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# RUMOUR PROPAGATION AMONG SCEPTICS: THE MARKOVIAN CASE

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We consider a model of the spread of rumour among sceptical individuals. Let $X_0, X_1, \ldots$ be a $\{0,1\}$-valued Markov chain and $\rho_0, \rho_1, \ldots$ a sequence of i.i.d. $\mathbb{N}$ valued random variables independent of the Markov chain. An individual located at site $i \in \mathbb{N}^* := \mathbb{N} \cup \{0\}$ spreads the rumour to the individuals located in the interval $[i, i+\rho_i]$ provided (i) $X_i = 1$ and (ii) if s/he has received the rumour from at least two distinct sources $j, k < i$ with $X_j = X_k = 1$. To start the process we place two individuals at locations $-1$ and $-2$, each of spread the rumour to a distance $\rho_{-1}$ and $\rho_{-2}$ respectively to the right of itself. Here $\rho_{-1}$ and $\rho_{-2}$ are i.i.d. copies of $\rho_0$. This extends the work of Sajadi and Roy [7] who considered the case when $X_0, X_1, \ldots$ is a sequence of i.i.d. $\{0,1\}$ valued random variables, i.e. the believers $\{i : X_i = 1\}$ and the disbelievers $\{i : X_i = 0\}$ are located in an i.i.d. fashion. Here we study the case when the the believers and the disbelievers are located in a Markovian fashion.

**Key words** : Rumour process; Markov chain; complete coverage; eventual coverage; firework process.

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## 1. Introduction

Sudbury [8] introduced a model to study the spread of a rumour among individuals. Subsequently Junior, Machado and Zuluaga [4] christened this model as the 'firework process'. For more recent work on this model see [2, 3, 5]. Sajadi and Roy [7] tweaked this model to include sceptics, i.e. individuals who require to hear the rumour from at least two sources before they spread it.

Athreya *et al*. [1] studied two different models, both related to rumor percolation. The first model arises from genomic analysis and the second complements the results about the complete coverage in stochastic geometry. In genomics, there is a method to determine the nucleic acid sequence of molecules. After obtaining many identical copies of the DNA sequence, each copy is fragmented into many random segments. Then to determine longer sequences, they are joined together by overlapping. The question of interest is what should be the random length of these segments to ensure that they cover the sequence significantly.

We now proceed to describe the model before we return to the discussion about the results.

Let $X_0, X_1, \ldots$ be a $\{0,1\}$-valued Markov chain and $\rho_0, \rho_1, \ldots$ a sequence of i.i.d. $\mathbb{N}$ valued random variables independent of the Markov chain. In addition let $\rho_{-1}$ and $\rho_{-2}$ be to i.i.d. copies of $\rho_0$. Let

$$C := \bigcup_{i \geq 0, X_i = 1} ([i + [0, \rho_i]) \cup [-1, -1 + \rho_{-1}] \cup [-2, -2 + \rho_{-2}]\,,$$

be the 'covered region', and

$$\begin{aligned} D := \{x \in [0, \infty) : &\text{there exist } j,\ k \geq -1 \text{ with } j \neq k,\ X_j = X_k = 1 \\ &\text{and } x \in ([j, j + \rho_j] \cap [k, k + \rho_k])\}, \end{aligned} \tag{1}$$

the 'doubly covered region'.

An individual located at site $i$ is one of two types. S/he is either a disbeliever, which is when $X_i = 0$, or a believer, which is when $X_i = 1$. S/he receives the rumour if $i \in C$. The construction of the random region $C$ assumes that on receiving the rumour, the believer at $i$ spreads it to a distance $\rho_i$ to the right. Whereas, for the construction of the random region $D$, we think of the believer at site $i$ being a sceptic. S/he will spread the rumour only if s/he heard the rumour from two different sources.

