

A VARIATION OF L^p UNCERTAINTY PRINCIPLES IN WEINSTEIN SETTING

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The Weinstein operator has several applications in pure and applied Mathematics especially in Fluid Mechanics and satisfies some uncertainty principles similar to the Euclidean Fourier transform. The aim of this paper is establish a generalization of uncertainty principles for Weinstein transform in L^p_α -norm. Firstly, we extend the Heisenberg-Pauli-Weyl uncertainty principle to more general case. Then we establish three continuous uncertainty principles of concentration type. The first and the second uncertainty principles are L^p_α versions and depend on the sets of concentration Ω and Σ , and on the time function φ . However, the third uncertainty principle is also L^p_α version depends on the sets of concentration and he is independent on the band limited function φ . These L^p_α -Donoho-Stark-type inequalities generalize the results obtained in the case $p = q = 2$.

Key words : Weinstein operator; Heisenberg-Pauli-Weyl-type inequality; concentration uncertainty principles; Donoho-Stark-type inequality.

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1. INTRODUCTION

Uncertainty principles (UP) appear in harmonic analysis and signal theory in a various different forms involving not only the signal φ and its Fourier transform $\mathcal{F}(\varphi)$, but fundamentally every representation of a signal in the time-frequency space. The (UP) are mathematical results that give limitations on the simultaneous concentration of a signal and its Fourier transform and they have implications in signal analysis and quantum physics. In quantum physics they tell us that a particle's speed and position cannot both of them be measured with infinite precision. In signal analysis they tell us that

if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consists of. Timelimited functions and bandlimited functions are basic tools of signal and image processing. Like, the simple form of the uncertainty principle tells us that a signal cannot be simultaneously time and bandlimited. This leads to the inquiry of the set of almost time and almost bandlimited functions, which has been initially by Landau, Pollak [8, 9] and then by Donoho, Stark [3].

Motivated by the work of Laeng and Morpurgo [7, 12] and the work of Soltani and Ghazwani [15, 16], we propose an extension of the techniques and results of Ghobber [5, 6] to establish a variation of the L^p uncertainty principles for the Weinstein transform.

Many uncertainty principles have already been proved for the Weinstein transform $\mathcal{F}_{W,\alpha}$, [10, 11, 14]. The authors have established in [11] the Heisenberg-Pauli-Weyl inequality for the Weinstein transform, by showing that, for every φ in $L^2_\alpha(\mathbb{R}^{d+1}_+)$

$$\|\varphi\|_{\alpha,2} \leq \frac{2}{2\alpha + d + 2} \|x|\varphi\|_{\alpha,2} \|y|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,2}. \quad (1.1)$$

Our first result will be the following variation of Heisenberg-Pauli-Weyl-type inequality.

Theorem A — Let $1 < p \leq 2$, $q = p/(p-1)$, $0 < s < (2\alpha + d + 2)/q$ and $t > 0$, then for all $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$,

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq K(s, t) \|x|^s \varphi\|_{\alpha,p}^{\frac{t}{s+t}} \|y|^t \mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^{\frac{s}{s+t}},$$

where $K(s, t)$ is a positive constant.

This theorem implies in particular that if φ is highly localized in the neighbourhood of $x = 0$, then $\mathcal{F}_{W,\alpha}(\varphi)$ cannot be concentrated in the neighbourhood of $y = 0$. In particular, for $p = q = 2$, we obtain the general case of Heisenberg-Pauli-Weyl-type inequality (1.1)

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,2} \leq K(s, t) \|x|^s \varphi\|_{\alpha,2}^{\frac{t}{s+t}} \|y|^t \mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,2}^{\frac{s}{s+t}}.$$

The second and third results are two continuous-time uncertainty principles of concentration type depend on the sets of concentration Ω and Σ , and on the time function φ .

Theorem B — Let Ω, Σ be a measurable subsets of $\mathbb{R}^{d+1}_+$ and $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1 \leq p \leq 2$. If φ is ε_Ω -concentrated to Ω in $L^p_\alpha(\mathbb{R}^{d+1}_+)$ -norm and $\mathcal{F}_{W,\alpha}(\varphi)$ is ε_Σ -concentrated to Σ in $L^q_\alpha(\mathbb{R}^{d+1}_+)$ -norm, $q = p/(p-1)$, then we have

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \frac{(\mu_\alpha(\Sigma))^{\frac{1}{q}} (\mu_\alpha(\Omega))^{\frac{1}{q}} + \varepsilon_\Omega}{1 - \varepsilon_\Sigma} \|\varphi\|_{\alpha,p}.$$

The statement of Theorem B depends on the time function φ . However although for $p = q = 2$, the continuous-time uncertainty principle becomes

$$1 - \varepsilon_\Omega - \varepsilon_\Sigma \leq \mu_\alpha(\Omega)^{\frac{1}{2}} \mu_\alpha(\Sigma)^{\frac{1}{2}}.$$

Theorem C — Let Ω, Σ be a measurable subsets of $\mathbb{R}_+^{d+1}$ and $\varphi \in (L_\alpha^1 \cap L_\alpha^p)(\mathbb{R}_+^{d+1})$, $1 \leq p \leq 2$. If φ is ε_Ω -concentrated to Ω in $L_\alpha^p(\mathbb{R}_+^{d+1})$ -norm and $\mathcal{F}_{W,\alpha}(\varphi)$ is ε_Σ -concentrated to Σ in $L_\alpha^q(\mathbb{R}_+^{d+1})$ -norm, $q = p/(p-1)$, then we have

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \frac{(\mu_\alpha(\Sigma))^{\frac{1}{q}} (\mu_\alpha(\Omega))^{\frac{1}{q}}}{(1 - \varepsilon_\Omega)(1 - \varepsilon_\Sigma)} \|\varphi\|_{\alpha,p}.$$

As in Theorem B, the statement of the pervious Theorem depends on the time function φ and we have for $p = q = 2$,

$$(1 - \varepsilon_\Omega)(1 - \varepsilon_\Sigma) \leq \mu_\alpha(\Omega)^{\frac{1}{2}} \mu_\alpha(\Sigma)^{\frac{1}{2}}.$$

The last result is a continuous-bandlimited uncertainty principle of concentration type depends on the sets of concentration Ω and Σ , but it is independent on the bandlimited function.

