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# TAUTOLOGICAL ALGEBRA OF THE MODULI SPACE OF SEMISTABLE BUNDLES ON AN ELLIPTIC CURVE

Arijit Mukherjee

*School of Mathematics and Statistics, University of Hyderabad,*
*Prof. C. R. Rao Road, P.O - Central University,*
*Gachibowli, Hyderabad* 500 046, *India*
*e-mails: mukherjee90.arijit@gmail.com*

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In this paper, our aim is to find the relations amongst the cohomology classes of Brill-Noether subvarieties of the moduli space of semistable bundles over an elliptic curve. We obtain results similar to the Poincaré relations on a Jacobian variety.

**Key words** : Jacobian varieties; elliptic curves; vector bundles; moduli spaces.

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## 1. INTRODUCTION

Suppose $C$ is a smooth projective variety of genus $g$, defined over the complex numbers. The Picard variety $Pic^d(C)$ is the moduli space of isomorphism classes of line bundles of degree $d$ on $C$. For $1 \leq i \leq g-1$, there are naturally defined subvarieties, namely the Brill-Noether subvarieties $W_i$ of $Pic^i(C)$, which parametrize line bundles $L$ with $h^0(L) \geq 1$. Let $\Theta$ denote the theta divisor $W_{g-1} \subset Pic^{g-1}(C)$. Choosing isomorphisms $Pic^i(C) \simeq J(C)$, the classical Poincaré relations determine the relations between the cohomological classes of $W_i$, in $J(C)$:

$$[W_i] = \frac{1}{(g-i)!}[\Theta]^{g-i} \in H^*(J(C), \mathbb{Q}).$$

See [1, Ch 1, §5, p. 25].

Our aim in this paper is to investigate relations between the cohomology classes of the Brill-Noether subvarieties in the moduli space of higher rank vector bundles on a curve. Brill-Noether subvarieties are more subtle in this situation and not much seems to be known in this direction. We consider genus one case in this paper.

Let $E$ denote a complex elliptic curve. The moduli space of semistable bundles over $E$ has been computed by Atiyah (cf. [2]). Tu (cf. [6]) explicitly described the Brill-Noether subvarieties in this moduli space. In this paper, a *tautological class* is the cohomology class of Brill-Noether subvariety in the cohomology ring and the subalgebra of the cohomology ring generated by those tautological classes is called *tautological algebra*. We describe the relations amongst the tautological classes in the cohomology ring of the moduli space in this paper.

We denote by $f$ the *strong equivalence class* of $F$, a vector bundle over $E$, in the sense of [5, §8, p. 333]. Let $\mathcal{M}_E(r,d)$ be the moduli space of strong equivalence classes of semistable bundles of rank $r$ and degree $d$ over $E$. Let $X$ be a subvariety of $\mathcal{M}_E(r,d)$(or of $\mathcal{SU}_E(r,L)$), then by $[X]$ we denote the cohomology class of $X$ in $H^*(\mathcal{M}_E(r,d))$ (respectively in $H^*(\mathcal{SU}_E(r,L))$).

By [6], we note that the Brill-Noether loci are trivial for positive degree vector bundles (either empty or the whole moduli space), and for line bundles of degree $0$ (either empty or singleton). Therefore, we consider the Brill-Noether loci when degree of a vector bundle is $0$ and the rank is more than $1$. Let $L$ be a line bundle of degree $0$ and let $i$ be any non-negative integer. The Brill-Noether loci inside $\mathcal{SU}_E(r,L)$ are defined as follows:

$$W^i_{r,L}(\exists) := \{f \in \mathcal{SU}_E(r,L) \mid h^0(F) \geq i+1 \text{ for some } F \in f\}.$$

We denote by $[W^i_{r,L}(\exists)]$, a tautological class in $H^*(\mathcal{SU}_E(r,L),\mathbb{Z})$. We also consider the Brill-Noether loci, denoted by $W^i_{r,0}(\exists)$, inside $\mathcal{M}_E(r,0)$, defined as follows:

$$W^i_{r,0}(\exists) := \{f \in \mathcal{M}_E(r,0) \mid h^0(F) \geq i+1 \ \text{ for some } F \in f\}.$$

(See [6]).

