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ON RAMANUJAN'S INCOMPLETE ELLIPTIC INTEGRAL IDENTITIES

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In his lost notebook, S. Ramanujan recorded incomplete elliptic integral identities of the first kind. In this paper, we give new proofs to Ramanujan's incomplete elliptic integrals of the first kind for level 5 using the parameter $k(q) = R(q)R^2(q^2)$, where $R(q)$ is Rogers-Ramanujan continued fraction. Also, we construct new identities for level 5 similar to ones found in Ramanujan's work. Further, we prove related identities of level 7, using theta function identities.

Key words : Incomplete elliptic integrals; modular equations; Dedekind eta-function; theta functions; Rogers-Ramanujan continued fraction.

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1. INTRODUCTION

In his lost notebook [14, pp. 51-53], Ramanujan recorded several unusual identities involving integrals of theta function and incomplete elliptic integral of the first kind. For example,

$$5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = \sqrt{5}\int_0^{2\tan^{-1}(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q))} \frac{d\theta}{\sqrt{1-\epsilon 5^{-1/2}\sin^2\theta}}, \tag{1.1}$$

where $\epsilon = \frac{\sqrt{5}+1}{2}$ and $\psi(q)$, $f(-q)$ are the Ramanujan's theta functions defined by (2.2) and (2.3) respectively below. These identities are uncommon in nature, where we have a integral evaluation of ratios of theta function equivalent to incomplete elliptic integrals. These type of elliptic integrals have been extensively studied in the development of elliptic functions. Elliptic integrals and elliptic functions were studied simultaneously on several occasions throughout history by L. Euler, J. L. Lagrange, C. F. Gauss, A. M. Legendre and C. G. J. Jacobi.

Raghavan and Rangachari [12] proved all these identities by employing the theory of modular forms. Berndt, Chan and Huang [4] found proofs of all these identities based entirely on results found in Ramanujan's notebook [14]. In fact their proofs depend upon some Ramanujan–Eisenstein series identities found in Chapter 21 of Ramanujan's second notebook [3] and certain Ramanujan's modular equations found in [3].

In this paper, we give quite different proof of some of the Ramanujan's identities involving integrals of theta function and incomplete elliptic integrals of the first kind. Further we derive certain new identities which are analogous to that of Ramanujan's.

Work is organized as follows: In Section 2, we recall some definitions, notations and certain theta-function identities which will be used to prove the main results. In Section 3, we prove our main results.

2. Preliminary Results

As usual, set for any complex number a and q

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1,$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

for any integer n. Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1.$$

By Jacobi's triple product identity [3, p. 35, Entry 19], we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The most important special cases of $f(a,b)$ are given by

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \tag{2.1}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.2}$$

and

$$f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.3}$$

Following Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty.$$

One can easily show that

$$\varphi(q) = \frac{{f_2}^5}{{f_1}^2{f_4}^2}, \qquad \varphi(-q) = \frac{{f_1}^2}{f_2}, \qquad \psi(q) = \frac{{f_2}^2}{f_1}, \qquad \psi(-q) = \frac{f_1 f_4}{f_2},$$

$$\chi(q) = \frac{f_2^2}{f_1 f_4}, \qquad \chi(-q) = \frac{f_1}{f_2} \qquad \text{and} \qquad f(q) = \frac{{f_2}^3}{f_1 f_4}, \tag{2.4}$$

where $f_n = f(-q^n)$.

The famous Rogers-Ramanujan continued fraction is defined by

$$R(q) = \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \frac{q^4}{1} \underset{+\cdots}{}, \qquad |q| < 1,$$

first appeared in the paper by Rogers [15] in 1894. Ramanujan later rediscovered the Rogers–Ramanujan continued fraction, and offered many beautiful theorems on $R(q)$ in his notebook [13]. In fact Rogers [15] proved that

$$R(q) = q^{1/5} \frac{f(-q;-q^4)}{f(-q^2;-q^3)} = q^{1/5} \frac{\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}}.$$

The function $R(q)$ has since been extensively studied and applied, for more details, one can see [1, 2, 5-7, 9-11, 13-16, 20, 21].

