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GENERALIZED GEOMETRIC POLYNOMIALS VIA STEFFENSEN'S GENERALIZED FACTORIALS AND TANNY'S OPERATORS

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Our purpose is to give a generalization of geometric polynomials by applying an appropriate linear transformation on the generalized factorial function. Some identities are investigated including explicit formula, generating function and recurrence relations. Furthermore, some relations with other polynomials are given.

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1. Introduction

1.1 *Factorial functions and Operators*

Factorial functions are of basic importance in combinatorics and many other fields of mathematics. There are two kinds of factorials which are usually used, firstly we have, for a nonnegative integer n and any real x, the falling factorials denoted $x^{\underline{n}}$ given by,

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1), \tag{1}$$

the second kind are the rising factorials denoted $x^{\overline{n}}$ given by,

$$x^{\overline{n}} = x(x+1)(x+2)\cdots(x+n-1). \tag{2}$$

There is also a third version introduced by Steffensen (see [23]), under the name of *central factorials*, denoted $x^{[n]}$, where,

$$x^{[n]} = x(x+n/2-1)(x+n/2-2)\cdots(x-n/2+1), \tag{3}$$

we note that, $x^{\overline{0}} = x^{\underline{0}} = x^{[0]} = 1$.

These functions are highly connected to the following operators,

shift operator $\mathbf{E}^{(a)}$,

$$\mathbf{E}^{(a)} f(x) = f(x+a),$$

difference operator Δ,

$$\Delta f(x) = f(x+1) - f(x),$$

central difference operator δ,

$$\delta f(x) = f(x+1/2) - f(x-1/2).$$

Let us consider $P_n(x)$ a polynomial of degree n. Using *factorial functions* ($x^{\overline{n}}, x^{\underline{n}}$ and $x^{[n]}$) and *operators* ($\mathbf{E}^{(a)}$, Δ and δ), the following expansions holds true [23],

$$P_n(x) = \sum_{k=0}^{n} \frac{x^{\underline{k}}}{k!} \Delta^k P_n(0), \tag{4}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{x^{\overline{k}}}{k!} \left(\mathbf{E}^{(-1)}\Delta\right)^k P_n(0), \tag{5}$$

$$P_n(x) = \sum_{k=0}^{n} \frac{x^{[k]}}{k!} \delta^k P_n(0). \tag{6}$$

These results correspond to Newton's formula.

1.2 *Geoemtric numbers and polynomials*

Stirling numbers of the second kind, denoted $S(n,k)$, are a main focus of research in enumerative combinatorics, they count the number of different ways to partition a set of n elements into k non-empty subsets.

Algebraically, Stirling numbers of the second kind are the coefficients that connect x^n by falling factorials as,

$$x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}. \tag{7}$$

Let us consider the linear transformation $\mathcal{L}$ that transforms $x^{\underline{n}}$ into $n!x^n$,

$$\mathcal{L}(x^{\underline{n}}) = n!x^n. \tag{8}$$

By applying the transformation $\mathcal{L}$ on Relation (1.7), Tanny [24] introduced the geometric polynomials (or Fubini polynomials or ordered Bell polynomials), denoted $F_n(x)$, as follows,

$$F_n(x) := \mathcal{L}(x^n) = \mathcal{L}\left(\sum_{k=0}^{n} S(n,k)x^{\underline{k}}\right) = \sum_{k=0}^{n} k!S(n,k)x^k. \tag{9}$$

Setting $x = 1$ in (1.9) leads to the nth geometric number (or ordered Bell number) F_n which counts the number of preferential arrangements [11, 27] of n-objects.

Recently, several authors have studied the geometric numbers and polynomials. In [2], Boyadzhiev obtains a transformation formula involving $F_n(x)$. Dil and Kurt in [9] used the Euler-Seidel matrix to obtain some recurrences and relations with derivatives. They also introduced in [10] a new class of geometric numbers and polynomials called *harmonic-geometric polynomials*. Mező in [18] studies the periodicity of F_n and gives some congruences properties. In [3] Boyadzhiev and Dil studied various properties of as those linked to Riemann zeta function.

