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SOME NEW FACTS ABOUT (PSEUDO) INVOLUTIONS IN THE RIORDAN GROUP

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We answer the question posed by L.W. Shapiro about a property of elements whose (pseudo) order in the Riordan group is equal to 2. We prove that if $(g(z), f(z))$ is such an element, then, with some restictions, f is uniquely determined by g.

Key words : Riordan matrix; pseudo-involution; involution.

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1. Introduction

A proper Riordan array is a pair $(g(z), f(z))$ of analytic functions of the forms $g(z) = g_0 + g_1z + g_2z^2 + \ldots$, $f(z) = f_1z + f_2z^2 + f_3z^3 + \ldots$ with $g_0, f_1 \neq 0$. Such pair can be identified with an infinite lower triangular matrix R whose entries are defined according to the rule

$$r_{nm} = [z^n][g(z)(f(z))^m].$$

The pairs are multiplied as follows:

$$(g_1(z), f_1(z)) * (g_2(z), f_2(z)) = (g_1(z)g_2(f_1(z)), f_2(f_1(z))).$$

It can be easily noticed that the identity element of this multiplication is $(1, z)$ and that

$$(g(z), f(z))^{-1} = \left(\frac{1}{g(\overline{f}(z))}, \overline{f}(z)\right),$$

where $\overline{f}$ is the (compositional) inverse of f. Thus, the set of all Riordan arrays together with $*$ forms a group $\mathcal{R}$ called the Riordan group (this terminology appeared first in [7]).

It was proved in [6] that a finite order of an element of $\mathcal{R}$ can be equal either to 1 or 2. Notice that when we consider Riordan arrays in a combinatorial context, their entries are positive, so only $(1, z)$ is an involution. Therefore, one introduces a definition of (pseudo) involution as follows.

Definition 1.1 — We call R a (pseudo) involution, if RM has order 2, where $M = (1, -z)$.

In [4] and [2] we can find characterization of (pseudo) involutions in terms of $A-$, $Z-$ and $\Delta-$sequences.

In [6] we can find open questions about (pseudo) involutions in $\mathcal{R}$. In particular

> **Q9.** If $D = (g(z), f(z))$ has order 2, is there a simple condition for $g(z)$ in terms of $f(z)$?

This question was studied in [3]. Namely, it was proved that if $(g(z), f(z))$ has (pseudo) order 2 in $\mathcal{R}$, then $g(z) = \pm e^{\Phi(z, zf(z))}$ for some antisymmetric Φ.

In this article we would like to present some new information in this field. We will prove the following.

***Theorem* 1.1** — *Suppose that $R = (g(z), f(z)) \in \mathcal{R}$ is a (pseudo) involution, has nonnegative terms and $g_1 \neq 0$. Then f is uniquely determined by g.*

To be more precise, when we say that f is determined by g we mean that all terms of f can be expressed as functions of terms of g.

Let us note that this theorem is true for any g, f such that $g_0 = f_1 = 1$ and $g_1, f_2 \geq 0$. If $g_0 = f_1 = 1$, but we do not put any assumptions on f_2, then f_2 can be determined with accuracy to the sign, and f_n for $n \geq 3$ can be uniquely determined by g. If $g_0 = f_1 = 1$, but $g_1 = 0$, then for the given g there might exist more functions f such that $(g(z), f(z))$ will be an involution.

2. PROOF

To prove our result we first discuss properties of Riordan matrices (Section 2.1) and then properties of involutions (Section 2.2). Next we join these results (in Section 2.3).

Let us underline that we assume that all Riordan matrices that appear in this paper satisfy the condition $g_0 = f_1 = 1$.

2.1 *Riordan arrays*

It is known that the entries of Riordan matrices can be written as linear combinations of some other entries lying in their neigbourhood (for more information see [5]). In our proof we would like to make use of a similar property. Namely, we would like to prove that the entries of R lying on the k-th diagonal depend only on the entries lying on the preceding diagonals. The lemmas presented in this section are true for all Riordan matrices (not only those with $g_0 = f_1 = 1$). First we discuss the first diagonal.