Let $k(q) := R(q)R^2(q^2)$, Ramanujan has recorded several identities involving the parameter k in his lost notebook [14]. For more details on the parameter k one may refer to [7, 9, 10]. On page 26 of his lost notebook [14], Ramanujan also defines $\mu(q) := R(q)R(q^4)$. For details one may refer to [10]. In this paper, we make use of the following identities involving parameter k and μ, whose proofs can be found in [9, 10]:

$$q\frac{\chi(-q)}{\chi^5(-q^5)} = \frac{k}{1-k^2} \tag{2.5}$$

$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)} = \frac{1-4k-k^2}{1-k^2}, \tag{2.6}$$

$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1+k-k^2}{k}, \tag{2.7}$$

$$q\frac{\chi(q)}{\chi^5(q^5)} = \frac{\mu}{(1-\mu)^2}, \tag{2.8}$$

$$\frac{\varphi(q)}{\varphi(q^5)} = \frac{1+\mu}{1-\mu} \tag{2.9}$$

and

$$\frac{\psi^2(-q)}{q\psi^2(-q^5)} = \frac{1-3\mu+\mu^2}{\mu}. \tag{2.10}$$

In an interesting paper [7], Cooper defines another parameter z by

$$z = q\frac{d}{dq}\log k. \tag{2.11}$$

and showed that

$$z = \frac{f_1 f_2^2 f_5^3}{f_{10}^2}. \tag{2.12}$$

Analogues to this parameter z, in [17] another parameter w, with respect to the parameter μ is defined by

$$w = q\frac{d}{dq}\log \mu. \tag{2.13}$$

Then

$$w = \frac{f_{10}^{12} f_1 f_4}{2 f_5^5 f_{20}^5}\sqrt{\frac{\varphi^4(q)}{\varphi^4(q^5)} - 2\frac{\varphi^2(q)}{\varphi^2(q^5)} + 5}. \tag{2.14}$$

Cooper and Ye in [8] have obtained the following Eisenstein series using the theory of modular forms:

$$\frac{1}{72}\left[-P_1 + P_2 + 7P_7 - 7P_{14}\right] - \frac{1}{3} q f_1 f_2 f_7 f_{14} = \frac{q^2 f_2^4 f_{14}^4}{f_1^2 f_7^2}, \tag{2.15}$$

$$\frac{1}{18}\left[-P_2 + 4P_4 + 7P_{14} - 28P_{28}\right] - \frac{8}{3} q f_1 f_2 f_7 f_{14} = \frac{f_1^4 f_7^4}{f_2^2 f_{14}^2}, \tag{2.16}$$

where, $P_n := P(q^n) = 1 - 24\sum_{k=1}^{\infty} \frac{kq^{nk}}{1-q^{nk}}$. For an elementary proof of the above identities one may refer [19].

3. INCOMPLETE ELLIPTIC INTEGRALS

***Theorem* 3.1** — *[1, p. 333], [4, 12], [14, p.52]. If* $v = \frac{q^{1/2} f_5^3}{f_1^3}$, *then*

$$5^{3/4} \int_0^q \frac{f^2(-t) f^2(-t^5)}{\sqrt{t}} dt = \int_0^{2\tan^{-1}(5^{3/4} v)} \frac{d\theta}{\sqrt{1 - \epsilon^{-5} 5^{-3/2} \sin^2\theta}},$$

where $\epsilon = \frac{\sqrt{5}+1}{2}$.

PROOF : From (2.4), (2.6) and (2.7), we have

$$\begin{aligned} v^2 &= \frac{q\varphi^4(-q^5)\psi^2(q^5)}{\varphi^4(-q)\psi^2(q)} \\ &= \left(\frac{1-k^2}{1-4k-k^2}\right)^2 \left(\frac{k}{1+k-k^2}\right). \end{aligned} \tag{3.1}$$

By logarithmic differentiation of the above identity and by using (2.11), we obtain

$$\frac{2}{v}\frac{dv}{dq} = \frac{z(1+k^2)[k^4 - 4k^3 + 6k^2 + 4k + 1]}{q(1-k^2)(1+k-k^2)(1-4k-k^2)}. \tag{3.2}$$

From (3.1), we have

$$\frac{1}{v^2} + 22 + 125v^2 = \frac{(1+k^2)^2[k^4 - 4k^3 + 6k^2 + 4k + 1]^2}{k(1-k^2)^2(1+k-k^2)(1-4k-k^2)^2}.$$