Thus the connected component of $C$ containing the site $0$ describes the set of individuals who received the rumour when each believer spreads the rumour once s/he receives it. While the connected component of $D$ containing the site $0$ describes the set of individuals who received the rumour when each believer is a sceptic and spreads the rumour once s/he receives it from two distinct sources. If this connected component of $C$ is finite, then we say that the rumour dies out among believers, and if the connected component of $D$ is finite, then we say that the rumour dies out among sceptics.

When $X_0, X_1, \ldots$ is a sequence of i.i.d Bernoulli ($p$) random variables, both models of spreading rumour among sceptics and non-sceptics (when individuals accept the rumour if they receive it at least from one individual) have been studied in the literature [4, 7].

Here we study the case when the sequence $X_0, X_1, X_2, \ldots$ is a $\{0,1\}$-valued time homogeneous Markov chain with the transition probabilities $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i), i, j = 0, 1$, for all $n \geq 0$ and $\rho, \rho_{-1}, \rho_{-2}, \rho_0, \rho_1, \ldots$ is a sequence of independent and identically distributed positive integer valued random variables, independent of the Markov chain.

For non-sceptical believers, in context of stochastic geometry, the model of rumour spread when the sequence $X_0, X_1, \ldots$ is a Markov chain, has been considered in [1]. They considered this as a problem of stochastic geometry and defined a notion of *eventual coverage* as follows: $\mathbb{N}$ is eventually covered by $C$ if there exists $t \geq 1$ such that $[t, \infty) \subseteq C$.

Sajadi and Roy [7], introduced the notion of rumour propagation among sceptics. They showed the following equivalence

$$\mathbb{P}(D = [0, \infty)) > 0 \Longleftrightarrow \mathbb{P}(D \supseteq [t, \infty), \text{ for some } t \geq 1) = 1. \tag{2}$$

Also, for a sequence of i.i.d. random variables $X_0, X_1, \ldots$ they showed that the rumour model survives among sceptics if and only if it survives among non-sceptic believers, i.e. $\mathbb{P}(D = [0, \infty)) > 0$ if and only if $\mathbb{P}(C = [0, \infty)) > 0$. In the next section we discuss the known results of this model and state our main result.

## 2. The Results

Let $l := \liminf_{j \to \infty} j\mathbb{P}(\rho > j) > 1$ and $L := \limsup_{j \to \infty} j\mathbb{P}(\rho > j) < \infty$.

In the case $X_0, X_1, \ldots$ is an i.i.d. sequence of random variables with $\mathbb{P}(X_0 = 1) = p = 1 - \mathbb{P}(X_0 = 0)$, Athreya *et al.* [1] and, independently, Junior *et al.* [4] showed that

***Theorem* 1** — *For* $0 < p < 1$, *we have*

$$\mathbb{P}(C\ is\ unbounded) \begin{cases} > 0 & if\ p > 1/l \\ = 0 & if\ p < 1/L. \end{cases}$$

*and Sajadi and Roy (2019) showed that*

***Theorem* 2** — *For* $0 < p < 1$, *we have*

$$\mathbb{P}(D\ is\ unbounded) \begin{cases} > 0 & if\ p > 1/l \\ = 0 & if\ p < 1/L. \end{cases}$$

In the case $X_0, X_1, \ldots$ is a Markov chain with $\mathbb{P}(X_{n+1} = j | X_n = i) = p_{ij}$, for $i, j \in \{0, 1\}$ and all $n \geq 0$ in [1], Athreya *et al.*, showed that

***Theorem* 3** — *For $0 < p_{00}, p_{10} < 1$, we have*

$$\mathbb{P}(C\ is\ unbounded) \begin{cases} > 0 & if\ \frac{p_{01}}{p_{10}+p_{01}} > 1/l \\ = 0 & if\ \frac{p_{01}}{p_{10}+p_{01}} < 1/L. \end{cases}$$

Here we consider the Markov case and show that the same condition as in Theorem 3 guarantees double coverage, in particular.