In §5, we prove the main theorems in $H^*(\mathcal{SU}_E(r,L),\mathbb{Z})$ and $H^*(\mathcal{M}_E(r,0),\mathbb{Z})$ on the relations amongst the tautological classes. We show:

***Theorem* 1.1** — *Let $r$ be any positive integer and let $L$ be a degree $0$ line bundle over $E$. Then $W^0_{r,L}(\exists)$ is a divisor inside $\mathcal{SU}_E(r,L)$. Moreover, in $H^*(\mathcal{SU}_E(r,L),\mathbb{Z})$ we have,*

$$[W^i_{r,L}(\exists)] = [W^0_{r,L}(\exists)]^{i+1} \tag{1}$$

*for all $0 \leq i \leq r-2$ and the tautological algebra of $\mathcal{SU}_E(r,L)$ is $\mathbb{Z}[\zeta]/\langle\zeta^r\rangle$, where $\zeta$ is the cohomology class of $W^0_{r,L}(\exists)$ in $H^*(\mathcal{SU}_E(r,L),\mathbb{Z})$.*

Furthermore, we show that the relations amongst $[W^i_{r,0}(\exists)]$'s in $H^*(\mathcal{M}_E(r,0),\mathbb{Z})$, is similar to (1). In particular, we show:

***Theorem* 1.2** — *The tautological algebra of $\mathcal{M}_E(r,0)$ is*

$$H^*(E)\otimes\mathbb{Z}[\xi]/\langle\xi^r\rangle.$$

*Here $\xi$ is the cohomology class of the divisor $W^0_{r,0}(\exists)$ on $\mathcal{M}_E(r,0)$ in $H^*(\mathcal{M}_E(r,0),\mathbb{Z})$.*

It is interesting to investigate if similar relations hold in higher genus case.

## 2. NOTATIONS

All the varieties are defined over the complex numbers.

1) $\mathcal{SU}_E(r,L)$ denotes the moduli space of strong equivalence classes of semistable bundles of rank $r$ and fixed determinant $L$ of degree $d$ over $E$.

2) $S^nE$ is the $n$th symmetric product of the curve $E$.

3) $J(E)$ is the Jacobian variety of isomorphism classes of line bundles of degree $0$ over $E$.

4) $J_d(E)$ is the moduli space of isomorphism classes of line bundles of degree $d$ over $E$.

5) $detF$ is the determinant bundle of a semistable vector bundle $F$ over $E$.

6) $\mathcal{O}(D)$ denotes the line bundle corresponding to a divisor $D$ on $E$.

7) $\mathcal{O}_E$ is the trivial line bundle over $E$.

8) $h^i(E,F)$ (or $h^i(F)$) denotes the dimension of $H^i(E,F)$ for a semistable bundle $F$ over $E$.

9) By $H^*(X)$ we mean $H^*(X,\mathbb{Z})$, the cohomology ring of a complex variety $X$ with integral coefficients.

## 3. BRILL-NOETHER LOCI OF SEMISTABLE BUNDLES OVER AN ELLIPTIC CURVE

### 3.1 *Bundles with positive degree*

In this section we recall a few results from [6] which will be relevant for the next section. The following lemma is proved in [6]. For the sake of completeness we include the lemma with proof here.

*Lemma* 3.1 — [6, Lemma 17, p. 13]. Any semistable bundle $F$ of positive degree over $E$ is non-special, that is, $h^1(F)=0$.