Lemma 2.1 — Let $R = (g(z), f(z)) \in \mathcal{R}$. Then

$$r_{n+1,n} = nf_2 + g_1.$$

PROOF : From the defintion we have

$$\begin{aligned} r_{n+1,n} = &\ [z^{n+1}][(1 + g_1z + g_2z^2 + \ldots)(z + f_2z^2 + f_3z^3 + \ldots)^n] \\ = &\ [z^{n+1}][(z + f_2z^2 + f_3z^3 + \ldots)^n] + g_1[z^n][(z + f_2z^2 + f_3z^3 + \ldots)^n] \\ = &\ nf_2 + g_1. \end{aligned}$$

Now we move to the second diagonal.

Lemma 2.2 — Let $R = (g(z), f(z)) \in \mathcal{R}$. Then

$$r_{n+2,n} = nf_3 + g_2 + \frac{1}{2}(n^2 - n)f_2^2 + ng_1f_2.$$

PROOF : Again from the definition

$$\begin{aligned} r_{n+2,n} = &\ [z^{n+2}][(1 + g_1z + g_2z^2 + \ldots)(z + f_2z^2 + f_3z^3 + \ldots)^n] \\ = &\ [z^{n+2}][(z + f_2z^2 + f_3z^3 + \ldots)^n] \\ &+ g_1[z^{n+1}][(z + f_2z^2 + f_3z^3 + \ldots)^n] \\ &+ g_2[z^n][(z + f_2z^2 + f_3z^3 + \ldots)^n] \\ = &\ nf_3 + \tbinom{n}{2}f_2^2 + g_1(nf_2) + g_2 = nf_3 + g_2 + \tfrac{n^2-n}{2}f_2^2 + ng_1f_2. \end{aligned}$$

We end with the k-th diagonal for $k \geq 3$.

Lemma 2.3 — If $R = (g(z), f(z)) \in \mathcal{R}$ and $k \geq 3$, then

$$r_{n+k,n} = n \cdot f_{k+1} + [(n^2 - n)f_2 + ng_1] \cdot f_k + \phi_{nk},$$

where ϕ_{nk} is a number that depends only on the terms f_2, f_3, $\ldots$, f_{k-1}, g_1, g_2, $\ldots$, g_{k-1}, g_k.

PROOF : We proceed analogously as in the previous lemmas.

$$\begin{aligned}
r_{n+k,n} &= [z^{n+k}][(1+g_1z+g_2z^2+\ldots)(z+f_2z^2+f_3z^3+\ldots)^n] \\
&= [z^{n+k}][(z+f_2z^2+f_3z^3+\ldots)^n] \\
&\quad +g_1[z^{n+k-1}][(z+f_2z^2+f_3z^3+\ldots)^n] \\
&\quad +g_2[z^{n+k-2}][(z+f_2z^2+f_3z^3+\ldots)^n] \\
&\quad + \quad \cdots \\
&\quad +g_{k-1}[z^{n+1}][(z+f_2z^2+f_3z^3+\ldots)^n] \\
&\quad +g_k[z^n][(z+f_2z^2+f_3z^3+\ldots)^n] \\
&= \binom{n}{1}f_{k+1}+\binom{n}{1}\binom{n-1}{1}f_2f_k+\phi_{n,k,n+k}(f_2,\ldots,f_{k-1},g_1,\ldots,g_{k-2}) \\
&\quad +g_1\left[\binom{n}{1}f_k+\binom{n}{1}\binom{n-1}{1}f_2f_{k-1}+\phi_{n,k,n+k-1}(f_2,\ldots,f_{k-1},g_1,\ldots,g_{k-2})\right] \\
&\quad +\sum_{i=2}^{k} g_i\cdot\phi_{n,k,n+k-i}(f_2,\ldots,f_{k-1},g_1,\ldots,g_k) \\
&= nf_{k+1}+[(n^2-n)f_2+ng_1]f_k+\phi_{nk}(f_2,\ldots,f_{k-1},g_1,\ldots,g_k)
\end{aligned}$$

2.2 *Involutions*

We start with citing a theorem describing the structure of involutions. The symbols $\mathrm{T}_n(K)$ and $\mathrm{T}_\infty(K)$ stand for the groups of upper triangular matrices, $n\times n$ and $\mathbb{N}\times\mathbb{N}$, respectively.