*Proposition* 4 — Assume that $0 < p_{00}, p_{10} < 1$. We have

$$\mathbb{P}(D \text{ is unbounded}) \begin{cases} > 0 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} > 1/l \\ = 0 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} < 1/L. \end{cases}$$

To conclude this section, just as in [7], the above proposition also holds for 'k-coverage' instead of just double coverage.

## 3. Proof of Proposition 4

From the equivalence given in (2) we need to show that

$$\mathbb{P}(D \supseteq [t,\infty) \text{ for some } t \geq 0) = \begin{cases} 1 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} > 1/l \\ 0 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} < 1/L. \end{cases}$$

Let $F$ be the cumulative distribution function of $\rho$, $\bar{F} = 1 - F$ and for each $k \in \mathbb{N}$, let $A_k := \{k \notin C\}, B_k := \{k \notin D\}$. For $k \geq 1$ let

$$\mathbb{P}_i(B_k) := \mathbb{P}(B_k|X_0 = i), i = 0, 1\,.$$

We first show that

$$\sum_k \mathbb{P}_1(B_k) < \infty \text{ and } \sum_k \mathbb{P}_0(B_k) < \infty, \text{whenever } \frac{p_{01}}{p_{10}+p_{01}} > 1/l\,. \tag{3}$$

A simple conditioning argument and Markov property yield the following relations:

$$\mathbb{P}_0(B_{k+1}) = p_{00}\mathbb{P}_0(B_{k+1}|X_1 = 0) + p_{01}\mathbb{P}_0(B_{k+1}|X_1 = 1)\,, \tag{4}$$

and

$$\mathbb{P}_1(B_{k+1}) = p_{10}\mathbb{P}_1(B_{k+1}|X_1 = 0) + p_{11}\mathbb{P}_1(B_{k+1}|X_1 = 1)\,. \tag{5}$$

Because of the Markov property, the conditional probability in the first term of the right hand side of equation (4), i.e., $\mathbb{P}_0(B_{k+1}|X_1 = 0)$ is same as the conditional probability of $B_k$ given that $X_0 = 0$:

$$\mathbb{P}_0(B_{k+1}|X_1 = 0) = \mathbb{P}_0(B_k)\,.$$

That means, when for both individuals located at sites $0$ and $1$, the value of random variable $X$ is $0$, the probability that the individual located at site $k+1$ is not in the doubly covered region $D$, is same as the probability that the individual located at site $k$ is not in $D$, provided $X_0 = 0$.

For the second conditional probability in the right side of equation (4), according to whether $\rho_1 \geq k$ or $\rho_1 < k$, we have

$$\mathbb{P}_0(B_{k+1}|X_1 = 1) = \mathbb{P}_0(B_{k+1}, \rho_1 \geq k|X_1 = 1) + \mathbb{P}_0(B_{k+1}, \rho_1 < k|X_1 = 1)\,.$$

Therefore, equation (4) can be written as

$$\begin{aligned} \mathbb{P}_0(B_{k+1}) &= p_{00}\mathbb{P}_0(B_{k+1}|X_1 = 0) + p_{01}\mathbb{P}_0(B_{k+1}|X_1 = 1) \\ &= p_{00}\mathbb{P}_0(B_k) + p_{01}\left[\mathbb{P}_0(B_{k+1}, \rho_1 \geq k|X_1 = 1) + \mathbb{P}_0(B_{k+1}, \rho_1 < k|X_1 = 1)\right] \\ &= p_{00}\mathbb{P}_0(B_k) + p_{01}\left[\mathbb{P}_1(A_k)\bar{F}(k-1) + \mathbb{P}_1(B_k)F(k-1)\right] \end{aligned} \tag{6}$$