An interesting generalization of Stiling numbers is that of Hsu and Shiue [12], authors defined the generalized Stiling numbers pair of three parameters using two linear transformations between generalized factorials, especially the Stirling numbers of the second kind are given as,

$$S(n,k;\alpha,\beta,r) = \frac{1}{k!\beta^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\beta j + r|\alpha)^{\underline{n}}, \tag{10}$$

where $(x|\alpha)^{\underline{n}} = x(x-\alpha)\cdots(x-n\alpha+\alpha)$.

Using this generalization, Kargın and Corcino [16] gave a new generalization of geometric polynomials by applying the generalized Mellin's derivative and they obtained their basic properties. The same author Kargın with Çekim [15], based on the previous work and using the r-Whitney numbers of the second kind [5, 17], defined by,

$$W_{\beta,r}(n,k) = S(n,k;0,\beta,r) = \frac{1}{k!\beta^k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(\beta j + r)^n, \tag{11}$$

they evaluated some close formulas and investigated the relationship between the generalized geometric polynomials of higher orders and some classical classes of polynomials (Bernoulli, Euler, ...).

In a recent work, Corcino *et al*. [7], studied two new families of geometric polynomials called r-*Whitney-Fubini* polynomials and the r-*Eulerian-Fubini* polynomials.

For more details we refer the reader to [2, 3, 7-11, 14-16, 18, 24, 27] and papers cited therein.

In the next sections we deal with a new generalized geometric polynomials which are closely related to Hsu and Shiue generalized Stirling pairs of three parameters.

1.3 *The generalized geometric polynomials*

An interesting generalization of factorial functions was proposed by Steffensen [23]; for any real α and β, let

$$x_n^{\{\alpha,\beta\}} := x(x-\alpha n-\beta)(x-\alpha n-2\beta)\cdots(x-\alpha n-(n-1)\beta). \tag{12}$$

Similarly to (4), (5) and (6), Steffensen obtained the following expansion

***Theorem* 1.1** — *[Steffensen, 1936]. Let f a function on x, for all real x we have,*

$$f(x) = \sum_{k=0}^{n} \frac{x_k^{\{\alpha,\beta\}}}{k!} \Theta^k f(x)|_{x=0}, \tag{13}$$

where $\Theta = \dfrac{1}{\beta}\left(\mathbf{E}^{(\alpha+\beta)} - \mathbf{E}^{(\alpha)}\right)$ *and* $\beta \neq 0$.

From (13) and (12) it's clear that we can obtain $x^{\underline{n}}$, $x^{\overline{n}}$ and $x^{[n]}$ and their expansion formulas by setting $x_n^{\{0,1\}}, x_n^{\{0,-1\}}$ and $x_n^{\{-1/2,1\}}$ respectively.

Now we define the generalized geometric polynomials as follows

Definition 1.2 — Let $\mathcal{Z}$ be a linear transformation defined by,

$$\mathcal{Z}(x_n^{\{\alpha,\beta\}}) = \beta^n n! x^n. \tag{14}$$

By setting $f(x) = x^n$ on Theorem 1.1 and applying the transformation $\mathcal{Z}$, the generalized geometric polynomials $F_n^{(\alpha,\beta)}(x)$ is given by the following explicit formula.

***Theorem* 1.3** — *The generalized Fubini polynomials satisfy,*

$$F_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha k + \beta(k-j))^n . \tag{15}$$

PROOF : Let us consider, for a given, $f(x) = x^n$, Formula (13) will be,

$$\begin{aligned}
x^n &= \sum_{k=0}^{n} \frac{x_k^{\{\alpha,\beta\}}}{k!} \Theta^k x^n|_{x=0} \\
&= \sum_{k=0}^{n} \frac{x_k^{\{\alpha,\beta\}}}{\beta^k k!} \left(\mathbf{E}^{\alpha+\beta} - \mathbf{E}^{\alpha}\right)^k x^n|_{x=0} \\
&= \sum_{k=0}^{n} \frac{x_k^{\{\alpha,\beta\}}}{\beta^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mathbf{E}^{(\alpha+\beta)(k-j)} \mathbf{E}^{\alpha j} x^n|_{x=0} \\
&= \sum_{k=0}^{n} \frac{x_k^{\{\alpha,\beta\}}}{\beta^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(x + (\alpha+\beta)(k-j) + \alpha j\right)^n |_{x=0} \\
&= \sum_{k=0}^{n} \frac{x_k^{\{\alpha,\beta\}}}{\beta^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left((\alpha+\beta)(k-j) + \alpha j\right)^n .
\end{aligned}$$