PROOF : Let $F$ be a semistable vector bundle of degree $d>0$ and $K_E$ be the canonical line bundle over $E$. Then $K_E\cong\mathcal{O}_E$ as $E$ is an elliptic curve. By Serre duality we have

$$h^1(F)=h^0(K_E\otimes F^*)=h^0(F^*).$$

As $F^*$ is also a semistable bundle and of negative degree, $h^0(F^*) = 0$. Therefore, $h^1(F) = 0$. Moreover by Riemann-Roch theorem, $h^0(F) = d$. □

*Remark* 3.2 : A consequence of Lemma 3.1 is that the map

$$h^0 : \mathcal{M}_E(r,d) \longrightarrow \mathbb{Z}_+ \cup \{0\}$$
$$f \mapsto h^0(F)$$

is well defined for $d > 0$, and is the constant function $d$.

*Definition* 3.3 — Let $d > 0$ and $i \geq 0$ be any two integers. The *Brill-Noether loci* are defined by

$$W^i_{r,d} := \{f \in \mathcal{M}_E(r,d) \mid h^0(F) \geq i+1\}.$$

This definition is well defined by Remark 3.2.

The following lemma is a direct consequence of Lemma 3.1. See [6, p. 13].

*Lemma* 3.4 — Let $d > 0$. Then

$$W^i_{r,d} \cong \begin{cases} \emptyset & \text{if } 1 \leq d \leq i; \\ \mathcal{M}_E(r,d) & \text{if } d \geq i+1. \end{cases}$$

Therefore Brill-Noether loci inside $\mathcal{M}_E(r,d)$ are not of much interest when $d > 0$. Moreover for degree $0$ line bundles over $E$ we have the following result.

*Lemma* 3.5 — The Brill-Noether loci for $d = 0$, $r = 1$ are

$$W^i_{1,0} \cong \begin{cases} \emptyset & \text{if } 1 \leq i; \\ \{\mathcal{O}_E\} & \text{if } i = 0. \end{cases}$$

PROOF : See [6, p. 13]. As $h^0(L) = 0$ or $1$ for a line bundle $L$ of degree zero over $E$ and moreover $h^0(L) = 1$ if and only if $L \cong \mathcal{O}_E$.

### 3.2 *Degree zero bundles*

When $d = 0$, $h^0 : \mathcal{M}_E(r,0) \longrightarrow \mathbb{Z}_+ \cup \{0\}$ is not well defined. For example, let $F_2$ be the *Atiyah's indecomposable bundle* of rank 2 and $I_2$ be the trivial bundle of rank 2. Then $F_2 \cong I_2$, but $h^0(F_2) = 1 \neq 2 = h^0(I_2)$, (cf. [2, Proof of Theorem 5, p. 432]).

In this case, Brill-Noether loci inside $\mathcal{SU}_E(r,L)$ and $\mathcal{M}_E(r,0)$ are defined, see (cf. [6, p. 5]). Here $L$ is a line bundle on $E$ of degree $0$.

*Definition* 3.6 —

$$W^i_{r,L}(\exists) := \{f \in \mathcal{SU}_E(r,L) \mid h^0(F) \geq i+1 \text{ for some } F \in f\}.$$
$$W^i_{r,0}(\exists) := \{f \in \mathcal{M}_E(r,0) \mid h^0(F) \geq i+1 \text{ for some } F \in f\}.$$

We have the equality:

$$W^i_{r,L}(\exists) = W^i_{r,0}(\exists) \cap \mathcal{SU}_E(r,L).$$

We now have the following isomorphism of moduli spaces.