Similarly, for the conditional probabilities in the right side of equation (5), we have

$$\begin{aligned} \mathbb{P}_1(B_{k+1}|X_1 = 0) &= \mathbb{P}_1(B_{k+1}, \rho_0 \geq k+1|X_1 = 0) + \mathbb{P}_1(B_{k+1}, \rho_0 < k+1|X_1 = 0) \\ &= \mathbb{P}_0(A_k)\bar{F}(k) + \mathbb{P}_0(B_k)F(k)\,, \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}_1(B_{k+1}|X_1 = 1) &= \mathbb{P}_1(B_{k+1}, \rho_0 < k+1, \rho_1 < k|X_1 = 1) \\ &\quad + \mathbb{P}_1(B_{k+1}, \rho_0 < k+1, \rho_1 \geq k|X_1 = 1) \\ &\quad + \mathbb{P}_1(B_{k+1}, \rho_0 \geq k+1, \rho_1 < k|X_1 = 1) \\ &= \left[\mathbb{P}_1(B_k)F(k)F(k-1) + \mathbb{P}_1(A_k)F(k)\bar{F}(k-1) + \mathbb{P}_1(A_k)\bar{F}(k)F(k-1)\right]\,. \end{aligned}$$

Rewriting equation (5), we have the following recurrence formula:

$$\begin{aligned} \mathbb{P}_1(B_{k+1}) &= p_{10}\left[\mathbb{P}_0(A_k)\bar{F}(k) + \mathbb{P}_0(B_k)F(k)\right] \\ &\quad + p_{11}\left[\mathbb{P}_1(B_k)F(k)F(k-1) + \mathbb{P}_1(A_k)F(k)\bar{F}(k-1) + \mathbb{P}_1(A_k)\bar{F}(k)F(k-1)\right]\,. \end{aligned} \tag{7}$$

For $k_0 \geq 1$ to be chosen later, let $\Phi_i(s) := \sum_{k=k_0}^{\infty} \mathbb{P}_i(B_k)s^k$and $\Psi_i(s) := \sum_{k=k_0}^{\infty} \mathbb{P}_i(A_k)s^k$, for $i = 0, 1$. To establish (3), we need to show that $\Phi_0(1) < \infty$ and $\Phi_1(1) < \infty$ whenever $\frac{p_{01}}{p_{10}+p_{01}} > 1/l$.

Multiplying (6) by $s^{k+1}$ and summing over $k \geq k_0$, we have

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_0(B_{k+1}) \\
&= p_{00} \sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_0(B_k) + p_{01} \sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_1(A_k)\bar{F}(k-1) + p_{01} \sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_1(B_k)F(k-1)\,. \qquad (8)
\end{aligned}$$

Rewriting equation (8) in terms of functions $\Phi$ and $\mathbb{P}si$, we get

$$\Phi_0(s) - s^{k_0}\mathbb{P}_0(B_{k_0}) = p_{00}s\Phi_0(s) + p_{01} \sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_1(A_k)\bar{F}(k-1) + p_{01} \sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_1(B_k)F(k-1)\,,$$

or equivalently,

$$\Phi_0(s) = \frac{s^{k_0}\mathbb{P}_0(B_{k_0}) + p_{01}\sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_1(A_k)\bar{F}(k-1) + p_{01}\sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_1(B_k)F(k-1)}{(1-p_{00}s)}\,. \qquad (9)$$

Since $\bar{F}(k-1) \leq 1$ and $F(k-1) \leq 1$, we get from (9),

$$(1-p_{00}s)\Phi_0(s) \leq s^{k_0}\mathbb{P}_0(B_{k_0}) + p_{01}s\mathbb{P}si_1(s) + p_{01}s\Phi_1(s),$$

and consequently

$$\Phi_0(s) \leq \frac{s^{k_0}\mathbb{P}_0(B_{k_0}) + p_{01}s\mathbb{P}si_1(s) + p_{01}s\Phi_1(s)}{1-p_{00}s}\,. \qquad (10)$$