Theorem D — Let Ω, Σ be a measurable subsets of $\mathbb{R}_+^{d+1}$ and let $\varphi \in L^p(\mathbb{R}_+^d)$ such that $1 \leq p \leq 2$. Then if φ is ε_Ω -concentrated to Ω and ε_Σ -bandlimited to Σ in L_α^p -norm, we have

$$1 - \varepsilon_\Omega - \varepsilon_\Sigma \leq (1 + \varepsilon_\Sigma) (\mu_\alpha(\Sigma))^{\frac{1}{p}} (\mu_\alpha(\Omega))^{\frac{1}{p}}.$$

The main body of the paper is organized as follows. In Section 2, we recall some harmonic analysis results related to the Weinstein operator. In Section 3, we prove a variation of Heisenberg-Pauli-Weyl uncertainty principle for the Weinstein operator. Finally in Section 4, we establish three continuous uncertainty principles of concentration type in L_α^p -norm. These estimations depend on the sets of concentration Ω and Σ , and on the band-limited function φ , only the last estimation is independent on φ .

2. PRELIMINAIRES

The Weinstein operator $\Delta_{W,\alpha}^d$ defined on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0, \infty)$, by

$$\Delta_{W,\alpha}^d = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha, \quad \alpha > -1/2,$$

where Δ_d is the Laplacian operator for the d first variables and L_α is the Bessel operator for the last variable defined on $(0, \infty)$ by

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}}.$$

The Weinstein operator $\Delta_{W,\alpha}^d$ has several applications in pure and applied mathematics, especially in fluid mechanics [1, 18].

For all $\lambda = (\lambda_1, \dots, \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the system

$$\begin{aligned} \frac{\partial^2 u}{\partial x_j^2}(x) &= -\lambda_j^2 u(x), \quad \text{if } 1 \leq j \leq d \\ L_\alpha u(x) &= -\lambda_{d+1}^2 u(x), \\ u(0) &= 1, \quad \frac{\partial u}{\partial x_{d+1}}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad \text{if } 1 \leq j \leq d \end{aligned} \tag{2.1}$$

has a unique solution denoted by $\Lambda_\alpha^d(\lambda, \cdot)$, and given by

$$\Lambda_\alpha^d(\lambda, x) = e^{-i\langle x', \lambda' \rangle} j_\alpha(x_{d+1} \lambda_{d+1}) \tag{2.2}$$

where $x = (x', x_{d+1})$, $\lambda = (\lambda', \lambda_{d+1})$ and j_α is the normalized Bessel function of index α defined by

$$j_\alpha(x) = \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! \Gamma(\alpha + k + 1)}.$$

The function $(\lambda, x) \mapsto \Lambda_\alpha^d(\lambda, x)$ is called the Weinstein kernel and has a unique extension to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$, and satisfied the following properties.

(i) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have

$$\Lambda_\alpha^d(\lambda, x) = \Lambda_\alpha^d(x, \lambda).$$

(ii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have

$$\Lambda_\alpha^d(\lambda, -x) = \Lambda_\alpha^d(-\lambda, x).$$

(iii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we get

$$\Lambda_\alpha^d(\lambda, 0) = 1.$$

(iv) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}^{d+1}$ and $\lambda \in \mathbb{C}^{d+1}$ we have

$$\left| D_\lambda^\nu \Lambda_\alpha^d(\lambda, x) \right| \leq \|x\|^{|\nu|} e^{\|x\| \|\Im \lambda\|}$$

where $D_\lambda^\nu = \partial^\nu / (\partial \lambda_1^{\nu_1} \dots \partial \lambda_{d+1}^{\nu_{d+1}})$ and $|\nu| = \nu_1 + \dots + \nu_{d+1}$. In particular, for all $(\lambda, x) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}$, we have

$$\left| \Lambda_\alpha^d(\lambda, x) \right| \leq 1. \quad (2.3)$$

In the following we denote by

- (i) $C_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- (ii) $S_*(\mathbb{R}^{d+1})$, the space of the C^∞ functions, even with respect to the last variable, and rapidly decreasing together with their derivatives.
- (iii) $L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}_+^{d+1}$ such that

$$\|f\|_{\alpha, p} = \left(\int_{\mathbb{R}_+^{d+1}} |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < \infty, \quad p \in [1, \infty),$$

$$\|f\|_{\alpha, \infty} = \text{ess sup}_{x \in \mathbb{R}_+^{d+1}} |f(x)| < \infty,$$

where $d\mu_\alpha(x)$ is the measure on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times (0, \infty)$ given by

$$d\mu_\alpha(x) = \frac{x_{d+1}^{2\alpha+1}}{(2\pi)^d 2^{2\alpha} \Gamma^2(\alpha+1)} dx.$$

For a radial function $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1})$ the function $\tilde{\varphi}$ defined on $\mathbb{R}_+$ such that $\varphi(x) = \tilde{\varphi}(|x|)$, for all $x \in \mathbb{R}_+^{d+1}$, is integrable with respect to the measure $r^{2\alpha+d+1} dr$, and we have

$$\int_{\mathbb{R}_+^{d+1}} \varphi(x) d\mu_\alpha(x) = a_\alpha \int_0^\infty \tilde{\varphi}(r) r^{2\alpha+d+1} dr, \quad (2.4)$$

where

$$a_\alpha = \frac{1}{2^{\alpha+\frac{d}{2}} \Gamma(\alpha + \frac{d}{2} + 1)}.$$

The Weinstein transform generalizing the usual Fourier transform, is given for $\varphi \in L_\alpha^1(\mathbb{R}_+^{d+1})$ and $\lambda \in \mathbb{R}_+^{d+1}$, by

$$\mathcal{F}_{W, \alpha}(\varphi)(\lambda) = \int_{\mathbb{R}_+^{d+1}} \varphi(x) \Lambda_\alpha^d(x, \lambda) d\mu_\alpha(x),$$

We list some known basic properties of the Weinstein transform are as follows. For the proofs, we refer [1, 13].

