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A NEW GENERALIZATION OF DELANNOY NUMBERS

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In this paper, a new extension for Delannoy numbers is presented. Also, several identities such as generating function and recurrence relations for these numbers are obtained.

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1. Introduction

For $n,m \in \mathbb{N}_0 = \{0,1,2,...\}$, the Delannoy number $d_n(m)$, from the name of Henri Auguste Delannoy, describes the number of paths from $(0,0)$ to (m,n) in which only east $(1,0)$, north $(0,1)$, and northeast (1.1) steps are allowed. It is known that (for instance, see [1])

$$d_n(m) = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} 2^k.$$

The number $d_n(m)$ satisfies the following recurrence relation

$$d_n(m) = d_n(m-1) + d_{n-1}(m-1) + d_{n-1}(m)$$

with initial values $d_n(0) = d_0(m) = d_0(0) = 1$. From the recurrence relation, one can obtain the generating function

$$\sum_{n,m\geq 0} d_n(m)\, x^m y^n = \frac{1}{1-x-y-xy}.$$

Since

$$\sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} t^k = \sum_{k=0}^{n} \binom{n}{k}\binom{x+k}{n} (t-1)^k, \quad \text{[2, Eq. 3.17]}$$

one can write for the polynomials $d_n(x)$, which is introduced in [10] as

$$d_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} \binom{x+k}{n}.$$

Sun [10] derived some supercongruences involving $d_n(x)$ and conjectured following congruences for $d_n(x)$

$$x(x+1) \sum_{k=0}^{n-1} (2k+1) d_k(x)^2 \equiv 0 \pmod{2n^2}$$

and

$$\sum_{k=0}^{n-1} \varepsilon^k (2k+1) d_k(x)^{2m} \equiv 0 \pmod{n}, \text{ for given } \varepsilon \in \{-1,1\} \text{ and } m \in \mathbb{N}.$$

Guo [3] achieved the following identity for $d_n(x)^2$

$$d_n(x)^2 = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{x}{k} \binom{x+k}{k} 4^k$$

by using the Maple and Zeilberger algorithm [4, 7] and proved the Sun's two conjectures above. Recently, several arithmetic properties and congruence relations for Delannoy numbers have been investigated (see [5, 6, 9, 11]).

Sun [8] presented a generalization of $d_n(x)$ as

$$d_n^{(r)}(x) = \sum_{k=0}^{n} \binom{x+r+k}{k} \binom{x-r}{n-k}$$

and established a number of identities. Moreover, a new orthogonal polynomial $D_n^{(r)}(x)$ is introduced and its properties are investigated.

In this paper, we offer a new extension of Delannoy numbers as in the following.

Definition 1.1 — Let $\left\{d_n^{(r)}(x,y)\right\}$ be the polynomials in x and y defined by

$$d_n^{(r)}(x,y) = \sum_{k=0}^{n} \binom{x+r+k}{k} \binom{x-r}{n-k} y^k \quad (\text{for } y \in \mathbb{R} \text{ and } n = 0,1,...)$$

with the convenience $d_{-1}^{(r)}(x,y) = 0$.

The first few $d_n^{(r)}(x,y)$ are as foollows:

$$d_0^{(r)}(x,y) = 1,$$

$$d_1^{(r)}(x,y) = x - r + y(x+r+1),$$

$$\begin{aligned} d_2^{(r)}(x,y) &= \frac{(x-r)(x-r-1)}{2} + (x+r+1)(x-r)\,y \\ &\quad + \frac{(x+r+2)(x+r+1)}{2} y^2. \end{aligned}$$

In particular for $y = 1$, our definition reduces to orthogonal polynomials $d_n^{(r)}(x)$, given by [8]. For $y = 1$ and $r = 0$, we have $d_n^{(0)}(x,1) = d_n(x)$. For $r = 0$, $d_n^{(0)}(x,y) = D_n(x,y)$ defined by [10, Def. 1.1].