***Theorem* 2.1** — *(Thm.1.1, [8]). Let K be a field of characteristic different from 2 and let G be either the group $\mathrm{T}_\infty(K)$ or $\mathrm{T}_n(K)$ for some $n\in\mathbb{N}$. A matrix $g\in G$ is an involution if and only if g is described by the following statements.*

1. *For all i we have $g_{ii}\in\{1,-1\}$.*

2. *For all pairs of indices $1\le i<j$ such that $g_{ii}=-g_{jj}$, the coefficients g_{ij} may be arbitrary.*

3. *For all pairs of indices $1\le i<j$ such that $g_{ii}=g_{jj}$ we have*

$$g_{ij}=\begin{cases}-(2g_{ii})^{-1}\sum\limits_{p=i+1}^{j-1}g_{ip}g_{pj} & if \quad j>i+1\\ 0 \qquad if \quad j=i+1.\end{cases} \tag{1}$$

Clearly, an analogous result is true for lower triangular matrices.

Since

$$M = \begin{bmatrix} -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \\ 0 & 0 & -1 & \\ \vdots & & & \ddots \end{bmatrix},$$

we have $(MR)_{nn} = (MR)_{mm}$ if and only if the difference $m-n$ is even. In this case, condition (1) looks as follows:

$$(-1)^n r_{n+2k,n} = \frac{(-1)^{n+1}}{2} \sum_{i=n+1}^{n+2k-1} (-1)^i r_{n+2k,i} \cdot (-1)^n r_{in}.$$

Thus, we have

$$r_{n+2k,n} = \frac{(-1)^{n+1}}{2} \sum_{i=n+1}^{n+2k-1} (-1)^i r_{n+2k,i} r_{in}. \tag{2}$$

2.3 *Proof of Theorem 1.1*

Now we will join the results of Sections 2.1 and 2.3 to prove the main theorem. First notice the following.

Lemma 2.4 — Suppose $R = (g(z), f(z)) \in \mathcal{R}$ is a (pseudo) involution. Then for any even k we have

$$r_{n+k,n} = \begin{cases} \frac{1}{2}(n^2+n)f_2^2 + \frac{1}{2}(2n+1)f_2 g_1 + \frac{1}{2}g_1^2 & \text{if } k=2 \\ \frac{1}{2}[(2n^2+nk)f_2 + (2n+1)g_1]f_k + \psi_{nk} & \text{if } k \geq 4, \end{cases}$$

where ψ_{nk} is a number depending only on the terms f_2, f_3, ..., f_{k-1}, g_1, g_2, ..., g_{k-1}.

PROOF : We apply results of Lemmas 2.1 and 2.3 to Eq. (2).

For $k=2$ we have

$$\begin{aligned} r_{n+2,n} = {} & \tfrac{1}{2} r_{n+2,n+1} r_{n+1,n} = \tfrac{1}{2}[(n+1)f_2 + g_1](nf_2 + g_1) \\ = {} & \tfrac{1}{2}(n^2+n)f_2^2 + \tfrac{1}{2}(2n+1)f_2 g_1 + \tfrac{1}{2}g_1^2 \end{aligned}$$

whereas for $k \geq 4$:

$$\begin{aligned} r_{n+k,n} = {} & \tfrac{(-1)^{n+1}}{2} \textstyle\sum_{i=n+1}^{n+k-1} (-1)^i r_{n+k,i} r_{in} \\ = {} & \tfrac{(-1)^{n+1}}{2} \Big\{ (-1)^{n+1}[(n+1)f_k + g_{k-1} + \psi_{n+k,n,n+1}((f_j)_{j=2}^{k-1}, (g_j)_{j=1}^{k-2})] \cdot (nf_2 + g_1) \\ & + \sum_{i=n+2}^{n+k-2} (-1)^i \psi_{n+k,n,i}((f_j)_{j=2}^{k-1}, (g_j)_{j=1}^{k-2}) \cdot \psi_{n+k,n,k-i}((f_j)_{j=2}^{k-1}, (g_j)_{j=1}^{k-2}) \\ & + (-1)^{n+k-1}[(n+k-1)f_2 + g_1] \cdot [nf_k + g_{k-1} + \psi_{n+k,n,n+k-1}((f_j)_{j=2}^{k-1}, (g_j)_{j=1}^{k-2})] \Big\} \\ = {} & \tfrac{1}{2}[(n^2+n)f_2 + (n+1)g_1]f_k + \tfrac{(-1)^k}{2}[(n^2+n)f_2 + (n+1)g_1]f_k + \psi_{nk}((f_j)_{j=2}^{k-1}, (g_j)_{j=1}^{k-2}). \end{aligned}$$

Since k is even, the latter is equal to

$$\frac{1}{2}[(2n^2+nk)f_2+(2n+1)g_1]f_k+\psi_{nk}(f_2,\dots,f_{k-1},g_1,\dots,g_{k-1})\square.$$

Now we are ready to prove our main result.

PROOF OF THEOREM 1.1 : The proof is inductive on the number of consecutive diagonals.

First we will prove that f_2 depend only on g_1 and that f_3 and g_2 depend only on f_2 and g_1.

From Lemma 2.2 we know that

$$r_{n+2,n}=nf_3+g_2+\frac{1}{2}(n^2-n)f_2^2+ng_1f_2, \tag{3}$$

whereas from Lemma 2.4 we have

$$r_{n+2,n}=\frac{1}{2}(n^2+n)f_2^2+\frac{1}{2}(2n+1)f_2g_1+\frac{(-1)^{n+1}}{2}g_1^2. \tag{4}$$

From (3) and (4) we get an infinite system of equations of the form

$$nf_3+g_2+\frac{n^2-n}{2}f_2^2+ng_1f_2=\frac{1}{2}(n^2+n)f_2^2+\frac{1}{2}(2n+1)f_2g_1+\frac{(-1)^{n+1}}{2}g_1^2$$

for $n\in\mathbb{N}$.

After rearranging we have

$$nf_3+g_2=nf_2^2+\frac{1}{2}f_2g_1+\frac{1}{2}g_1^2 \quad \text{for } n\in\mathbb{N}, \tag{5}$$

i.e.

$$\left[\begin{array}{cc|c} 1 & 1 & f_2^2+\frac{1}{2}f_2g_1+\frac{1}{2}g_1^2 \\ 2 & 1 & 2f_2^2+\frac{1}{2}f_2g_1+\frac{1}{2}g_1^2 \\ 3 & 1 & 3f_2^2+\frac{1}{2}f_2g_1+\frac{1}{2}g_1^2 \\ \vdots & \vdots & \vdots \\ n & 1 & nf_2^2+\frac{1}{2}f_2g_1+\frac{1}{2}g_1^2 \\ \vdots & \vdots & \vdots \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & f_2^2+\frac{1}{2}f_2g_1+\frac{1}{2}g_1^2 \\ 1 & 0 & f_2^2 \\ 2 & 0 & 2f_2^2 \\ \vdots & \vdots & \vdots \\ n-1 & 0 & (n-1)f_2^2 \\ \vdots & \vdots & \vdots \end{array}\right]$$

One can see that system (5) has one solution

$$f_3=\frac{1}{2}f_2g_1+\frac{1}{2}g_1^2, \quad g_2=f_2^2.$$

Clearly, this means that f_2 is equal to either $\sqrt{g_2}$ or $-\sqrt{g_2}$, so as we assume that the terms of g and f are nonnegative, f_2 is uniquely determined by g_2. As f_3 depends on f_2 and g_1, f_3 is determined by g_1 and g_2.