(i) For all $\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$, the function $\mathcal{F}_{W,\alpha}(\varphi)$ is continuous on $\mathbb{R}^{d+1}_+$ and we have

$$\|\mathcal{F}_{W,\alpha}\varphi\|_{\alpha,\infty} \leq \|\varphi\|_{\alpha,1}. \quad (2.5)$$

(ii) The Weinstein transform is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1}_+)$ onto itself. The inverse transform is given by

$$\mathcal{F}_{W,\alpha}^{-1}\varphi(\lambda) = \mathcal{F}_{W,\alpha}\varphi(-\lambda), \text{ for all } \lambda \in \mathbb{R}^{d+1}_+. \quad (2.6)$$

(iii) Parseval formula: For all $\varphi, \phi \in \mathcal{S}_*(\mathbb{R}^{d+1}_+)$, we have

$$\int_{\mathbb{R}^{d+1}_+} \varphi(x) \overline{\phi(x)} d\mu_\alpha(x) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_{W,\alpha}(\varphi)(x) \overline{\mathcal{F}_{W,\alpha}(\phi)(x)} d\mu_\alpha(x).$$

(v) Plancherel formula: For all $\varphi \in L^2_\alpha(\mathbb{R}^{d+1}_+)$, we have

$$\|\mathcal{F}_{W,\alpha}\varphi\|_{\alpha,2} = \|\varphi\|_{\alpha,2}. \quad (2.7)$$

(vi) Plancherel Theorem: The Weinstein transform $\mathcal{F}_{W,\alpha}$ extends uniquely to an isometric isomorphism on $L^2_\alpha(\mathbb{R}^{d+1}_+)$.

(vii) Inversion formula: Let $\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$ such that $\mathcal{F}_{W,\alpha}\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)$, then we have

$$\varphi(\lambda) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_{W,\alpha}\varphi(x) \Lambda_\alpha^d(-\lambda, x) d\mu_\alpha(x), \text{ a.e. } \lambda \in \mathbb{R}^{d+1}_+. \quad (2.8)$$

Using relations (2.5) and (2.7) with Marcinkiewicz's interpolation theorem [17] we deduce that for every $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ for all $1 \leq p \leq 2$, the function $\mathcal{F}_{W,\alpha}(\varphi) \in L^q_\alpha(\mathbb{R}^{d+1}_+)$, $q = p/(p-1)$, and

$$\|\mathcal{F}_{W,\alpha}\varphi\|_{\alpha,q} \leq \|\varphi\|_{\alpha,p}. \quad (2.9)$$

3. L^p -HEISENBERG-PAULI-WEYL INEQUALITY

In this section, we extend the Heisenberg-Pauli-Weyl uncertainty principle (1.1) to more general case for the Weinstein operator. We need to use the method of Ciatti *et al.* [3], which is the counterpart in the Euclidean case. In the following we need this lemma.

Lemma 3.1 — Let $1 < p \leq 2$, $q = p/(p-1)$ and $0 < s < (2\alpha + d + 2)/q$. Then for all $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ and $z > 0$,

$$\left\| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi) \right\|_{\alpha,q} \leq \left(1 + \frac{K_\alpha}{(2q)^{(\alpha + \frac{1}{d} + 1)\frac{1}{q}}} \right) z^{-s/2} \| |x|^s \varphi \|_{\alpha,p}, \quad (3.1)$$

where

$$K_\alpha = \left((2\alpha + d + 2 - qs) 2^{\alpha + \frac{d}{2}} \Gamma(\alpha + \frac{d}{2} + 1) \right)^{-1/q}$$

PROOF : Let $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$. The inequality (3.1) holds if $\| |x|^s \varphi \|_{\alpha,p} = \infty$. Let us now assume that $\| |x|^s \varphi \|_{\alpha,p} < \infty$. For $\rho > 0$, we put $B_\rho = \{x \in \mathbb{R}^{d+1}_+ : |x| < \rho\}$ and $B_\rho^c = \mathbb{R}^{d+1}_+ \setminus B_\rho$. Denote by χ_{B_ρ} and $\chi_{B_\rho^c}$ the characteristic functions. Let $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$, $1 < p \leq 2$ and $q = p/(p-1)$. Like that

$$|\varphi \chi_{B_\rho^c}(x)| \leq \rho^{-s} |x|^s |\varphi(x)|,$$

then, by inequality (2.9), we get

$$\begin{aligned} \left\| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi \chi_{B_\rho^c}) \right\|_{\alpha,q} &\leq \left\| e^{-z|y|^2} \right\|_{\alpha,\infty} \left\| \mathcal{F}_{W,\alpha}(\varphi \chi_{B_\rho^c}) \right\|_{\alpha,q} \\ &\leq \left\| e^{-z|y|^2} \right\|_{\alpha,\infty} \left\| \varphi \chi_{B_\rho^c} \right\|_{\alpha,p} \\ &\leq \left\| e^{-z|y|^2} \right\|_{\alpha,\infty} \rho^{-s} \| |x|^s \varphi \|_{\alpha,p}. \end{aligned}$$