Using this in (3.2), we obtain

$$\frac{2}{v}\frac{dv}{dq} = \frac{z\sqrt{k}\sqrt{v^{-2} + 22 + 125v^2}}{q\sqrt{1+k-k^2}}.$$

Using (2.7) above and then using (2.4) and (2.12), we find that

$$2\frac{dv}{dq} = \frac{f_1^2 f_5^2}{q^{1/2}}\sqrt{1 + 22v^2 + 125v^4},$$

or

$$\int_0^q \frac{f^2(-t) f^2(-t^5)}{\sqrt{t}} dt = \int_0^{v(q)} \frac{2}{\sqrt{1 + 22v^2 + 125v^4}} dv. \tag{3.3}$$

Setting $v = 5^{-3/4}\tan\frac{\theta}{2}$ in the right-hand side of the above equation, we have

$$\begin{aligned}\frac{2dv}{\sqrt{1+22v^2+125v^4}} &= \frac{\sec^2(\theta/2)d\theta}{5^{3/4}\left[1+\frac{22}{5^{3/2}}\tan^2(\theta/2)+\tan^4(\theta/2)\right]^{1/2}}\\ &= \frac{d\theta}{5^{3/4}\left[\cos^4(\theta/2)+\sin^4(\theta/2)+\frac{22}{5^{3/2}}\sin^2(\theta/2)\cos^2(\theta/2)\right]^{1/2}}\\ &= \frac{d\theta}{5^{3/4}\left[1-[2-\frac{22}{5^{3/2}}]\sin^2(\theta/2)\cos^2(\theta/2)\right]^{1/2}}\\ &= \frac{d\theta}{5^{3/4}\sqrt{1-\frac{\epsilon^{-5}}{5^{3/2}}\sin^2\theta}}.\end{aligned}$$

Using this in (3.3), we obtain the required result.

***Theorem* 3.2** — *[1, p. 333], [4, 12], [14, p. 52]. If $v = q^{1/2}\frac{\psi(q^5)}{\psi(q)}$, then*

$$5^{1/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = \int_0^{2\tan^{-1}(5^{1/4}v)} \frac{d\theta}{\sqrt{1-\epsilon 5^{-1/2}\sin^2\theta}},$$

where ϵ is as in Theorem 3.1.

PROOF : From (2.7), we have

$$v^2 = \frac{k}{1+k-k^2}.$$

By logarithmic differentiation of v and then using (2.11), we obtain

$$\frac{2}{v}\frac{dv}{dq} = \frac{z}{q}\left(\frac{k}{1+k-k^2}\right)\frac{1+k^2}{k}. \tag{3.4}$$

From (2.7), we have

$$\frac{1+k^2}{k} = \frac{1}{v}\sqrt{\frac{1}{v^2}-2+5v^2},$$

or

$$\left(\frac{k}{1+k-k^2}\right)\frac{1+k^2}{k} = \sqrt{1-2v^2+5v^4}.$$

Using this in (3.4) and then employing (2.12), we find that

$$2\frac{dv}{dq} = \frac{f_1^2 f_5^2}{\sqrt{q}}\sqrt{1-2v^2+5v^4}.$$

Or

$$\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = \int_0^{v(q)} \frac{2}{\sqrt{1-2v^2+5v^4}}dv.$$

Setting $v = 5^{-1/4}\tan(\theta/2)$ in the right-hand side of the above and then proceeding as in the proof of previous theorem, we obtain the required result.

***Theorem* 3.3** — *If $v = q^{1/2}\frac{\psi(-q^5)}{\psi(-q)}$, then*

$$\sqrt[4]{5}\int_0^q \frac{f^2(t)f^2(t^5)}{\sqrt{t}}dt = \int_0^{2\tan^{-1}(5^{1/4}v)} \frac{d\theta}{\sqrt{1-\epsilon^{-1}5^{-1/2}\sin^2\theta}},$$

where ϵ is as in Theorem 3.1.