Furthermore by applying the operator $\mathcal{Z}$ on the two sides of the equality, we get,

$$F_n^{(\alpha,\beta)}(x) = \mathcal{Z}(x^n) = \sum_{k=0}^{n} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\alpha k + \beta(k-j)\right)^n . \square$$

Remark 1.4 :

- We can rewrite the generalized geometric polynomials as

  $$F_n^{(\alpha,\beta)}(x) = \sum_k k! S^{\alpha,\beta}(n,k) x^k,$$

  where,

  $$S^{\alpha,\beta}(n,k) = \frac{1}{k!\beta^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left(\alpha k + \beta(k-j)\right)^n ,$$

  which can be considered as a generalization of Stirling numbers of the second kind.

- Setting $\alpha = 0, \beta = 1$ we get the geometric polynomials (9).
- Setting $x = 1$ we can define the generalized geometric polynomials $F_n^{(\alpha,\beta)}$.

2. PROPERTIES

We begin by establishing the exponential generating function of the generalized geometric polynomials.

2.1 *Exponential generating function*

***Theorem* 2.1** — *$F_n^{(\alpha,\beta)}(x)$ have the following exponential generating function,*

$$\sum_{n=0}^{\infty} F_n^{(\alpha,\beta)}(x)\frac{t^n}{n!} = \frac{1}{1 - xe^{\alpha t}\left(e^{\beta t}-1\right)}. \tag{16}$$

PROOF : Using Formula (15), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n^{(\alpha,\beta)}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} x^k \sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\alpha j + (\alpha+\beta)(k-j)\right)^n \frac{t^n}{n!} \\
&= \sum_{k=0}^{n} x^k \sum_{j=0}^{k}(-1)^j\binom{k}{j} e^{(\alpha+\beta)(k-j)t}e^{\alpha j t} \\
&= \sum_{k=0}^{n} x^k \left(e^{(\alpha+\beta)t} - e^{\alpha t}\right)^k \\
&= \frac{1}{1 - xe^{\alpha t}\left(e^{\beta t}-1\right)}. \square
\end{aligned}$$

2.2 *Derivatives of higher order and recurrences*

For all $(\alpha,\beta) \in \mathbb{R}$ with $\beta \neq 0$, let $G^{\alpha,\beta}(x,t)$ be the exponential generating function of $F_n^{(\alpha,\beta)}(x)$, then we have

Proposition 2.2 — For all integer r, the rth derivative of $G^{\alpha,\beta}(x,t)$ with respect to x, satisfy

$$\frac{\mathrm{d}^r}{\mathrm{d}^r x}G^{\alpha,\beta}(x,t) = \frac{r!}{x^r}G^{\alpha,\beta}(x,t)\left(G^{\alpha,\beta}(x,t)-1\right)^r. \tag{17}$$

PROOF : We use induction on r, denote $G^{\alpha,\beta}(x,t) := G(x,t)$. For $r=1$ we have,

$$\frac{\mathrm{d}}{\mathrm{d}x}G(x,t) = \frac{e^{\alpha t}\left(e^{\beta t}-1\right)}{\left(1 - xe^{\alpha t}\left(e^{\beta t}-1\right)\right)^2}, \tag{18}$$

and

$$\frac{G(x,t)}{x}\left(G(x,t)-1\right) = \frac{e^{\alpha t}\left(e^{\beta t}-1\right)}{\left(1 - xe^{\alpha t}\left(e^{\beta t}-1\right)\right)^2}. \tag{19}$$

From (18) and (19) the proposition is true for $r=1$.