On the other hand, according to (2.5) and Hölder's inequality, we obtain

$$\begin{aligned} \left\| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi \chi_{B_\rho}) \right\|_{\alpha,q} &\leq \left\| e^{-z|y|^2} \right\|_{\alpha,q} \left\| \mathcal{F}_{W,\alpha}(\varphi \chi_{B_\rho}) \right\|_{\alpha,\infty} \\ &\leq \left\| e^{-z|y|^2} \right\|_{\alpha,q} \left\| \varphi \chi_{B_\rho} \right\|_{\alpha,1} \\ &\leq \left\| e^{-z|y|^2} \right\|_{\alpha,q} \| |x|^{-s} \chi_{B_\rho} \|_{\alpha,q} \| |x|^s \varphi \|_{\alpha,p}. \end{aligned}$$

According to integral relationship (2.4), we have the following identity

$$\left\| e^{-z|y|^2} \right\|_{\alpha,q} = \frac{1}{(2q)^{(\alpha + \frac{d}{2} + 1)}} z^{-(\alpha + \frac{d}{2} + 1)} \quad \text{and} \quad \| |x|^{-s} \chi_{B_\rho} \|_{\alpha,q} = K_\alpha \rho^{-s + (2\alpha + d + 2)/q}.$$

Hence, we get

$$\left\| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi \chi_{B_\rho}) \right\|_{\alpha,q} \leq \frac{C_\alpha}{(2q)^{(\alpha + \frac{d}{2} + 1)}} z^{-(\alpha + \frac{d}{2} + 1)} \rho^{-s + (2\alpha + d + 2)/q} \| |x|^s \varphi \|_{\alpha,p},$$

and

$$\begin{aligned} \left\| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi) \right\|_{\alpha,q} &\leq \left\| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi \chi_{B_\rho}) \right\|_{\alpha,q} + \left\| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi \chi_{B_\rho^c}) \right\|_{\alpha,q} \\ &\leq \rho^{-s} \left(1 + \frac{K_\alpha}{(2q)^{(\alpha + \frac{d}{2} + 1)\frac{1}{q}}} \rho^{(2\alpha + d + 2)/q} z^{-(\alpha + \frac{d}{2} + 1)\frac{1}{q}} \right) \| |x|^s \varphi \|_{\alpha,p}. \end{aligned}$$

By choosing $\rho = z^{1/2}$ we get the result.

Theorem 3.2 — Let $1 < p \leq 2$, $q = p/(p-1)$, $0 < s < (2\alpha + d + 2)/q$ and $t > 0$, then for all $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$,

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq K(s,t) \| |x|^s \varphi \|_{\alpha,p}^{\frac{t}{s+t}} \| |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}^{\frac{s}{s+t}}, \quad (3.2)$$

where $K(s,t)$ is a positive constant.

PROOF : Let $\varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+)$ and $1 < p \leq 2$, such that

$$\| |x|^s \varphi \|_{\alpha,p} + \| |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} < \infty.$$

Suppose that $0 < s < (2\alpha + d + 2)/q$ and $t < 2$. Then, by Lemma 3.1 we have, for all $z > 0$,

$$\begin{aligned} \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} &\leq \| e^{-z|y|^2} \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} + \| (1 - e^{-z|y|^2}) \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} \\ &\leq \left(1 + \frac{K_\alpha}{(2q)^{(\alpha+\frac{1}{d}+1)\frac{1}{q}}} \right) z^{-s/2} \| |x|^s \varphi \|_{\alpha,p} + \| (1 - e^{-z|y|^2}) \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}. \end{aligned}$$

on the other hand, we have

$$\| (1 - e^{-z|y|^2}) \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} = z^{t/2} \| (z|y|^2)^{-t/2} (1 - e^{-z|y|^2}) |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}.$$

Take into account $(1 - e^{-u})u^{-s/2}$ is bounded for all $u \geq 0$, if $t \leq 2$. Consequently,

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq K \left(z^{s/2} \| |x|^s \varphi \|_{\alpha,p} + z^{t/2} \| |y|^s \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} \right). \quad (3.3)$$

By taking

$$z = \left(\frac{s \| |x|^s \varphi \|_{\alpha,p}}{t \| |y|^s \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}} \right)^{\frac{2}{s+t}},$$

we get the result for all $t \leq 2$.

It remains to show the result for $t > 2$. Since, we have for all $\varepsilon > 0$, $|y| \leq \varepsilon + \varepsilon^{1-t}|y|^t$, then it follows that

$$\| |y| \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} \leq \varepsilon \| \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} + \varepsilon^{1-t} \| |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}. \quad (3.4)$$

By choosing,

$$\varepsilon = (t-1)^{\frac{1}{t}} \left(\frac{\| |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}}{\| \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}} \right)^{\frac{1}{t}},$$

we get

$$\| |y| \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q} \leq \frac{t}{t-1} (t-1)^{\frac{1}{t}} \| \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}^{\frac{t-1}{t}} \| |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}^{\frac{1}{t}}. \quad (3.5)$$

Combining the inequalities (3.3) and (3.4), we obtain

$$\begin{aligned} \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} &\leq K \| |x|^s \varphi \|_{\alpha,p}^{\frac{1}{1+s}} \| |y| \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}^{\frac{s}{1+s}} \\ &\leq K \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^{\frac{s(t-1)}{t(s+1)}} \| |x|^s \varphi \|_{\alpha,p}^{\frac{1}{1+s}} \| |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}^{\frac{s}{t(s+1)}}. \end{aligned}$$

which implies

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q}^{\frac{s+t}{t(s+1)}} \leq K \| |x|^s \varphi \|_{\alpha,p}^{\frac{1}{1+s}} \| |y|^s \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,q}^{\frac{s}{t(s+1)}}.$$

which gives the result for $t > 0$.

Remark 3.3 : Let $q = 2$. According to Placherel formula (2.7), we get

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,2} \leq K(s, t) \| |x|^s \varphi \|_{\alpha,2}^{\frac{t}{s+t}} \| |y|^t \mathcal{F}_{W,\alpha}(\varphi) \|_{\alpha,2}^{\frac{s}{s+t}},$$

which is the general case of the Heisenberg-Pauli-Weyl inequality (1.1) proved by Mejjaoli [11].