PROOF : From (2.10), we have

$$v^2 = \frac{\mu}{1-3\mu+\mu^2}.$$

By logarithmic differentiation of the above and the use of (2.13), we obtain

$$2\frac{q}{v}\frac{dv}{dq} = \frac{w(1-\mu^2)}{1-3\mu+\mu^2}.$$

Upon using (2.8), (2.10) and (2.14) above, we obtain

$$2\frac{q}{v}\frac{dv}{dq} = v^2\left(\frac{1-\mu^2}{\mu}\right)\frac{f_{10}^{12}f_1f_4}{2f_5^5f_{20}^5}\sqrt{\frac{\varphi^4(q)}{\varphi^4(q^5)} - 2\frac{\varphi^2(q)}{\varphi^2(q^5)} + 5}. \tag{3.5}$$

We have from [9, eq. (2.8),(2.11)]

$$\frac{\varphi^2(q)}{\varphi^2(q^5)} = 1 + 4q\frac{\chi(q)}{\chi^5(q^5)}$$
$$\frac{\psi^2(-q)}{q\psi^2(-q^5)} + 1 = \frac{\chi^5(q^5)}{q\chi(q)}.$$

From the above two identities, we deduce that

$$\frac{\varphi^2(q)}{\varphi^2(q^5)} = \frac{5v^2+1}{v^2+1}.$$

Using above in (3.5) and then employing (2.4) and (2.10), we obtain

$$2\frac{dv}{dq} = \frac{f^2(q)f^2(q^5)}{q^{1/2}}\sqrt{5v^4+2v^2+1},$$

or

$$\int_0^q \frac{f^2(t)f^2(t^5)}{\sqrt{t}}dt = 2\int_0^{v(q)} \frac{dv}{\sqrt{5v^4+2v^2+1}}.$$

Setting $v = 5^{-1/4}\tan(\theta/2)$ in the right-hand side of the above, we obtain the required result.

***Theorem* 3.4** — *If* $v = \frac{qf_{10}^3}{f_2^3}$, *then*

$$2\,5^{3/4}\int_0^q f^2(-t^2)f^2(-t^{10})dt = \int_0^{2\tan^{-1}(5^{3/4}v)} \frac{d\theta}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\theta}}.$$

where ϵ *is as in Theorem 3.1.*

PROOF : From (2.4), (2.6) and (2.7), we have

$$\begin{aligned} v^2 &= q^2\frac{\varphi^2(-q^5)\psi^4(q^5)}{\varphi^2(-q)\psi^4(q)} \\ &= \frac{1-k^2}{1-4k-k^2}\left(\frac{k}{1+k-k^2}\right)^2. \end{aligned}$$

By logarithmic differentiation of above and the use of (2.11), we obtain

$$\frac{q}{v}\frac{dv}{dq} = z\frac{(1+k^2)(k^4+2k^3-2k+1)}{(1-k^2)(1+k-k^2)(1-4k-k^2)}. \tag{3.6}$$

One can see that

$$\frac{1}{v^2}+22+125v^2 = \frac{(1+k^2)^2(k^4+2k^3-2k+1)^2}{k^2(1-k^2)(1+k-k^2)^2(1-4k-k^2)}.$$

Using this in (3.6), we obtain

$$q\frac{dv}{dq} = z\left(\frac{k}{1-k^2}\right)\sqrt{\frac{1-k^2}{1-4k-k^2}}\sqrt{125v^4+22v^2+1}.$$

Upon using (2.5), (2.7) above and then using (2.4) and (2.12), we find that

$$\frac{dv}{dq} = f_2^2f_{10}^2\sqrt{125v^4+22v^2+1},$$

or

$$\int_0^q f^2(-t^2)f^2(-t^{10})dt = \int_0^{v(q)} \frac{dv}{\sqrt{125v^4+22v^2+1}}.$$

Setting $v = 5^{-3/4}\tan(\theta/2)$ in the right-hand side of the above, we obtain the required result.