Assume that Proposition 2.2 holds for a fixed r, i.e.,

$$\frac{\mathrm{d}^r}{\mathrm{d}^r x}G(x,t) = \frac{r!}{x^r}G(x,t)\left(G(x,t)-1\right)^r,$$

then, for $r+1$,

$$\begin{aligned}\frac{\mathrm{d}^{r+1}}{\mathrm{d}^{r+1}x}G(x,t) =& \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{r}}{\mathrm{d}^{r}x}G(x,t)\right)\\ =& \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{r!}{x^r}G(x,t)\left(G(x,t)-1\right)^r\right)\\ =& \frac{-rr!}{x^r+1}G(x,t)(G(x,t)-1)^r+\frac{r!}{x^r}(G(x,t)-1)^r\frac{\mathrm{d}}{\mathrm{d}x}G(x,t)\\ &+\frac{rr!}{x^r}G(x,t)(G(x,t)-1)^{r-1}\frac{\mathrm{d}}{\mathrm{d}x}G(x,t),\end{aligned}$$

but $\frac{\mathrm{d}}{\mathrm{d}x}G(x,t)=\frac{G(x,t)}{x}\left(G(x,t)-1\right)$, so

$$\begin{aligned}\frac{\mathrm{d}^{r+1}}{\mathrm{d}^{r+1}x}G(x,t) =& \frac{-rr!}{x^{r+1}}G(x,t)(G(x,t)-1)^r+\frac{r!}{x^{r+1}}(G(x,t)-1)^{r+1}\\ &+\frac{rr!}{x^{r+1}}G^2(x,t)(G(x,t)-1)^r\\ =& \frac{r!}{x^{r+1}}G(x,t)(G(x,t)-1)^r\left[-r+(G(x,t)-1)+rG(x,t)\right]\\ =& \frac{(r+1)!}{x^{r+1}}G(x,t)\left(G(x,t)-1\right)^{r+1}.\square\end{aligned}$$

Using Proposition 2.2 we can establish the multinomial convolution below:

***Theorem* 2.2** — *The rth derivative of $F_n^{\alpha,\beta}(x)$ is given by,*

$$\begin{aligned}\frac{\mathrm{d}^{r}}{\mathrm{d}^{r}x}F_n^{\alpha,\beta}(x) =& \frac{r!}{x^r}\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}\\ &\times\sum_{j_1+\cdots+j_{k+1}=n}\binom{n}{j_1,\ldots,j_{k+1}}F_{j_1}^{\alpha,\beta}(x)\cdots F_{j_{k+1}}^{\alpha,\beta}(x).\end{aligned} \tag{20}$$

PROOF : Applying Cauchy product on the right hand side of Proposition 2.2 and identifying the coefficients of $t^n/n!$ we get the result. $\square$

Corollaary 2.3 — The first derivative of $F_n^{\alpha,\beta}(x)$ satisfy the binomial convolution bellow,

$$x\frac{\mathrm{d}}{\mathrm{d}x}F_n^{\alpha,\beta}(x)=\sum_{k=0}^{n-1}\binom{n}{k}F_k^{\alpha,\beta}(x)F_{n-k}^{\alpha,\beta}(x) \tag{21}$$

PROOF : It holds by setting $r=1$ in Theorem 2.2. $\square$

Let D^n be the nth derivative with respect to t, i.e., $D^n=\frac{\mathrm{d}^n}{\mathrm{d}t^n}$

Proposition 2.4 — $F_n^{\alpha,\beta}(x)$ satisfy,

$$F_n^{\alpha,\beta}(x) = \sum_{k=0}^{\infty} D^n \left(xe^{\alpha t}\left(e^{\beta t}-1\right)\right)^k \Big|_{t=0}. \tag{22}$$

