}\in S_\varphi$. It is obvious, $T_{1-|z|^2}$ does not belong to S_1 but $S_p,\ p>1$.

Generally speaking, it is difficulty to calculate the norm of the $\|T_{1-|z|^2}\|_\varphi$ since the Orlicz function is abstract. However, if $\varphi\in\delta_2(0)$, by [1] of Theorem 3.1, the norm of $T_{1-|z|^2}$ satisfies

$$1=\sum_{n\geq 0}\varphi\left(\frac{1}{(n+2)\|T_{1-|z|^2}\|_\varphi}\right).$$

Especially, if $\varphi=|x|^p,\ p>1$, one can get

$$\|T_{1-|z|^2}\|_p=\left[\sum_{n\geq 0}\frac{1}{(n+2)^p}\right]^{\frac{1}{p}}=\left(\zeta(p)-1\right)^{\frac{1}{p}},$$

where $\zeta(p)$ is the Riemann function.

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