***Theorem* 3.5 —** *[1, pp. 356-361], [4, 12]. Let* $v = q\frac{\chi^4(-q)}{\chi^4(-q^7)}$*, then*

$$\frac{dv}{dq} = f_1 f_2 f_7 f_{14}\sqrt{v^4 - 14v^3 + 19v^2 - 14v + 1}.$$

PROOF : From (2.4), we have

$$v = q\frac{f_1^4 f_{14}^4}{f_2^4 f_7^4}.$$

By logarithmic differentiation of above and then using the definition of P_n, we find that

$$\frac{q}{v}\frac{dv}{dq} = \frac{1}{6}[P_1 - 2P_2 - 7P_7 + 14P_{14}]. \tag{3.7}$$

From [19, eq. 3.19, 3.20], we have

$$P_1 - 2P_2 - 7P_7 + 14P_{14} = 6[\varphi^2(-q)\varphi^2(-q^7) - 8q^2\psi^2(q)\psi^2(q^7)].$$

Substituting the above in (3.7) gives

$$\frac{q}{v}\frac{dv}{dq} = \varphi^2(-q)\varphi^2(-q^7) - 8q^2\psi^2(q)\psi^2(q^7).$$

From (2.4) and above equation, we have

$$\frac{q}{v}\frac{dv}{dq} = qf_1f_2f_7f_{14}\left[\frac{f_1^3f_7^3}{qf_2^3f_{14}^3} - 8q\frac{f_2^3f_{14}^3}{f_1^3f_7^3}\right]. \tag{3.8}$$

From [19, eq. 3.14], we have

$$-q\frac{\chi^4(-q)}{\chi^4(-q^7)} - \frac{\chi^4(-q^7)}{q\chi^4(-q)} + 7 = \frac{-\chi^3(-q)\chi^3(-q^7)}{q} - \frac{8q}{\chi^3(-q)\chi^3(-q^7)}.$$

Upon using (2.4) in left-hand side of the above equation, we obtain

$$\frac{f_1^3f_7^3}{qf_2^3f_{14}^3} + 8q\frac{f_2^3f_{14}^3}{f_1^3f_7^3} = v + \frac{1}{v} - 7.$$

This implies

$$\frac{f_1^3f_7^3}{qf_2^3f_{14}^3} - 8q\frac{f_2^3f_{14}^3}{f_1^3f_7^3} = \frac{\sqrt{v^4 - 14v^3 + 19v^2 - 14v + 1}}{v}.$$

Using the above in (3.8), we obtain the required result.

Remark : Setting $\cos\theta = c\left(\frac{1+v}{1-v}\right)$, where $c = \frac{\sqrt{13+16\sqrt{2}}}{7}$ in Theorem 3.7, Berndt *et al*. [4, Theorem 8.5], have obtained the following Ramanujan's identity:

$$\int_0^q f(-t)f(-t^2)f(-t^7)f^2(-t^{14})dt = \frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c} \frac{d\theta}{\sqrt{1-\frac{16-\sqrt{2}-13}{32\sqrt{2}}\sin^2\theta}}.$$

***Theorem* 3.6** — *If* $v^2 = \frac{q\chi(-q)}{\chi^5(-q^5)}$, *then*

$$\int_0^q \left(\frac{f^3(-t)f^3(-t^2)f(-t^5)f(-t^{10})}{t}\right)^{1/2} dt = \frac{1}{\sqrt{2}}\int_0^{2\tan^{-1}(\sqrt{2}v)} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}.$$

PROOF : From (2.5), we have

$$v^2 = \frac{k}{1-k^2}.$$

By logarithmic differentiation of above and the use of (2.11), we find that

$$\frac{2q}{v}\frac{dv}{dq} = z\left(\frac{1+k^2}{1-k^2}\right),$$

or

$$\frac{2q}{v}\frac{dv}{dq} = zv^2\left(\frac{1+k^2}{k}\right). \tag{3.9}$$

From (2.5), we have

$$\frac{1+k^2}{k} = \frac{\sqrt{4v^4+1}}{v^2}.$$

Using the above in (3.9) and employing (2.4) and (2.12), we obtain

$$2\frac{dv}{dq} = \left(\frac{f_1^3f_2^3f_5f_{10}}{q}\right)^{1/2}\sqrt{4v^4+1},$$

or

$$\int_0^q \left(\frac{f_1^3f_2^3f_5f_{10}}{q}\right)^{1/2} dq = \int_0^{v(q)} \frac{2dv}{\sqrt{4v^4+1}}.$$

Setting $v = 2^{-1/2}\tan(\theta/2)$ in the right-hand side of the above, we obtain the required result.