PROOF : We have

$$\begin{aligned}
\sum_{k=0}^{\infty} D^n \left(xe^{\alpha t}\left(e^{\beta t}-1\right)\right)^k \Big|_{t=0} &= \sum_{k=0}^{\infty} x^k D^n e^{\alpha k t} \sum_{j=0}^{k} (-1)^j \binom{k}{j} e^{(k-j)\beta t}\Big|_{t=0} \\
&= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} D^n e^{(\alpha k+\beta(k-j))t}\Big|_{t=0} \\
&= \sum_{k=0}^{\infty} x^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\alpha k+\beta(k-j))^n \\
&= F_n^{\alpha,\beta}(x). \square
\end{aligned}$$

Lemma 2.5 — Let $p_n(x)$ be a polynomial of degree n,

$$\mathcal{Z}(p_n(x)) = \beta x \mathcal{Z}(\Theta p_n(x)) \tag{23}$$

PROOF : It suffices to prove the result for $x_n^{\{\alpha,\beta\}}$ and it holds for all polynomials $p_n(x)$,

$$\mathcal{Z}(x_n^{\{\alpha,\beta\}}) = n!(\beta)^n = n\beta(n-1)!(\beta x)^{n-1} = \beta x \mathcal{Z}\left(nx_{n-1}^{\{\alpha,\beta\}}\right),$$

and from [23] we have $nx_{n-1}^{\{\alpha,\beta\}} = \Theta x_n^{\{\alpha,\beta\}}$, which gives the lemma. $\square$

Theorem 2.6 — *For any nonnegative integer $n \geq 1$ and real variable x, we have*

$$F_n^{\alpha,\beta}(x) = x\sum_{k=0}^{n-1} \binom{n}{k} F_k^{\alpha,\beta}(x)\Theta\left(x^{n-k}\right)|_{x=0}, \tag{24}$$

where $\Theta(x^n) = \dfrac{1}{\beta}\left((x+\alpha+\beta)^n - (x+\alpha)^n\right).$

PROOF : We have,

$$\begin{aligned}
F_n^{\alpha,\beta}(x) &= \mathcal{Z}(x^n) = \beta x \mathcal{Z}(\Theta x^n) = x\mathcal{Z}\left((x+\alpha+\beta)^n - (x+\alpha)^n\right) \\
&= x\mathcal{Z}\left(\sum_{j=0}^{n-1}\binom{n}{j}(x+\alpha)^j\beta^{n-j}\right) \\
&= x\mathcal{Z}\left(\sum_{j=0}^{n-1}\binom{n}{j}\beta^{n-j}\sum_{k=0}^{j}\binom{j}{k}\alpha^{j-k}x^k\right) \\
&= x\sum_{k=0}^{n-1}\binom{n}{k}F_k^{\alpha,\beta}(x)\sum_{j=k}^{n-1}\binom{n-k}{j-k}\beta^{n-j}\alpha^{j-k} \\
&= x\sum_{k=0}^{n-1}\binom{n}{k}F_k^{\alpha,\beta}(x)\left((\alpha+\beta)^{n-k} - (\alpha)^{n-k}\right) \\
&= x\sum_{k=0}^{n-1}\binom{n}{k}F_k^{\alpha,\beta}(x)\Theta\left(x^{n-k}\right)|_{x=0}.\square
\end{aligned}$$

Remark 2.7 : We obtain the binomial convolution given by Tanny [24] for the classical geometric polynomials by setting $(\alpha,\beta) = (0,1)$.

3. UMBRAL FORMULA

In the following result we use the umbral notation $F_n^{\alpha,\beta}(x) \equiv \mathbf{F}^n_{\mathrm{x},\alpha,\beta}$, the corollary bellow holds true,

Corollary 3.1 — Let n be a nonnegative integer, for any real variable x, we have

$$\mathbf{F}^n_{\mathrm{x},\alpha,\beta} = (\mathbf{F}_{\mathrm{x},\alpha,\beta} + \alpha + \beta)^n - (\mathbf{F}_{\mathrm{x},\alpha,\beta} + \alpha)^n + \delta_{n,0}, \tag{25}$$

where $\delta_{n,0}$ is the Kronecker delta.

PROOF : From Theorem 2.6 and using the umbral notation $F_n^{\alpha,\beta}(x) \equiv \mathbf{F}^n_{\mathrm{x},\alpha,\beta}$, a simple calculation gives the umbral representation result. $\square$

Remark 3.2 : We can use this representation to get the exponential generating function easily.