From the fact that $\binom{x}{k} = (-1)^k \binom{-x+k-1}{k}$, we have

$$\begin{aligned} d_n^{(r)}(x,y) &= \sum_{k=0}^{n} \binom{x+r+k}{k} \binom{x-r}{n-k} y^k \\ &= \sum_{k=0}^{n} (-1)^k \binom{-1-x-r}{k} \binom{x-r}{n-k} y^k. \end{aligned} \tag{1}$$

Now, we give a relation, which generalizes the formula

$$d_n^{(r)}(-1-x) = (-1)^n\, d_n^{(r)}(x),$$

given by [10, Eq. 2.3], and we establish the generating function for $d_n^{(r)}(x,y)$, which reduces to [8, Theorem 2.1] for $y = 1$.

***Theorem* 1.2** — *For $d_n^{(r)}(x,y)$, we have*

$$d_n^{(r)}\left(-1-x, \frac{1}{y}\right) = \left(-\frac{1}{y}\right)^n d_n^{(r)}(x,y) \tag{2}$$

and

$$\sum_{n=0}^{\infty} d_n^{(r)}(x,y)\, t^n = \frac{(1+t)^{x-r}}{(1-yt)^{x+r+1}} \quad (\textit{for } |t| < 1). \tag{3}$$

With the help of the generating function (3), we reach the following identity:

Corollary 1.3 — For $n \in \mathbb{N}$, we have

$$\sum_{k=0}^{n} \binom{r+\frac{1}{2}}{k} (-y)^k\, d_{n-k}^{(r)}\left(-\frac{1}{2}, y\right) = \binom{-\frac{1}{2}-r}{n}.$$

Now, let us derive further properties of $d_n^{(r)}(x,y)$ as follows:

***Theorem* 1.4** — *For $n \in \mathbb{N}$, the following recurrence relation holds:*

$$(n+1)\,d_{n+1}^{(r)}\,(x,y) = (x-r-n+y(x+r+n+1))\,d_{n}^{(r)}\,(x,y) + y\,(n+2r)\,d_{n-1}^{(r)}\,(x,y)\,.$$

***Theorem* 1.5** — *For $n \in \mathbb{N}_0$, the following relations are valid:*

$$d_{n}^{(r+1)}\,(x,y) - yd_{n-1}^{(r+1)}\,(x,y) = \sum_{k=0}^{n} (-1)^k\, d_{n-k}^{(r)}\,(x,y)\,,$$

$$d_{n}^{(r+1)}\,(x,y) + d_{n-1}^{(r+1)}\,(x,y) = \sum_{k=0}^{n} y^k d_{n-k}^{(r)}\,(x,y)$$

and

$$d_{n}^{(r)}\,(x,y) = d_{n}^{(r+1)}\,(x,y) + (1-y)d_{n-1}^{(r+1)}\,(x,y) - yd_{n-2}^{(r+1)}\,(x,y)\,.$$

2. PROOFS

1.1 *Proofs of Theorem 1.2 and Corollary 1.3*

From the definition of $d_n^{(r)}\,(x,y)$ and the known property $\binom{x}{k} = (-1)^k \binom{-x+k-1}{k}$, one can write

$$\begin{aligned} d_n^{(r)}\left(-1-x,\frac{1}{y}\right) &= \sum_{k=0}^{n} \binom{-1-x+r+k}{k}\binom{-1-x-r}{n-k} y^{-k} \\ &= \sum_{k=0}^{n} (-1)^k \binom{x-r}{k}\binom{-1-x-r}{n-k} y^{-k} \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{x-r}{n-k}\binom{-1-x-r}{k} y^{k-n} \\ &= \sum_{k=0}^{n} (-1)^{n} \binom{x-r}{n-k}\binom{x+r+k}{k} y^{k-n} \\ &= \left(-\frac{1}{y}\right)^n d_n^{(r)}\,(x,y)\,, \end{aligned}$$

which completes the proof of (2).

Newton's binomial theorem states that, for $\alpha \in \mathbb{C}$ and $|t| < 1$, $(1+t)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n$, where the principal value of $(1+t)^\alpha$ is taken. Applying this formula, Cauchy product and the identity (1) gives that

$$
\begin{aligned}
(1+t)^{x-r}(1-yt)^{-x-r-1} &= \left(\sum_{n=0}^{\infty}\binom{x-r}{n}t^n\right)\left(\sum_{n=0}^{\infty}\binom{-x-r-1}{n}(-yt)^n\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{-x-r-1}{k}(-1)^k\binom{x-r}{n-k}y^k\right)t^n\\
&= \sum_{n=0}^{\infty}d_n^{(r)}(x,y)\,t^n,
\end{aligned}
$$

which gives the desired identity (3).