***Theorem* 3.7** — *If* $v^2 = \frac{q\chi(q)}{\chi^5(q^5)}$, *then*

$$\int_0^q \left(\frac{f^{12}(-t^2)f^4(-t^{10})}{t\, f^3(-t)f^3(-t^4)f(-t^5)f(-t^{20})}\right)^{1/2} dt = \frac{1}{16}\int_0^{2\tan^{-1}(\sqrt{2}v)} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}.$$

PROOF : From (2.8), we have

$$v^2 = \frac{\mu}{(1-\mu)^2}. \tag{3.10}$$

By logarithmic differentiation and the use of (2.13), we have

$$\frac{2q}{v}\frac{dv}{dq} = w\frac{1+\mu}{1-\mu}.$$

Upon using (2.9) and (2.14), above we obtain

$$\frac{2q}{v}\frac{dv}{dq} = \frac{\varphi(q)}{\varphi(q^5)}\left(\frac{f_{10}^{12}f_1f_4}{2f_5^5f_{20}^5}\right)\sqrt{\frac{\varphi^4(q)}{\varphi^4(q^5)} - 2\frac{\varphi^2(q)}{\varphi^2(q^5)} + 5}. \tag{3.11}$$

From [9, eq. 2.8], we have

$$\frac{\varphi^2(q)}{\varphi^2(q^5)} = 1 + 4v^2.$$

By using above in (3.11) and the use of (3.10) and (2.4), we obtain

$$\frac{dv}{dq} = \frac{1}{4}\left(\frac{f_2^{12}f_{10}^4}{qf_1^3f_4^3f_5f_{20}}\right)^{1/2}\sqrt{4v^4+1}.$$

Setting $v = 2^{-1/2}\tan(\theta/2)$ in the right-hand side of the above, we obtain the required result.

***Theorem* 3.8** — *If* $v = q\frac{f_1^6f_{10}^6}{f_2^6f_5^6}$, *then*

$$\int_0^q \left(\frac{f^5(-t)f^5(-t^{10})}{t^2\, f(-t^2)f(-t^5)}\right)^{1/2} dt = \frac{1}{\sqrt{2}5^{1/4}}\int_{\sin^{-1}\sqrt{\frac{8\sqrt{5}}{9+4\sqrt{5}}}}^{\sin^{-1}\sqrt{\frac{8\sqrt{5}}{9+4\sqrt{5}-v}}} \frac{d\theta}{\sqrt{1-\left(\frac{2+\sqrt{5}}{2\sqrt{5}}\right)\sin^2\theta}}.$$

PROOF : From (2.6) and (2.7), we have

$$v = \frac{1-4k-k^2}{1-k^2}\left(\frac{k}{1+k-k^2}\right).$$

By logarithmic differentiation of the above and the use of (2.11), we have

$$\frac{q}{v}\frac{dv}{dq} = z\frac{(1+k^2)\left[k^4+8k^3-6k^2-8k+1\right]}{(1-k^2)(1+k-k^2)(1-4k-k^2)}. \tag{3.12}$$

One can easily see that

$$\frac{v^2-18v+1}{v} = \frac{\left[k^4+8k^3-6k^2-8k+1\right]^2}{k(1-k^2)(1+k-k^2)(1-4k-k^2)}$$

and

$$1-v = \frac{(1+k^2)^2}{(1-k^2)(1+k-k^2)}.$$

Using above in (3.12) and then employing (2.12), we obtain

$$q\frac{dv}{dq} = \left(\frac{qf_2^5f_5^5}{f_1f_{10}}\right)^{1/2}\sqrt{v(1-v)(v^2-18v+1)},$$

or

$$\left(\frac{qf_2^5f_5^5}{f_1f_{10}}\right)^{1/2} dq = \frac{dv}{\sqrt{v(1-v)(v^2-18v+1)}}. \tag{3.13}$$

Setting

$$\sin^2\theta = \frac{-8\sqrt{5}}{v-9-4\sqrt{5}}, \tag{3.14}$$

it follows that

$$\frac{v-1}{v-9-4\sqrt{5}} = 1-\left(\frac{2+\sqrt{5}}{2\sqrt{5}}\right)\sin^2\theta$$

and

$$\frac{d\theta}{dv} = \frac{(-2\sqrt{5})^{1/2}}{(v-9-4\sqrt{5})\sqrt{v-9+4\sqrt{5}}}.$$

Using the above in the right-hand side of (3.13), we obtain

$$\frac{dv}{\sqrt{(1-v)(v^2-18v+1)}} = \frac{d\theta}{\sqrt{2}5^{1/4}\sqrt{1-\left(\frac{2+\sqrt{5}}{2\sqrt{5}}\right)\sin^2\theta}}.$$

Which is the required result.