Suppose now that for some even k we have proved that for all even l, $2 \leq l \leq k-1$, the terms f_{l+1}, f_l depend only on $f_2, f_3, \ldots, f_{l-1}, g_1, g_2, \ldots, g_l$, and consequently only on $g_1, \ldots, g_l$. Consider $r_{n+k,n}$ and equations from Lemmas 2.3 and 2.4. After equating them we get

$$nf_{k+1} + \left[\left(-n - \frac{nk}{2}\right) f_2 - \frac{1}{2}g_1\right] f_k = \psi_{nk} - \phi_{nk} = \alpha_{nk},$$

where α_{nk} depends only on $f_2, \ldots, f_{k-1}$ and $g_1, \ldots, g_k$.

Hence, we have the following system of equations

$$\left[\begin{array}{cc|c} 1 & \left(-1-\frac{k}{2}\right) f_2 - \frac{1}{2}g_1 & \alpha_{1k} \\ 2 & \left(-2-k\right) f_2 - \frac{1}{2}g_1 & \alpha_{2k} \\ 3 & \left(-3-\frac{3k}{2}\right) f_2 - \frac{1}{2}g_1 & \alpha_{3k} \\ \vdots & \vdots & \vdots \\ n & \left(-n-\frac{nk}{2}\right) f_2 - \frac{1}{2}g_1 & \alpha_{nk} \\ \vdots & \vdots & \vdots \end{array}\right]$$

which is equivalent to

$$\left[\begin{array}{cc|c} 1 & \left(-1-\frac{k}{2}\right) f_2 - \frac{1}{2}g_1 & \alpha_{1k} \\ 0 & \frac{1}{2}g_1 & \alpha_{2k} - 2\alpha_{1k} \\ 0 & g_1 & \alpha_{3k} - 3\alpha_{3k} \\ \vdots & \vdots & \vdots \\ 0 & \frac{n-1}{2}g_1 & \alpha_{nk} - n\alpha_{1k} \\ \vdots & \vdots & \vdots \end{array}\right].$$

As we can see, since $g_1 \neq 0$,this system may have at most one solution. Therefore, f_{k+1}, f_k are uniquely determined by $f_2, \ldots, f_{k-1}, g_1, \ldots, g_k$, and consequently, from the induction it follows that all the terms of f are uniquely determined by the terms of g. □

Let us note that in particular we have

Corollary 2.1 — Suppose that $R = (g(z), f(z)) \in \mathcal{R}$ has only integer entries. If $2 \nmid g_1$, $2|f_2$, then R is not a (pseudo) involution.

PROOF : From Lemma 2.1 and our assumptions it follows that for all n the entries $r_{n+1,n}$ are odd. However, from (1) we know that $r_{n+2,n} = \frac{1}{2}r_{n+2,n+1}r_{n+1,n}$. The latter implies $r_{n+2,n} \notin \mathbb{Z}$ – a contradiction.

3. Closing comments

At the end of the paper let us attract the attention to some interesting fact. We have proved that if $(g(z), f(z))$ is an involution, then f is uniquely determined by g. Hence, it seems to make sense that g should be uniquely determined by f. However, when we try to prove it the same way as we did in the paper, it turns out to be impossible. Namely, we can generalize Lemma 2.3 and get

$$r_{n+k,n} = n \cdot f_{k+1} + [(n^2 - n)f_2 + ng_1] \cdot f_k + g_k + nf_2 \cdot g_{k-1} + \phi_{nk}.$$

After generalizing Lemma 2.4 we get

$$r_{n+k,n} = \frac{1}{2}\left[(2n^2 + nk)f_2 + (2n+1)g_1\right] f_k + \left[\frac{1}{2}(2n + k - 1)f_2 + g_1\right] g_{k-1}.$$

When we equate the two above we get the following system:

$$nf_{k+1} + \left[\left(-n - \frac{nk}{2}\right) f_2 - \frac{1}{2}g_1\right] f_k + g_k + \left[\frac{1}{2}(1-k)f_2 - g_1\right] g_{k-1} = \beta_{nk},$$

so g_k and g_{k-1} satsify the system

$$g_k + \left[\frac{1}{2}(1-k)f_2 - g_1\right] g_{k-1} = \gamma_{nk}, \quad n \in \mathbb{N}.$$

One can see that all the rows of the matrix of our system are the same, so we can not say anything about the number of its solutions without thorough investigation of γ_{nk}.

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