*Proposition* 3.7 — Let $r > 1$ be a positive integer and $L$ a degree $0$ line bundle over $E$. Then we have the isomorphisms:

$$\begin{aligned} \mathcal{M}_E(r,0) &\cong S^r E \\ \mathcal{SU}_E(r,L) &\cong \mathbb{P}^{r-1} \\ J(E) &\cong E \\ W^i_{r,0}(\exists) &\cong S^{r-i-1}E \\ W^i_{r,L}(\exists) &\cong \mathbb{P}^{r-i-2}. \end{aligned}$$

PROOF : See [6, Theorem 2, 3, 4 and 5; Corollary 15; p. 4-5, 10]. □

## 4. TAUTOLOGICAL ALGEBRA OF SEMISTABLE BUNDLES OF DEGREE ZERO OVER AN ELLIPTIC CURVE

Consider the map:

$$\begin{aligned} \pi : J(E) \times \mathcal{SU}_E(r,L) &\to \mathcal{M}_E(r,d) \\ (l,F) &\mapsto l \otimes F. \end{aligned} \tag{2}$$

See [3, p. 338] and [4, p. 348].

This map as in (2) suggests that the cohomology subalgebra generated by $[W^i_{r,0}(\exists)]$'s is the same as that generated by $[W^i_{r,L}(\exists)]$'s with coefficients lying in $H^*(J(E))$. We make this precise in the next section.

Let $det : \mathcal{M}_E(r,0) \to J(E)$ be the *determinant* map which sends $F$ to $detF$ and $\alpha : S^r E \to J_r(E)$ be the *Abel-Jacobi* map defined as $x_1 + x_2 + \cdots + x_r \mapsto \mathcal{O}(x_1 + x_2 + \cdots + x_r)$. Then we have the following commutative diagram (cf. [6, p. 12]), which says that the *determinant* map and the *Abel-Jacobi* map can be identified.

$$\begin{array}{ccc} \mathcal{M}_E(r,0) & \xrightarrow{\cong} & S^rE \\ \big\downarrow{\scriptstyle det} & & \big\downarrow{\scriptstyle \alpha} \\ J(E) & \xrightarrow{\cong} & J_r(E) \end{array} \tag{3}$$

*Remark* 4.1 : By *Abel*'s theorem (cf. [1, p. 18]), fiber of the map $\alpha$ over any line bundle $L \in J_r(E)$ is the complete linear system $\mid D \mid$ of a divisor $D$ on $E$ with $\mathcal{O}(D) = L$. Now if $r > 0$, then by Serre duality $h^1(E, \mathcal{O}(D)) = 0$ and by Riemann-Roch theorem $h^0(E, \mathcal{O}(D)) = r$. Therefore each fiber of the map $\alpha : S^rE \to J_r(E)$ is isomorphic to $\mathbb{P}^{r-1}$ if $r > 0$. This is also another reason to work in $\mathcal{M}_E(r, 0)$ with $r > 0$.

Let $det : W^i_{r,0}(\exists) \to J(E)$ be the restriction of the *determinant* map $det : \mathcal{M}_E(r,0) \to J(E)$. Then we have the following commutative diagram (cf. [6, p. 16]) similar to (3).

$$\begin{array}{ccc} W^i_{r,0}(\exists) & \xrightarrow{\cong} & S^{r-i-1}E \\ \big\downarrow{\scriptstyle det} & & \big\downarrow{\scriptstyle \alpha} \\ J(E) & \xrightarrow{\cong} & J_{r-i-1}(E) \end{array} \tag{4}$$

Here we assume $0 \leq i \leq r - 2$. So by Remark 4.1, the fiber of $\alpha$ is isomorphic to $\mathbb{P}^{r-i-2}$.

## 5. Main Theorems

In this section we define the tautological subalgebra of $H^*(\mathcal{M}_E(r,0), \mathbb{Z})$ and $H^*(\mathcal{SU}_E(r,L), \mathbb{Z})$ and prove our main theorems.

*Definition* 5.1 — The cohomology classes $[W^i_{r,0}(\exists)] \in H^*(\mathcal{M}_E(r,0), \mathbb{Z})$ are called the *tautological classes*. The subalgebra of $H^*(\mathcal{M}_E(r,0), \mathbb{Z})$ generated by these tautological classes is called the *tautological subalgebra* of $H^*(\mathcal{M}_E(r,0), \mathbb{Z})$.