4. L^p -DONOHO-STRAK UNCERTAINTY PRINCIPLES

Definition 4.1 — Let Ω and Σ be measurable subsets of $\mathbb{R}_+^{d+1}$. We define the timelimiting operator P_Ω by $P_\Omega \varphi := \varphi \chi_\Omega$ and define the Weinstein integral operator Q_Σ by $\mathcal{F}_{W,\alpha}(Q_\Sigma \varphi) = \mathcal{F}_{W,\alpha}(\varphi) \chi_\Sigma$.

Proposition 4.2 — Let $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \leq p \leq 2$. If $\mu_\alpha(\Sigma) < \infty$, then we have

$$Q_\Sigma \varphi(x) = \int_\Sigma \Lambda_\alpha^d(x, \lambda) \mathcal{F}_{W,\alpha}(\varphi)(\lambda) d\mu_\alpha(\lambda).$$

PROOF : Let $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \leq p \leq 2$ and put $q = p/(p-1)$. According to inequalities (2.3) and (2.9) we obtain

$$\begin{aligned} \|\mathcal{F}_{W,\alpha}(\varphi) \chi_\Sigma\|_{\alpha,1} &= \int_\Sigma |\mathcal{F}_{W,\alpha}(\varphi)(x)| d\mu_\alpha(x) \\ &\leq (\mu_\alpha(\Sigma))^{1/p} \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \\ &\leq (\mu_\alpha(\Sigma))^{1/p} \|\varphi\|_{\alpha,q}. \end{aligned}$$

In the same way, we get

$$\begin{aligned} \|\mathcal{F}_{W,\alpha}(\varphi) \chi_\Sigma\|_{\alpha,2} &= \left(\int_\Sigma |\mathcal{F}_{W,\alpha}(\varphi)(x)|^2 d\mu_\alpha(x) \right)^{\frac{1}{2}} \\ &\leq (\mu_\alpha(\Sigma))^{\frac{q-2}{2q}} \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \\ &\leq (\mu_\alpha(\Sigma))^{\frac{q-2}{2q}} \|\varphi\|_{\alpha,q}. \end{aligned}$$

Hence, $\mathcal{F}_{W,\alpha}(\varphi)\chi_\Sigma \in L_\alpha^1 \cap L_\alpha^2(\mathbb{R}_+^{d+1})$ and by the definition of the Weinstein integral operator Q_Σ we obtain

$$Q_\Sigma \varphi = \mathcal{F}_{W,\alpha}^{-1}(\mathcal{F}_{W,\alpha}(\varphi)\chi_\Sigma).$$

Finally, the inversion formula (2.8) gives the result. $\square$

It is easy to see the following result by inequality (2.9).

Lemma 4.3 — Let $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 < p \leq 2$ and let $q = p/(p-1)$. Then we have

$$\|\mathcal{F}_{W,\alpha}(Q_\Sigma \varphi)\|_{\alpha,q} \leq \|\varphi\|_{\alpha,p}.$$

Theorem 4.4 — Let Ω and Σ be measurable subsets of $\mathbb{R}_+^{d+1}$. Let $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 < p \leq 2$ and $q = p/(p-1)$. Then we have

$$\|\mathcal{F}_{W,\alpha}(Q_\Sigma P_\Omega \varphi)\|_{\alpha,q} \leq (\mu_\alpha(\Sigma))^{\frac{1}{q}} (\mu_\alpha(\Omega))^{\frac{1}{q}} \|\varphi\|_{\alpha,p}.$$

PROOF : Assume that $\mu_\alpha(\Sigma) < \infty$ and $\mu_\alpha(\Omega) < \infty$. Let $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \leq p \leq 2$ and $q = p/(p-1)$. From the definition of the Weinstein integral operator Q_Σ , we get

$$\mathcal{F}_{W,\alpha}(Q_\Sigma P_\Omega \varphi) = \mathcal{F}_{W,\alpha}(P_\Omega \varphi)\chi_\Sigma.$$

Therefore

$$\|\mathcal{F}_{W,\alpha}(Q_\Sigma P_\Omega \varphi)\|_{\alpha,q} = \left(\int_\Sigma |\mathcal{F}_{W,\alpha}(P_\Omega \varphi)(\lambda)|^q d\mu_\alpha(\lambda) \right)^{\frac{1}{q}}. \quad (4.1)$$

Since

$$\mathcal{F}_{W,\alpha}(P_\Omega \varphi)(\lambda) = \int_\Omega \varphi(x) \Lambda_\alpha^d(x, \lambda) d\mu_\alpha(x),$$

then according to Holder's inequality and (2.3), we obtain

$$\begin{aligned} |\mathcal{F}_{W,\alpha}(P_\Omega \varphi)| &\leq \left(\int_\Omega |\Lambda_\alpha^d(x, \lambda)|^q d\mu_\alpha(x) \right)^{\frac{1}{q}} \left(\int_\Omega |\varphi(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}} \\ &\leq (\mu_\alpha(\Omega))^{\frac{1}{q}} \|\varphi\|_{\alpha,q}. \end{aligned}$$

Hence, by (4.1), we get

$$\|\mathcal{F}_{W,\alpha}(Q_\Sigma \varphi)\|_{\alpha,q} \leq (\mu_\alpha(\Sigma))^{\frac{1}{q}} (\mu_\alpha(\Omega))^{\frac{1}{q}} \|\varphi\|_{\alpha,p}. \square$$

4.1 Concentration uncertainty principle

In this section we present two continuous-time uncertainty principles of concentration type and we show there are depend on the sets of concentration Ω and Σ , and on the time function φ .

Definition 4.5 — Let Ω, Σ be a measurable subsets of $\mathbb{R}_+^{d+1}$ and $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 \leq p \leq 2$.