3.1 *Dobiński formula*

To calculate the nth geometric polynomials there exists several formulas, one of them is Dobiński-like formula, in the following Theorem we give the generalized Dobiński-like formula for $F_n^{\alpha,\beta}(x)$.

***Theorem* 3.3** — *For all nonnegative integer n and for all $x \in \mathbb{R} - \{-1\}$, we have*

$$F_n^{\alpha,\beta}(x) = \frac{1}{1+x}\sum_{k\geq 0}\left(\frac{x}{1+x}\right)^k\sum_{j=0}^{k}\binom{k}{j}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}(\alpha j + \beta i)^n, \tag{26}$$

or, equivalently, from Remark 1.4,

$$F_n^{\alpha,\beta}(x) = \frac{1}{1+x}\sum_{k\geq 0}\left(\frac{x}{1+x}\right)^k \sum_{j=0}^{k} j!\beta^j \binom{k}{j} S^{\alpha,\beta}(n,j). \tag{27}$$

PROOF : We have,

$$\begin{aligned}
\frac{1}{1-xe^{\alpha t}(e^{\beta t}-1)} &= \frac{1}{(1+x)(1-\frac{x}{1+x}\left(e^{\alpha t}(e^{\beta t}-1)+1\right))} \\
&= \frac{1}{1+x}\sum_{k\geq 0}\left(\frac{x}{1+x}\right)^k \left(e^{\alpha t}(e^{\beta t}-1)+1\right)^k \\
&= \frac{1}{1+x}\sum_{k\geq 0}\left(\frac{x}{1+x}\right)^k \sum_{j=0}^{k}\binom{k}{j} e^{\alpha j t}(e^{\beta t}-1)^j \\
&= \frac{1}{1+x}\sum_{k\geq 0}\left(\frac{x}{1+x}\right)^k \sum_{j=0}^{k}\binom{k}{j} \sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i} e^{(\alpha j+\beta i)t},
\end{aligned}$$

by comparing the coefficients of $t^n/n!$ in both sides we get the result. □

Corollary 3.4 — For $\alpha = 0$, for all β, we have

$$F_n^{0,\beta}(x) = \frac{1}{1+x}\sum_{k\geq 0}\left(\frac{x}{1+x}\right)^k (\beta k)^n . \tag{28}$$

By setting $\beta = 1$ and $x = 1$, we get the well known Dobiński formula for the classical geometric numbers (see [11]),

$$F_n = \sum_{k\geq 0}\frac{k^n}{2^{k+1}}.$$

3.2 *Parity*

A function $f(x)$ is said to be even if $f(x) = f(-x)$ and it's said to be odd if $f(x) = -f(-x)$,

***Theorem* 3.5** — *For any integer n and real x, if $\beta = -2\alpha$, then*

$$F_n^{\alpha,\beta}(x) = (-1)^n F_n^{\alpha,\beta}(-x), \tag{29}$$

i.e.,

$$F_n^{\alpha,\beta}(x) \textit{ is odd if and only if } n \textit{ is odd.}$$

PROOF : Let us consider $h(t) = e^{\alpha t}(e^{\beta t} - 1)$, it is easy to verify that $h(t)$ is odd if and only if $\beta = -2\alpha$, hence we can write,

$$\frac{1}{1 - xe^{\alpha t}(e^{\beta t} - 1)} = \frac{1}{1 + xe^{-\alpha t}(e^{-\beta t} - 1)},$$

the comparison of $t^n/n!$ coefficients gives the results. □

4. Link with Other Numbers and Polynomials

In this section we establish some links with other classes of numbers and polynomials as applications of the exponential generating function.