Now, let us present the proof of Corollary 1.3. Using (3), one arrives that

$$
\sum_{n=0}^{\infty}d_n^{(r)}\left(-\frac{1}{2},y\right)=\frac{(1+t)^{-\frac{1}{2}-r}}{(1-yt)^{\frac{1}{2}+r}}.
$$

That is, one has

$$
(1-yt)^{\frac{1}{2}+r}\sum_{n=0}^{\infty}d_n^{(r)}\left(-\frac{1}{2},y\right)=(1+t)^{-\frac{1}{2}-r}.
$$

Applying the Newton's binomial theorem yields

$$
\left(\sum_{n=0}^{\infty}\binom{\frac{1}{2}+r}{n}(-yt)^n\right)\left(\sum_{n=0}^{\infty}d_n^{(r)}\left(-\frac{1}{2},y\right)\right)=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}-r}{n}t^n.
$$

Now, using Cauchy product and comparing the coefficients of t^n on both sides complete the proof.

2.2 *Proof of Theorem 1.4*

By (3), one has

$$
\begin{aligned}
&\sum_{n=0}^{\infty}(n+1)\,d_{n+1}^{(r)}(x,y)\,t^n-y\sum_{n=0}^{\infty}nd_n^{(r)}(x,y)\,t^n\\
&=\left(\sum_{n=0}^{\infty}d_{n+1}^{(r)}(x,y)\,t^{n+1}\right)'-yt\left(\sum_{n=0}^{\infty}d_n^{(r)}(x,y)\,t^n\right)'\\
&=\left((1+t)^{x-r}(1-yt)^{-x-r-1}\right)'-yt\left((1+t)^{x-r}(1-yt)^{-x-r-1}\right)'\\
&=(1-yt)\left((1+t)^{x-r}(1-yt)^{-x-r-1}\right)'\\
&=(1-yt)\left((x-r)(1+t)^{x-r-1}(1-yt)^{-x-r-1}+y(x+r+1)(1+t)^{x-r}(1-yt)^{-x-r-2}\right)\\
&=(1-yt)(1+t)^{x-r}(1-yt)^{-x-r-1}\left(\frac{x-r}{1+t}+\frac{y(x+r+1)}{1-yt}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(1+t)\left(\sum_{n=0}^{\infty}(n+1)\,d_{n+1}^{(r)}(x,y)\,t^n - y\sum_{n=0}^{\infty} n d_n^{(r)}(x,y)\,t^n\right)\\
&=((x-r)(1-yt)+y(x+r+1)(1+t))\sum_{n=0}^{\infty} d_n^{(r)}(x,y)\,t^n,
\end{aligned}
$$

which yields the result.

2.3 *Proof of Theorem 1.5*

From the generating function given by (3), one can write

$$
\begin{aligned}
\sum_{n=0}^{\infty} d_n^{(r+1)}(x,y)\,t^n &= \frac{(1+t)^{x-r-1}}{(1-yt)^{x+r+2}} = \frac{1}{(1+t)(1-yt)}(1+t)^{x-r}(1-yt)^{-x-r-1}\\
&= \frac{1}{(1+t)(1-yt)}\sum_{n=0}^{\infty} d_n^{(r)}(x,y)\,t^n.
\end{aligned}
$$

Then,

$$
(1-yt)\sum_{n=0}^{\infty} d_n^{(r+1)}(x,y)\,t^n = \left(\sum_{n=0}^{\infty}(-t)^n\right)\left(\sum_{n=0}^{\infty} d_n^{(r)}(x,y)\,t^n\right)
$$

which gives the first identity by applying the Cauchy product on the right hand-side. The second identity can be obtain by a similar way by writing the equation above as

$$
(1+t)\sum_{n=0}^{\infty} d_n^{(r+1)}(x,y)\,t^n = \frac{1}{1-yt}\sum_{n=0}^{\infty} d_n^{(r)}(x,y)\,t^n.
$$

The last relation comes from the equality

$$
\sum_{n=0}^{\infty} d_n^{(r+1)}(x,y)\,t^n = \frac{1}{1-(1-y)\,t-yt^2}\sum_{n=0}^{\infty} d_n^{(r)}(x,y)\,t^n.
$$

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