Now multiplying (7) by $s^{k+1}$ and summing over $k \geq k_0$, we have that

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} s^{k+1}\mathbb{P}_1(B_{k+1}) \\
&= p_{10} \sum_{k=k_0}^{\infty} s^{k+1}\bar{F}(k)\mathbb{P}_0(A_k) + p_{10} \sum_{k=k_0}^{\infty} s^{k+1}F(k)\mathbb{P}_0(B_k) \\
&+ p_{11} \sum_{k=k_0}^{\infty} s^{k+1}\bar{F}(k-1)F(k)\mathbb{P}_1(A_k) + p_{11} \sum_{k=k_0}^{\infty} s^{k+1}\bar{F}(k)F(k-1)\mathbb{P}_1(A_k) \\
&+ p_{11} \sum_{k=k_0}^{\infty} s^{k+1}F(k)F(k-1)\mathbb{P}_1(B_k)\,. \qquad (11)
\end{aligned}$$

Each of the above power series is uniformly convergent for $|s| < 1$ and hence differentiating (11) term by term with respect to $s$ we obtain that, when $|s| < 1$,

$$\begin{aligned}\sum_{k=k_0}^{\infty}(k+1)s^k\mathbb{P}_1(B_{k+1}) &= p_{10}\sum_{k=k_0}^{\infty}(k+1)s^k\bar{F}(k)\mathbb{P}_0(A_k) + p_{10}\sum_{k=k_0}^{\infty}(k+1)s^kF(k)\mathbb{P}_0(B_k)\\ &+ p_{11}\sum_{k=k_0}^{\infty}(k+1)s^k\bar{F}(k-1)F(k)\mathbb{P}_1(A_k)\\ &+ p_{11}\sum_{k=k_0}^{\infty}(k+1)s^kF(k-1)\bar{F}(k)\mathbb{P}_1(A_k)\\ &+ p_{11}\sum_{k=k_0}^{\infty}(k+1)s^kF(k-1)F(k)\mathbb{P}_1(B_k)\,. \end{aligned} \tag{12}$$

Let $\epsilon > 0$ be given such that $C := l - \epsilon > 0$, where $l$ is as in the statement of the theorem. There exists a $k_0$ such that,

$$k_0 + (1-C) > 0, \quad P_0(A_{k_0}) > 0, \quad P_1(A_{k_0}) > 0, \quad P_0(B_{k_0}) > 0, \quad P_1(B_{k_0}) > 0,$$

and for all $k \geq k_0$,

$$F(k-1) \leq F(k) \leq 1 - \frac{C}{k+1} \text{ and } F(k-1)F(k) \leq 1 - \frac{C}{k+1}.$$

Using these bounds for $F(k-1), F(k)$ and $F(k-1)F(k)$ in (12), we have

$$\begin{aligned}\sum_{k=k_0}^{\infty}(k+1)s^k\mathbb{P}_1(B_{k+1}) &\leq p_{10}\sum_{k=k_0}^{\infty}(k+1)s^k\mathbb{P}_0(A_k)\\ &+ p_{10}\sum_{k=k_0}^{\infty}(k+1)s^k[1-\frac{C}{k+1}]\mathbb{P}_0(B_k)\\ &+ p_{11}\sum_{k=k_0}^{\infty}(k+1)s^k[1-\frac{C}{k+1}]\mathbb{P}_1(A_k)\\ &+ p_{11}\sum_{k=k_0}^{\infty}(k+1)s^k[1-\frac{C}{k+1}]\mathbb{P}_1(A_k)\\ &+ p_{11}\sum_{k=k_0}^{\infty}(k+1)s^k[1-\frac{C}{k+1}]\mathbb{P}_1(B_k). \end{aligned}$$