*Definition* 5.2 — Let $L$ be a degree zero line bundle over $E$. Then the cohomology classes $[W^i_{r,L}(\exists)] \in H^*(\mathcal{SU}_E(r,L), \mathbb{Z})$ are called the *tautological classes*. The subalgebra of $H^*(\mathcal{SU}_E(r,L), \mathbb{Z})$ generated by these tautological classes is called the *tautological subalgebra* of $H^*(\mathcal{SU}_E(r,L), \mathbb{Z})$.

Following theorem shows that the tautological class $\zeta := [W^0_{r,L}(\exists)]$ is the generator of the tautological subalgebra of $H^*(\mathcal{SU}_E(r,L), \mathbb{Z})$.

***Theorem* 5.3** — *Let $r$ be any positive integer and let $L$ be a degree $0$ line bundle over $E$. Then $W^0_{r,L}(\exists)$ is a divisor inside $\mathcal{SU}_E(r,L)$. Moreover, in $H^*(\mathcal{SU}_E(r,L), \mathbb{Z})$ we have,*

$$[W^i_{r,L}(\exists)] = [W^0_{r,L}(\exists)]^{i+1}$$

*for all* $0 \leq i \leq r-2$ *and the tautological algebra of* $\mathcal{SU}_E(r,L)$ *is* $\mathbb{Z}[\zeta]/\langle\zeta^r\rangle$*, where* $\zeta$ *is the cohomology class of* $W^0_{r,L}(\exists)$ *in* $H^*(\mathcal{SU}_E(r,L),\mathbb{Z})$*.*

PROOF : We have the following stratification inside $\mathcal{SU}_E(r,L)$ by Proposition 3.7.

$$\begin{array}{ccccc}
\mathcal{SU}_E(r,L) & \cong & \mathbb{P}^{r-1} & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^0_{r,L}(\exists) & \cong & \mathbb{P}^{r-2} & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^1_{r,L}(\exists) & \cong & \mathbb{P}^{r-3} & & \\
\cup\!\mid & & \cup\!\mid & & \\
. & & . & & \\
. & & . & & \\
. & & . & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^{r-3}_{r,L}(\exists) & \cong & \mathbb{P}^{1} & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^{r-2}_{r,L}(\exists) & \cong & \mathbb{P}^{0} & \cong & \{\cdot\}
\end{array}$$

So, $W^0_{r,L}(\exists)$ is a subvariety of $\mathcal{SU}_E(r,L)$ of codimension 1 and hence a divisor. We can calculate relations between $[\mathbb{P}^i]$'s as follows. Inside $\mathbb{P}^{r-1}$ we have the following stratification:

$$\{\cdot\} \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq \cdots \subseteq \mathbb{P}^{r-2} \subseteq \mathbb{P}^{r-1}.$$

Then we have:

$$H^*(\mathbb{P}^{r-1},\mathbb{Z}) = \frac{\mathbb{Z}[\zeta]}{\langle\zeta^r\rangle} \tag{5}$$

where $\zeta$ is the cohomology class of $\mathbb{P}^{r-2}$, that is, $\zeta = [\mathbb{P}^{r-2}] = c_1(\mathcal{O}(1))$, $c_1(\mathcal{O}(1))$ being the first Chern class of $\mathcal{O}(1)$ over $\mathbb{P}^{r-1}$. Moreover in $H^*(\mathbb{P}^{r-1},\mathbb{Z})$, we have:

$$[\mathbb{P}^{r-1-k}] = \zeta^k. \tag{6}$$

Therefore, by (5) and Proposition 3.7 we get:

$$H^*(\mathcal{SU}_E(r,L),\mathbb{Z}) = \frac{\mathbb{Z}[\zeta]}{\langle\zeta^r\rangle}.$$

Furthermore, by (6) and Proposition 3.7 we get the following equality in $H^*(\mathcal{SU}_E(r,L),\mathbb{Z})$:

$$[W^i_{r,L}(\exists)] = [\mathbb{P}^{r-i-2}] = [\mathbb{P}^{r-1-(i+1)}] = \zeta^{i+1} = [\mathbb{P}^{r-2}]^{i+1} = [W^0_{r,L}(\exists)]^{i+1}.$$

Hence the theorem follows.