(i) We say that φ is ε_Ω -concentrated to Ω in $L_\alpha^p(\mathbb{R}_+^{d+1})$ -norm, if

$$\|\varphi - P_\Omega \varphi\|_{\alpha,p} \leq \varepsilon_\Omega \|\varphi\|_{\alpha,p}. \quad (4.2)$$

(ii) $\mathcal{F}_{W,\alpha}(\varphi)$ is ε_Σ -concentrated to Σ in $L_\alpha^q(\mathbb{R}_+^{d+1})$ -norm, $q = p/(p-1)$, if

$$\|\mathcal{F}_{W,\alpha}(\varphi) - \mathcal{F}_{W,\alpha}(Q_\Sigma \varphi)\|_{\alpha,q} \leq \varepsilon_\Sigma \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q}. \quad (4.3)$$

The following theorem, states the first continuous-time uncertainty principle of concentration type for the L_α^p -theory.

Theorem 4.6 — Let Ω, Σ be a measurable subsets of $\mathbb{R}_+^{d+1}$ and $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 < p \leq 2$. If φ is ε_Ω -concentrated to Ω in $L_\alpha^p(\mathbb{R}_+^{d+1})$ -norm and $\mathcal{F}_{W,\alpha}(\varphi)$ is ε_Σ -concentrated to Σ in $L_\alpha^q(\mathbb{R}_+^{d+1})$ -norm, $q = p/(p-1)$, then we have

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \frac{(\mu_\alpha(\Sigma))^{\frac{1}{q}} (\mu_\alpha(\Omega))^{\frac{1}{q}} + \varepsilon_\Omega}{1 - \varepsilon_\Sigma} \|\varphi\|_{\alpha,p}.$$

PROOF : Let $\varphi \in L_\alpha^p(\mathbb{R}_+^{d+1})$, $1 < p \leq 2$. According to (4.2), (4.3) and Theorem 4.4 it follows that

$$\begin{aligned} \|\mathcal{F}_{W,\alpha}(\varphi) - \mathcal{F}_{W,\alpha}(Q_\Sigma P_\Omega \varphi)\|_{\alpha,q} &\leq \|\mathcal{F}_{W,\alpha}(\varphi) - \mathcal{F}_{W,\alpha}(Q_\Sigma \varphi)\|_{\alpha,q} \\ &\quad + \|\mathcal{F}_{W,\alpha}(Q_\Sigma \varphi) - \mathcal{F}_{W,\alpha}(Q_\Sigma P_\Omega \varphi)\|_{\alpha,q} \\ &\leq \varepsilon_\Sigma \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} + \|\varphi - P_\Omega \varphi\|_{\alpha,p} \\ &\leq \varepsilon_\Sigma \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} + \varepsilon_\Omega \|\varphi\|_{\alpha,p}. \end{aligned}$$

Applying the triangle inequality and think to Theorem 4.4, we show that

$$\begin{aligned} \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} &\leq \|\mathcal{F}_{W,\alpha}(Q_\Sigma P_\Omega \varphi)\|_{\alpha,q} + \|\mathcal{F}_{W,\alpha}(\varphi) - \mathcal{F}_{W,\alpha}(Q_\Sigma P_\Omega \varphi)\|_{\alpha,q} \\ &\leq \left((\mu_\alpha(\Sigma))^{\frac{1}{q}} (\mu_\alpha(\Omega))^{\frac{1}{q}} + \varepsilon_\Omega \right) \|\varphi\|_{\alpha,p} + \varepsilon_\Sigma \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q}. \end{aligned}$$

which gives the desired result.

The following result gives the second continuous-time uncertainty principle of concentration type for the $L^1_\alpha \cap L^p_\alpha$ theory.

Theorem 4.7 — *Let Ω, Σ be a measurable subsets of $\mathbb{R}^{d+1}_+$ and $\varphi \in (L^1_\alpha \cap L^p_\alpha)(\mathbb{R}^{d+1}_+)$, $1 < p \leq 2$. If φ is ε_Ω -concentrated to Ω in $L^p_\alpha(\mathbb{R}^{d+1}_+)$ -norm and $\mathcal{F}_{W,\alpha}(\varphi)$ is ε_Σ -concentrated to Σ in $L^q_\alpha(\mathbb{R}^{d+1}_+)$ -norm, $q = p/(p-1)$, then we have*

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \frac{(\mu_\alpha(\Sigma))^{\frac{1}{q}} (\mu_\alpha(\Omega))^{\frac{1}{q}}}{(1 - \varepsilon_\Omega)(1 - \varepsilon_\Sigma)} \|\varphi\|_{\alpha,p}.$$

PROOF : Let Ω, Σ be a measurable subsets of $\mathbb{R}^{d+1}_+$ such that $\mu_\alpha(\Omega) < \infty$ and $\mu_\alpha(\Sigma) < \infty$. Assume that $\varphi \in (L^1_\alpha \cap L^p_\alpha)(\mathbb{R}^{d+1}_+)$, $1 < p \leq 2$. Like that $\mathcal{F}_{W,\alpha}(\varphi)$ is ε_Σ -concentrated to Σ in $L^q_\alpha(\mathbb{R}^{d+1}_+)$ -norm, $q = p/(p-1)$, then we get

$$\begin{aligned} \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} &\leq \varepsilon_\Sigma \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} + \|\mathcal{F}_{W,\alpha}(Q_\Sigma \varphi)\|_{\alpha,q} \\ &\leq \varepsilon_\Sigma \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} + (\mu_\alpha(\Sigma))^{\frac{1}{q}} \|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,\infty}. \end{aligned}$$

According to inequality (2.5), it follows

$$\|\mathcal{F}_{W,\alpha}(\varphi)\|_{\alpha,q} \leq \frac{(\mu_\alpha(\Sigma))^{\frac{1}{q}}}{1 - \varepsilon_\Sigma} \|\varphi\|_{\alpha,1}. \quad (4.4)$$