Let $\mathbb{P}si_0', \mathbb{P}si_1', \Phi_0'$ and $\Phi_1'$ be the derivatives of $\mathbb{P}si_0, \mathbb{P}si_1, \Phi_0$ and $\Phi_1$ with respect to $s$. Therefore, the above inequality can be written as

$$\begin{aligned}\Phi_1'(s) - k_0 s^{k_0-1}\mathbb{P}_1(B_{k_0}) &\leq p_{10} s \mathbb{P}si_0'(s) + p_{10}\mathbb{P}si_0(s) \\ &+ p_{10}(s\Phi_0'(s) + (1-C)\Phi_0(s)) + p_{11}(s\mathbb{P}si_1'(s) + (1-C)\mathbb{P}si_1(s)) \\ &+ p_{11}(s\mathbb{P}si_1'(s) + (1-C)\mathbb{P}si_1(s)) + p_{11}(s\Phi_1'(s) + (1-C)\Phi_1(s)).\end{aligned} \tag{13}$$

Using (9) we have,

$$\begin{aligned}\Phi_0'(s) &= \frac{k_0 s^{k_0-1}\mathbb{P}_0(B_{k_0})}{(1-p_{00}s)^2} \\ &+ \frac{\left[p_{01}\sum_{k=k_0}^{\infty}(k+1)s^k\mathbb{P}_1(A_k)\bar{F}(k-1) + p_{01}\sum_{k=k_0}^{\infty}(k+1)s^k\mathbb{P}_1(B_k)F(k-1)\right](1-p_{00}s)}{(1-p_{00}s)^2} \\ &+ \frac{p_{00}\left[s^{k_0}\mathbb{P}_0(B_{k_0}) + p_{01}\sum_{k=k_0}^{\infty}s^{k+1}\mathbb{P}_1(A_k)\bar{F}(k-1) + p_{01}\sum_{k=k_0}^{\infty}s^{k+1}\mathbb{P}_1(B_k)F(k-1)\right]}{(1-p_{00}s)^2}.\end{aligned} \tag{14}$$

Since $F(k-1) < 1$ and $\bar{F}(k-1) < 1$, we obtain

$$\begin{aligned}\Phi_0'(s) &\leq \frac{[k_0 s^{k_0-1}\mathbb{P}_0(B_{k_0}) + p_{01}\mathbb{P}si_1(s) + p_{01}s\mathbb{P}si_1'(s) + p_{01}\Phi_1(s) + p_{01}s\Phi_1'(s)](1-p_{00}s)}{(1-p_{00}s)^2} \\ &+ \frac{p_{00}\left[s^{k_0}\mathbb{P}_0(B_{k_0}) + p_{01}s\mathbb{P}si_1(s) + p_{01}s\Phi_1(s)\right]}{(1-p_{00}s)^2}.\end{aligned} \tag{15}$$

Substituting the upper bound of $\Phi_0$ (see the equation (10)) and the upper bound for $\Phi_0'$ (see equation (15)) in (13), we get

$$P(s)\Phi_1'(s) \leq Q(s)\Phi_1(s) + R(s), \tag{16}$$

where

$$P(s) := 1 - \frac{p_{01}p_{10}s^2(1-p_{00}s)}{(1-p_{00}s)^2} - p_{11}s,$$

$$Q(s) := \frac{p_{01}p_{10}s(1-p_{00}s)}{(1-p_{00}s)^2} + \frac{p_{00}p_{01}p_{10}s^2}{(1-p_{00}s)^2} + \frac{(1-C)p_{01}p_{10}s}{(1-p_{00}s)} + (1-C)p_{11}$$

and

$$\begin{aligned}
R(s) := &\; k_0 s^{k_0-1}\mathbb{P}_1(B_{k_0}) + p_{10}s\mathbb{P}si_0'(s) + p_{10}\mathbb{P}si_0(s) \\
&+ \frac{p_{10}k_0 s^{k_0}\mathbb{P}_0(B_{k_0}) + p_{10}sp_{01}\mathbb{P}si_1(s) + p_{10}p_{01}s^2\mathbb{P}si_1'(s)}{1-p_{00}s} \\
&+ \frac{p_{10}p_{00}s^{k_0+1}\mathbb{P}_0(B_{k_0}) + p_{01}p_{10}s^2 p_{00}\mathbb{P}si_1(s)}{(1-p_{00}s)^2} \\
&+ (1-C)\frac{p_{10}s^{k_0}\mathbb{P}_0(B_{k_0}) + p_{10}p_{01}s\mathbb{P}si_1(s)}{1-p_{00}s} \\
&+ 2p_{11}s\mathbb{P}si_1'(s) + 2p_{11}(1-C)\mathbb{P}si_1(s).
\end{aligned}$$