4.1 *The r-Whitney numbers of the second kind*

The nth r-Whitney number of the second kind $W_{m,r}(n,k)$ was introduced by Mező [17], as

$$\sum_{n\geq 0} W_{m,r}(n,k)\frac{t^n}{n!} = \frac{e^{rt}}{k!}\left(\frac{e^{mt}-1}{m}\right)^k. \tag{30}$$

Using the previous generating function, we have,

***Theorem* 4.1** — *For any integer n and any real variable x the following identity holds true,*

$$F_n^{\alpha,\beta}(x) = \sum_{k\geq 0} k!(\beta x)^k \sum_{j=0}^{n} \binom{n}{j} (\alpha(k-1))^{n-j} W_{\beta,\alpha}(j,k). \tag{31}$$

PROOF : Using the exponential generating of $F_n^{\alpha,\beta}(x)$ we get,

$$\begin{aligned}
\frac{1}{1 - xe^{\alpha t}\left(e^{\beta t} - 1\right)} &= \sum_{k\geq 0} (x)^k e^{\alpha k t}\left(e^{\beta t} - 1\right)^k \\
&= \sum_{k\geq 0} k!(\beta x)^k e^{\alpha(k-1)t}\frac{e^{\alpha t}}{k!}\left(\frac{e^{\beta t}-1}{\beta}\right)^k \\
&= \sum_{k\geq 0} k!(\beta x)^k e^{\alpha(k-1)t}\frac{e^{\alpha t}}{k!}\left(\frac{e^{\beta t}-1}{\beta}\right)^k \\
&= \sum_{k\geq 0} k!(\beta x)^k \sum_{n\geq 0} (\alpha(k-1))^n \frac{t^n}{n!} \sum_{n\geq 0} W_{\beta,\alpha}(n,k)\frac{t^n}{n!} \\
&= \sum_{n\geq 0}\sum_{k\geq 0} k!(\beta x)^k \sum_{j=0}^{n} \binom{n}{j} (\alpha(k-1))^{n-j} W_{\beta,\alpha}(j,k)\frac{t^n}{n!}.
\end{aligned}$$

by comparing the coefficients of $t^n/n!$ in both sides, the result holds true. □

Remark 4.2 : • For $(\alpha, \beta) = (0, 1)$ we get the definition of the classical Fubini polynomials.

• For $\alpha = 1$ we get the link with translated Whitney numbers defined by Belbachir and Bousbaa in [1].

4.2 *Apostol-Euler polynomials of higher order*

One of the analogues of the classical Euler polynomials are Apostol-Euler polynomials $\mathcal{E}_n(x, \lambda)$ which is given by,

$$\sum_{n\geq 0} \mathcal{E}_n(x,\lambda)\frac{t^n}{n!} = \left(\frac{2}{1+\lambda e^t}\right) e^{xt}, \quad (|t+\log\lambda| < \pi). \tag{32}$$

In a recent work, Luo and Srivastava [22], introduce the Apostol-Euler polynomials of higher order by,

$$\sum_{n\geq 0} \mathcal{E}_n^{(\alpha)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2}{1+\lambda e^t}\right)^{\alpha} e^{xt}; \quad (|t+\log\lambda| < \pi; 1^{\alpha} := 1), \tag{33}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0, \lambda)$ denotes the so-called Apostol-Euler numbers of order α.

The following theorem gives a connection between $F_n^{\alpha,\beta}(x)$ and $E_n^{(\alpha)}(x, \lambda)$.

***Theorem* 4.3** — *The generalized geometric polynomials is expressed in terms of Apostol-Euler polynomials of higher order, for $\alpha \neq 0$, as*

$$F_n^{\alpha,\beta}(x) = \frac{\alpha^n}{2}\sum_{k\geq 0}\left(\frac{x}{2}\right)^k \mathcal{E}_n^{(k+1)}\left(\frac{(\alpha+\beta)k}{\alpha}, x\right). \tag{34}$$