The next theorem is about some relations between the generators of the tautological subalgebra of $H^*(\mathcal{SU}_E(r,L),\mathbb{Z})$ and $H^*(\mathcal{M}_E(r,0),\mathbb{Z})$. The theorem says that tautological algebra of $\mathcal{M}_E(r,0)$ is generated by the cohomology class of the Brill-Noether subvariety $W^0_{r,0}(\exists)$ as an $H^*(E)$-algebra.

***Theorem* 5.4** — *The tautological algebra of $\mathcal{M}_E(r,0)$ is*

$$H^*(E)\otimes\mathbb{Z}[\xi]/\langle\xi^r\rangle.$$

*Here $\xi$ is the cohomology class of the divisor $W^0_{r,0}(\exists)$ on $\mathcal{M}_E(r,0)$ in $H^*(\mathcal{M}_E(r,0),\mathbb{Z})$.*

PROOF : We have the following stratification inside $\mathcal{M}_E(r,0)$ by Proposition 3.7.

$$\begin{array}{ccccc}
\mathcal{M}_E(r,0) & \cong & S^rE & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^0_{r,0}(\exists) & \cong & S^{r-1}E & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^1_{r,0}(\exists) & \cong & S^{r-2}E & & \\
\cup\!\mid & & \cup\!\mid & & \\
. & & . & & \\
. & & . & & \\
. & & . & & \\
\cup\!\mid & & \cup\!\mid & & \\
W^{r-2}_{r,0}(\exists) & \cong & S^1E & \cong & E
\end{array}$$

So, $W^0_{r,0}(\exists)$ is a subvariety of $\mathcal{M}_E(r,0)$ of codimension $1$ and hence a divisor. By (3) and by Remark 4.1, the determinant morphism

$$\mathcal{M}_E(r,0)\to J(E)$$

is a projective bundle $\mathbb{P}^{r-1}_{J(E)}\to J(E)$.

Hence, by projective bundle formula,

$$H^*(\mathcal{M}_E(r,0),\mathbb{Z})\;=\;H^*(J(E))\otimes\mathbb{Z}[\xi]/\langle\xi^r\rangle. \tag{7}$$

Here $\xi$ is the first Chern class of $\mathcal{O}(1)$ on $\mathbb{P}^{r-1}_{J(E)}$.

Therefore by (7) and Proposition 3.7 we get,

$$H^*(\mathcal{M}_E(r,0),\mathbb{Z})\;=\;H^*(E)\otimes\mathbb{Z}[\xi]/\langle\xi^r\rangle.$$

However, by (4), we have the equality of the cohomology classes:

$$[W^i_{r,0}(\exists)] = \xi^{i+1}$$

for all $0 \leq i \leq r-2$. This gives the assertion.

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## REFERENCES

1. E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, *Geometry of Algebraic Curves.* Vol. I, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, New York., **267** (1985).
2. M. F. Atiyah, Vector bundles over an elliptic curve, *Proc. London Math. Soc. (3)*, **7** (1957), 414-452.
3. M. Bigas and L. Tu, Theta divisors for vector bundles, *Contemp. Math.*, **136** (1992), 327-342.
4. R. Donagi and L. Tu, Theta functions for $SL(n)$ versus $GL(n)$, *Math. Res. Lett. 1*, **3** (1994), 345-357.
5. C. S. Seshadri, Space of unitary vector bundles on a compact Riemann surface, *Ann. of. Math. (2)*, **85** (1967), 303-336.
6. L. Tu, Semistable bundles over an elliptic curve, *Adv. Math. 98*, **1** (1993), 1-26.