Next two identities are due to Berndt *et al*. [4], where they have proved the identities using the theory of modular forms.

***Theorem* 3.9** — *[4]. If* $v = q\left(\frac{f(-q^2)f(-q^{10})}{f(-q)f(-q^5)}\right)^4$, *then*

$$\frac{dv}{dq} = q^{-1/2} f^2(-q) f^2(-q^5)\sqrt{v(4v+1)(16v^2+12v+1)}.$$

PROOF : For $u = \frac{qf_1^6 f_{10}^6}{f_2^6 f_5^6}$, we have from (3.13)

$$\frac{du}{dq} = \left(\frac{f_2^5 f_5^5}{qf_1 f_{10}}\right)^{1/2} \sqrt{u(1-u)(u^2-18u+1)}. \tag{3.15}$$

From [18], we have for $P = \frac{f(-q)}{q^{1/24} f(-q^2)}$ and $Q = \frac{f(-q^5)}{q^{5/24} f(-q^{10})}$.

$$P^2Q^2 + \frac{4}{P^2Q^2} = \frac{Q^3}{P^3} - \frac{P^3}{Q^3}, \tag{3.16}$$

This can be written as

$$\frac{1}{\sqrt{v}} + 4\sqrt{v} = \frac{1}{\sqrt{u}} - \sqrt{u}. \tag{3.17}$$

Which implies,

$$\frac{1}{v} + 16v = \frac{1}{u} + u - 10 \tag{3.18}$$

$$16v - \frac{1}{v} = \frac{(1-u)\sqrt{u^2-18u+1}}{u} \tag{3.19}$$

and

$$u - \frac{1}{u} = \frac{(1+4v)\sqrt{16v^2+12v+1}}{v}. \tag{3.20}$$

Differentiating (3.17) with respect to q, we obtain

$$\frac{dv}{dq} = \frac{v(u-\frac{1}{u})}{u(16v-\frac{1}{v})}\frac{du}{dq}.$$

Upon using (3.15), (3.17) and (3.18) above, we obtain the required result

***Theorem* 3.10** — *If* $v = \frac{qf_2^3 f_{14}^3}{f_1^3 f_7^3}$, *then*

$$\frac{dv}{dq} = f_1 f_2 f_7 f_{14}\sqrt{(8v+1)(v+1)(8v^2+5v+1)}.$$

PROOF : From Theorem 3.7, we have for $w = q\frac{f_1^4 f_{14}^4}{f_2^4 f_7^4}$.

$$\frac{dw}{dq} = f_1 f_2 f_7 f_{14}\sqrt{w^4 - 14w^3 + 19w^2 - 14w + 1}. \tag{3.21}$$

From [18], we have for $P = \frac{f_1}{q^{1/24} f_2}$ and $Q = \frac{f_7}{q^{1/24} f_{14}}$,

$$P^3Q^3 + \frac{8}{P^3Q^3} + 7 = \frac{P^4}{Q^4} + \frac{Q^4}{P^4}.$$

This implies

$$w + \frac{1}{w} = 8v + \frac{1}{v} + 7, \tag{3.22}$$

$$w - \frac{1}{w} = \frac{1}{v}\sqrt{(8v+1)(v+1)(8v^2+5v+1)} \tag{3.23}$$

and

$$8v - \frac{1}{v} = \frac{1}{w}\sqrt{w^4 - 14w^3 + 19w^2 - 14w + 1}. \tag{3.24}$$

Differentiating (3.22) with respect to q, we obtain

$$\frac{dv}{dq} = \frac{v\left(w - \frac{1}{w}\right)}{w\left(8v - \frac{1}{v}\right)}\frac{dw}{dq}.$$

Using (3.21), (3.22), (3.23) and (3.24) in the above and simplifying, we obtain the required result.

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