On the other hand, since φ is ε_Ω -concentrated to Ω in $L^p_\alpha(\mathbb{R}^{d+1}_+)$ -norm, then

$$\begin{aligned} \|\varphi\|_{\alpha,1} &\leq \varepsilon_\Omega \|\varphi\|_{\alpha,1} + \|P_\Sigma \varphi\|_{\alpha,1} \\ &\leq \varepsilon_\Omega \|\varphi\|_{\alpha,q} + (\mu_\alpha(\Omega))^{\frac{1}{p}} \|\varphi\|_{\alpha,p}. \end{aligned}$$

Hence

$$\|\varphi\|_{\alpha,1} \leq \frac{(\mu_\alpha(\Omega))^{\frac{1}{p}}}{1 - \varepsilon_\Omega} \|\varphi\|_{\alpha,q}. \quad (4.5)$$

Combining (4.4) and (4.5) we obtain the result of this theorem.

Remark 4.8 : We observe at first that the statement of Theorem 4.6 depends on the time function φ . However although for $p = q = 2$, the continuous-time uncertainty principle becomes

$$1 - \varepsilon_\Omega - \varepsilon_\Sigma \leq \mu_\alpha(\Omega)^{\frac{1}{2}} \mu_\alpha(\Sigma)^{\frac{1}{2}},$$

and is independent on the time function φ . Likewise we observe that the statement of Theorem 4.7 depends on the time function φ . Like the first continuous-time uncertainty principle the statement of Theorem 4.7 is independent on the time function φ for $p = q = 2$, and we have

$$(1 - \varepsilon_\Omega)(1 - \varepsilon_\Sigma) \leq \mu_\alpha(\Omega)^{\frac{1}{2}} \mu_\alpha(\Sigma)^{\frac{1}{2}}.$$

4.2 Continuous-bandlimited uncertainty principle

In this section, we establish continuous-bandlimited uncertainty principle of concentration. This principle depends on the sets of concentration Ω and Σ , but he is independent on the bandlimited function φ .

Definition 4.9 — Let $1 \leq p \leq 2$ and $\psi \in L^p(\mathbb{R}_+^d)$.

(i) We say that ψ is bandlimited to Σ if $Q_\Sigma \psi = \psi$ and denote by $\mathcal{B}_\alpha^p(\Sigma)$ the set of functions $\psi \in L^p(\mathbb{R}_+^d)$ that are bandlimited to Σ .

(ii) We say that φ is ε_Σ -bandlimited to Σ in L_α^p -norm if there exists $\psi \in \mathcal{B}_\alpha^p(\Sigma)$ such that

$$\|\varphi - \psi\|_{\alpha,p} \leq \varepsilon \|\varphi\|_{\alpha,p}.$$

The space of bandlimited functions $\mathcal{B}_\alpha^p(\Sigma)$ satisfies the following property.

Proposition 4.10 — Let Ω, Σ be a measurable subsets of $\mathbb{R}_+^{d+1}$. For all $\psi \in \mathcal{B}_\alpha^p(\Sigma)$, $1 \leq p \leq 2$, we have

$$\|P_\Omega \psi\|_{\alpha,p} \leq (\mu_\alpha(\Sigma))^{\frac{1}{p}} (\mu_\alpha(\Omega))^{\frac{1}{p}} \|\psi\|_{\alpha,p}.$$

PROOF : The estimation is trivial if the measure of Ω or Σ is infinite. Suppose that $\mu_\alpha(\Sigma) < \infty$ and $\mu_\alpha(\Omega) < \infty$. Let $\psi \in \mathcal{B}_\alpha^p(\Sigma)$ such that $1 \leq p \leq 2$, then from Propostion 4.2, we have

$$\psi(x) = \int_\Sigma \Lambda_\alpha^d(x, \lambda) \mathcal{F}_{W,\alpha}(\psi)(\lambda) d\mu_\alpha(\lambda).$$

From the property (2.3) and Hölder's inequality, we obtain for all $q = p/(p-1)$

$$\begin{aligned} \psi(x) &\leq (\mu_\alpha(\Sigma))^{\frac{1}{p}} \|\mathcal{F}_{W,\alpha}(\psi)\|_{\alpha,q} \\ &\leq (\mu_\alpha(\Sigma))^{\frac{1}{p}} \|\psi\|_{\alpha,p}. \end{aligned}$$

Applying now the time-limiting operator P_Ω to the bandlimited function ψ and observing the L_α^p -norm, we obtain

$$\begin{aligned} \|P_\Omega \psi\|_{\alpha,p} &= \left(\int_\Omega |\psi(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}} \\ &\leq (\mu_\alpha(\Omega))^{\frac{1}{p}} (\mu_\alpha(\Sigma))^{\frac{1}{p}} \|\psi\|_{\alpha,p}. \end{aligned}$$

which yields the desired result.

Theorem 4.11 — *Let Ω, Σ be measurable subsets of $\mathbb{R}_+^{d+1}$ and let $\varphi \in L^p(\mathbb{R}_+^d)$ such that $1 \leq p \leq 2$. Then if φ is ε_Σ -bandlimited to Σ in L_α^p -norm, we have*

$$\|P_\Omega \psi\|_{\alpha,p} \leq \left((1 + \varepsilon_\Sigma) (\mu_\alpha(\Sigma))^{\frac{1}{p}} (\mu_\alpha(\Omega))^{\frac{1}{p}} + \varepsilon_\Sigma \right) \|\psi\|_{\alpha,p}.$$