PROOF : Using the exponential generating function of $F_n^{\alpha,\beta}(x)$ we get,

$$\begin{aligned}
\frac{1}{1-xe^{\alpha t}\left(e^{\beta t}-1\right)} &= \left(\frac{1}{1+xe^{\alpha t}}\right)\left(\frac{1}{1-\left(\frac{x}{1+xe^{\alpha t}}\right)e^{(\alpha+\beta)t}}\right) \\
&= \sum_{k\geq 0} x^k e^{(\alpha+\beta)kt}\left(\frac{1}{1+xe^{\alpha t}}\right)^{k+1} \\
&= \frac{1}{2}\sum_{k\geq 0}\left(\frac{x}{2}\right)^k e^{(\alpha+\beta)kt}\left(\frac{2}{1+xe^{\alpha t}}\right)^{k+1} \\
&= \frac{1}{2}\sum_{k\geq 0}\left(\frac{x}{2}\right)^k \sum_{n\geq 0}\alpha^n \mathcal{E}_n^{(k+1)}\left(\frac{(\alpha+\beta)k}{\alpha}, x\right)\frac{t^n}{n!}.
\end{aligned}$$

Cauchy product and the comparison of coefficients of $t^n/n!$ give the result. □

4.3 *Eulerian numbers*

Eulerian numbers $a(n,k)$ are studied widely in literature, these numbers satisfy (see [4, 19]),

$$\sum_{n\geq 0}\sum_{k=0}^{n} a(n,k)x^k\frac{t^n}{n!} = \frac{x-1}{x-e^{(x-1)t}}. \tag{35}$$

Before given the connection between $F_n^{\alpha,\beta}(x)$ and Eulerian numbers $a(n,k)$ we give the following Lemma,

Lemma 4.4 — The Apostol-Euler numbers $\mathcal{E}_n(\lambda)$ satisfy,

$$\mathcal{E}_n(\lambda) = \frac{2}{1+\lambda}\left(\frac{-\lambda}{1+\lambda}\right)^n \sum_{k=0}^{n}\left(\frac{-1}{\lambda}\right)^k a(n,k), \tag{36}$$

PROOF : The generating function of the right hand side in Lemma 4.4 is,

$$\sum_{n\geq 0}\frac{2}{1+\lambda}\left(\frac{-\lambda}{1+\lambda}\right)^n \sum_{k=0}^{n}\left(\frac{-1}{\lambda}\right)^k a(n,k)\frac{t^n}{n!} = \frac{2}{1+\lambda e^t}. \tag{37}$$

from Identity 33 the result holds true.

***Theorem* 4.5** — *The generalized geometric polynomials $F_n^{\alpha,\beta}(x)$ satisfy,*

$$F_n^{\alpha,\beta}(x) = \sum_{k\geq 0} x^k \sum_{i_1+\cdots+i_{k+2}=n}\binom{n}{i_1,\ldots,i_{k+2}}((\alpha+\beta)k)^{i_1}\alpha^{n-i_1}\mathcal{E}_{i_2}(x)\cdots\mathcal{E}_{i_{k+2}}(x) \tag{38}$$

PROOF : From the exponential generating function, we have

$$\begin{aligned}
\frac{1}{1-xe^{\alpha t}\left(e^{\beta t}-1\right)} &= \left(\frac{1}{1+xe^{\alpha t}}\right)\left(\frac{1}{1-\left(\frac{x}{1+xe^{\alpha t}}\right)e^{(\alpha+\beta)t}}\right)\\
&= \sum_{k\geq 0} x^k e^{(\alpha+\beta)kt}\left(\frac{1}{1+xe^{\alpha t}}\right)^{k+1}\\
&= \sum_{k\geq 0} x^k \sum_{n\geq 0}((\alpha+\beta)k)^n\frac{t^n}{n!}\left(\sum_{n\geq 0}\mathcal{E}_n(x)\frac{t^n}{n!}\right)^{k+1}\\
&= \sum_{k\geq 0} x^k \sum_{n\geq 0}\sum_{i_1+\cdots+i_{k+2}=n}\binom{n}{i_1,\cdots,i_{k+2}}((\alpha+\beta)k)^{i_1}\\
&\quad\times\alpha^{n-i_1}\mathcal{E}_{i_2}(x)\ldots\mathcal{E}_{i_{k+2}}(x)\frac{t^n}{n!},
\end{aligned}$$

by identifying the coefficients of $t^n/n!$ we get the result. □

Remark 4.6 : From Lemma 4.4 and Theorem 4.5 we get the link between the generalized geometric polynomials and the Eulerian numbers.

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