Multiplying both sides of (16) by $(1-p_{00}s)^2$, we have that

$$\bar{P}(s)\Phi_1'(s) \leq \bar{Q}(s)\Phi_1(s) + \bar{R}(s)\,, \tag{17}$$

where

$$\begin{aligned}
&\bar{P}(s) = (1-p_{00}s)(1-s)[1-s(1-p_{01}-p_{10})]\,, \\
&\bar{Q}(s) = (1-p_{00}s)^2(1-C)p_{11} + (1-C)p_{10}p_{01}s(1-p_{00}s) \\
&\;+ p_{10}sp_{01}(1-p_{00}s) + p_{10}p_{00}p_{01}s^2\,, \\
&\bar{R}(s) = (1-p_{00}s)^2 R(s).
\end{aligned}$$

To solve the differential equation (17), we follow the method in [1]. For any $0 < t < 1$, we have

$$\Phi_1(t) \leq \mathrm{e}^{\int_0^t \frac{\bar{Q}(s)}{\bar{P}(s)}ds} \int_0^t \mathrm{e}^{\int_0^t \frac{\bar{Q}(r)}{\bar{P}(r)}dr}\frac{\bar{R}(s)}{\bar{P}(s)}ds\,. \tag{18}$$

Observe that for $s < 1$,

$$\frac{\bar{Q}(s)}{\bar{P}(s)} = \frac{U}{1-p_{00}s} + \frac{V}{1-s} + \frac{W}{1-s(1-p_{01}-p_{10})}\,,$$

for some real numbers $U$, $V$ and $W$ and since $p_{00} < 1$ and $-1 < 1-(p_{01}+p_{10}) < 1$, we get

$$\int_0^t \frac{\bar{Q}(s)}{\bar{P}(s)}ds = \ln\left\{(1-p_{00}t)^{\frac{-U}{p_{00}}}(1-t)^{-V}(1-t(1-p_{01}-p_{10}))^{\frac{-W}{1-p_{01}-p_{10}}}\right\}\,.$$

Now if we show that $\bar{P}(s) > 0$ and $\bar{R}(s)$ is bounded above by some constant, then by result in [1], we have for all $0 < \alpha \leq s < t < 1$,

$$\Phi_1(t) \leq \text{(constant)}[\frac{(1-t)^{-V}(1-\alpha)^V}{V} - \frac{1}{V}]\,. \tag{19}$$

Since $0 < p_{00} < 1$ and $-1 < 1-(p_{01}+p_{10}) < 1$ whenever $0 < s < 1$, both $1-p_{00}$ and $1-s(1-p_{01}-p_{10})$ are bounded above by $1$ and below by a strictly positive constant. Hence, $\bar{P}(s) > 0$. For $s < 1$,

$$\bar{R}(s) \leq h + m(\mathbb{P}si_0(s) + \mathbb{P}si_0'(s) + \mathbb{P}si_1(s) + \mathbb{P}si_1'(s)),$$

where $h$ and $m$ are positive constants. However, $\mathbb{P}si_0, \mathbb{P}si_1$ and their derivatives are finite. From [1], we have

$$\bar{P}(s)\mathbb{P}si_1'(s) \leq \bar{Q}(s)\mathbb{P}si_1(s) + T(s)\,,$$

where

$$\begin{aligned} T(s) &= (1-p_{00}s)^2 k_0 s^{k_0-1}\mathbb{P}_1(A_{k_0}) + (k_0+1-C)p_{10}s^{k_0}(1-p_{00}s)\mathbb{P}_0(A_{k_0}) \\ &\quad + p_{10}s^{k_0+1}p_{00}\mathbb{P}_0(A_{k_0})\,. \end{aligned}$$