PROOF : Let $\varphi \in L^p(\mathbb{R}_+^d)$ such that $1 \leq p \leq 2$. Like that φ is ε_Σ -bandlimited to Σ in L_α^p -norm, then there exists $\psi \in \mathcal{B}_\alpha^p(\Sigma)$ such that

$$\|\varphi - \psi\|_{\alpha,p} \leq \varepsilon \|\varphi\|_{\alpha,p}$$

and we have

$$\|P_\Omega \varphi\|_{\alpha,p} \leq \|P_\Omega \psi\|_{\alpha,p} + \|P_\Omega(\varphi - \psi)\|_{\alpha,p} \leq \|P_\Omega \psi\|_{\alpha,p} + \varepsilon_\Sigma \|\varphi\|_{\alpha,p}. \quad (4.6)$$

Observe now the result of Proposition 4.10 and the fact that

$$\|\psi\|_{\alpha,p} \leq (1 + \varepsilon_\Sigma) \|\varphi\|_{\alpha,p}$$

we obtain the desired result. $\square$

Theorem 4.12 — *Let Ω, Σ be measurable subsets of $\mathbb{R}_+^{d+1}$ and let $\varphi \in L^p(\mathbb{R}_+^d)$ such that $1 \leq p \leq 2$. Then if φ is ε_Ω -concentrated to Ω and ε_Σ -bandlimited to Σ in L_α^p -norm, we have*

$$1 - \varepsilon_\Omega - \varepsilon_\Sigma \leq (1 + \varepsilon_\Sigma) (\mu_\alpha(\Sigma))^{\frac{1}{p}} (\mu_\alpha(\Omega))^{\frac{1}{p}}.$$

PROOF : Let $\varphi \in L^p(\mathbb{R}_+^d)$ such that $1 \leq p \leq 2$. Like that φ is ε_Ω -concentrated to Ω in L_α^p -norm, then by estimation (4.2) we obtain

$$\|\varphi\|_{\alpha,p} \leq \varepsilon_\Omega \|\varphi\|_{\alpha,p} + \|P_\Omega \varphi\|_{\alpha,p}.$$

Therefore

$$\|\varphi\|_{\alpha,p} \leq \frac{1}{1 - \varepsilon_\Omega} \|P_\Omega \varphi\|_{\alpha,p}.$$

Finally, we deduce the desired estimation by inequality (4.5) and Theorem 4.11. $\square$

Remark 4.13 : The continuous bandlimited uncertainty principle of concentration type for the L_α^p -norm given by pervious Corollary is independent on the bandlimited function φ for every $1 \leq p \leq 2$.

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REFERENCES

1. M. Brelot, Equation de weinstein et potentiels de Marcel Riesz, In *Séminaire de Théorie du Potentiel Paris*, No. 3 (1978), 18-38. Springer.
2. P. Ciatti, F. Ricci, and M. Sundari, Heisenberg-Pauli-Weyl uncertainty inequalities and polynomial volume growth, *Advances in Mathematics*, **215**(2) (2007), 616-625.
3. D. L. Donoho and P. B. Stark, Uncertainty principles and signal recovery, *SIAM Journal on Applied Mathematics*, **49**(3) (1989), 906-931.
4. S. Ghobber, Uncertainty principles involving L^1 -norms for the Dunkl transform, *Integral Transforms and Special Functions*, **24**(6) (2013), 491-501.
5. S. Ghobber, Variations on uncertainty principles for integral operators, *Applicable Analysis*, **93**(5) (2014), 1057-1072.
6. E. Laeng and C. Morpurgo, An uncertainty inequality involving L^1 -norms, *Proceedings of the American Mathematical Society*, **127**(12) (1999), 3565-3572.
7. H. Landau, On Szegő's eigenvalue distribution theorem and non-Hermitian kernels, *Journal d'Analyse Mathématique*, **28**(1) (1975), 335-357.
8. H. J. Landau and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty-iii: The dimension of the space of essentially time-and band-limited signals, *Bell Labs Technical Journal*, **41**(4) (1962), 1295-1336.
9. H. Mejjaoli and A. O. A. Salem, Weinstein Gabor transform and applications, *Advances in Pure Mathematics*, **2**(03) (2012), 203.

10. H. Mejjaoli and M. Salhi, Uncertainty principles for the Weinstein transform, *Czechoslovak mathematical journal*, **61**(4) (2011), 941-974.
11. C. Morpurgo, Extremals of some uncertainty inequalities, *Bulletin of the London Mathematical Society*, **33**(1) (2001), 52-58.
12. Z. B. Nahia and N. B. Salem, On a mean value property associated with the Weinstein operator, In *Proceedings of the International Conference on Potential Theory held in Kouty, Czech Republic (ICPT'94)*, (1996), 243-253.
13. Z. Ben Nahia and N. Ben Salem, Spherical harmonics and applications associated with the Weinstein operator. In: *Proceedings of the International Conference on Potential Theory held in Kouty, Czech Republic (ICPT '94)*; Berlin: de Gruyter; 1994. p. 223-241.
14. N. B. Salem and A. R. Nasr. Heisenberg-type inequalities for the Weinstein operator. *Integral Transforms and Special Functions*, **26**(9) (2015), 700-718.
15. F. Soltani, L^p Donoho-Stark uncertainty principles for the Dunkl transform on $\mathbb{R}^d$. *J. Phys. Math.*, **5**(4), 2014.
16. F. Soltani and J. Ghazwani, A variation of the L^p uncertainty principles for the Fourier transform, *Proc. Inst. Math. Mech., Natl. Acad. Sci. Azerb.*, **42**(1) (2016), 10-24.
17. E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Mathematical Series, Princeton, N. J.: Princeton University Press. X, 297, 1971.
18. A. Weinstein, Singular partial differential equations and their applications, *Fluid Dynamics and Applied Mathematics*, **67** (1962), 29-49.
19. A. Saoudi and I. A. Kallel, L2-uncertainty principle for the Weinstein-multiplier operators, *Int. J. Anal. Appl.*, **17**(1) (2019), 64-75.
20. F. Soltani, A general form of Heisenberg-Pauli-Weyl uncertainty inequality for the Dunkl transform, *IntegrTransf Spec Funct.*, **24**(5) (2013), 401-409.