Observe that $T(s)$ is bounded above by some constant $t_1 < \infty$ for $s < 1$. Therefore,

$$\mathbb{P}si_1'(s) \leq \frac{\bar{Q}(s)}{\bar{P}(s)}\mathbb{P}si_1(s) + \frac{t_1}{\bar{P}(s)}\,.$$

Athreya *et al.* [1], proved that for $s < 1, \mathbb{P}si_1(s) \leq \mathbb{P}si_1(1) < \infty$. Thus from above inequality, we get $\mathbb{P}si_1'(s) < \infty$ since $\frac{\bar{Q}}{\bar{P}}$ and $T$ are bounded from above. On the other hand, we have

$$\mathbb{P}si_0(s) = \frac{s^{k_0}\mathbb{P}_0(A_{k_0}) + p_{01}s\mathbb{P}si_1(s)}{1-p_{00}s},$$

and

$$\mathbb{P}si_0'(s) = \frac{[k_0 s^{k_0-1}\mathbb{P}_0(A_{k_0}) + p_{01}\mathbb{P}si_1(s) + p_{01}s\mathbb{P}si_1'(s)](1-p_{00}s) + p_{00}(s^{k_0}\mathbb{P}_0(A_{k_0}) + p_{01}s\mathbb{P}si_1(s))}{(1-p_{00}s)^2}\,.$$

Therefore both $\mathbb{P}si_0(s)$ and $\mathbb{P}si_0'(s)$ are finite (since they include finite $\mathbb{P}si_1(s)$ and $\mathbb{P}si_1'(s)$) and hence we have that $\bar{R}(s)$ is bounded above by some constant.

Note that $\Phi_1(1)$ in (19) is finite whenever $V < 0$. But,

$$\begin{aligned} V &= \frac{\bar{Q}(1)}{(1-p_{00})(p_{01}+p_{10})} \\ &= \frac{(1-p_{00})(p_{10}+(1-C)p_{01})}{(1-p_{00})(p_{01}+p_{10})} \\ &= \frac{p_{10}+(1-C)p_{01}}{p_{01}+p_{10}}\,. \end{aligned}$$

Therefore $V < 0$ if and only if $p_{10} + (1 - C)p_{01} < 0$. That is $V < 0$ if and only if

$$\frac{p_{01}}{p_{01} + p_{10}} > \frac{1}{C} = \frac{1}{l - \epsilon}.$$

Since $\epsilon$ is arbitrary, we have $\Phi_1(1) < \infty$ whenever $\frac{p_{01}}{p_{01}+p_{10}} > \frac{1}{l}$. It should be noted that the condition cannot be relaxed because to study $A_k$ we need the full force of the condition. From the Proof of Theorem 1.1(a) in [1] we have $\mathbb{P}si_1(1) < \infty$ whenever $\frac{p_{01}}{p_{01}+p_{10}} > \frac{1}{l}$. Now from (10) we have that $\Phi_0(1)$ is also finite. Therefore the Borel-Cantelli lemma implies that $\mathbb{P}(D \supseteq [t, \infty)$ for some $t \geq 0) = 1$ whenever $\frac{p_{01}}{p_{10}+p_{01}} > 1/l$.

For the proof of second part of proposition, we use the result of Athreya *et al*. [1] that says whenever $\frac{p_{01}}{p_{01}+p_{10}} < \frac{1}{L}$ the single coverage of $\mathbb{N}$ does not happen and therefore double coverage also will not happen, i.e., $\mathbb{P}(D \supseteq [t, \infty)$ for some finite $t) = 0$.

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