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SPECTRUM OF GALLAI GRAPH OF SOME GRAPHS

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The Gallai graph $\Gamma(G)$ of a graph G, has the edges of G as its vertices and two distinct vertices are adjacent in $\Gamma(G)$ if they are adjacent edges in G, but do not lie on a triangle. In this paper we find the adjacency spectrum of Gallai graph of some graphs derived from simple graphs by certain operations, the characteristic polynomial for some graphs and exhibit the adjacency matrix of some other graphs.

Key words : Gallai graph, adjacency spectrum.

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1. Introduction

Let G be a simple graph on p vertices $\{1, 2, ..., p\}$ with an adjacency matrix $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1, & \text{if the vertex } i \text{ is adjacent to the vertex } j \\ 0, & \text{otherwise.} \end{cases}$$

Then $A(G)$ is a real symmetric matrix with zero as the diagonal entries. The eigen values of A are the roots of $det(xI - A) = 0$, the characteristic equation of A or G. The set of eigen values of G is called the spectrum of G and is denoted by $spec(G)$ [5]. In this paper we denote the characteristic polynomial of G by $\phi(G(x))$. Since $A(G)$ is a real symmetric matrix all of its eigen values are real and can be ordered as $\lambda_1 \geq \lambda_2, ..., \geq \lambda_p$. The energy of a graph G denoted by E_G is defined by $E_G = \Sigma_{i=1}^{p} |\lambda_i|$. The $(i,j)^{th}$ entry of A^k is the number of $i - j$ walks of length k [5]. Two graphs are cospectral if they have the same spectrum. A graph G is said to be characterized by its spectrum, if the only graphs cospectral with G are those isomorphic to G [5].

The line graph [7] $L(G)$ of a graph G has the edges of G as its vertices and two distinct edges of G are adjacent in $L(G)$ if they are adjacent in G. The adjacency spectrum of line graph of many classes of graphs has been studied in [4-6]. The Gallai graph $\Gamma(G)$ of a graph G, has the edges of G as its vertices and two distinct edges are adjacent in $\Gamma(G)$ if they are adjacent in G, but do not lie on a triangle.

Clearly Gallai graph is a spanning subgraph of line graph. Though $L(G)$ has a forbidden subgraph characterization, Gallai graphs do not have the vertex hereditary property and hence, cannot be characterized using forbidden subgraphs [2]. Motivated from the results regarding the spectrum of line graph of a graph, we study the spectrum of Gallai graph of some classes of graphs for the first time to add them to the literature on the spectrum of graphs, here we give an upper bound for energy of $\Gamma(G)$ and give some characterization regarding the spectrum of some classes of graphs. In the class of Gallai graphs we mean by saying that a graph $\Gamma(G)$ is characterized by the spectrum if, $spec(\Gamma(G)) \cong spec(\Gamma(H))$ then $G \cong H$.

Let G_1 and G_2 be two vertex-disjoint graphs. Then the *join*, $G_1 \vee G_2$, of G_1 and G_2 is the supergraph of $G_1 + G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2 [2]. If $\mathscr{G}$ is any class of graphs then the class of graphs $\{G \vee K_1 / G \in \mathscr{G}\}$ is called a $\mathscr{G} - \mathscr{P}$. All graph theoretic notations and terminology not mentioned here are from [2, 4].

2. Spectrum of Gallai Graph of Some Graphs

If the graph is a triangle free graph then $\Gamma(G) \cong L(G)$. If G is regular, $L(G)$ is regular; but $\Gamma(G)$ need not be regular. So, we cannot calculate the spectrum of $\Gamma(G)$ like the spectrum of $L(G)$. Here we find the spectrum of Gallai graph of some classes of graphs. In this section we use the following theorems.

***Theorem* 2.1** — *[3]. Let G_j denote the graph obtained from G by adding a pendant edge at the vertex j. Then, $\Phi(G_j, x) = x\Phi(G, x) - \Phi(G - j, x)$*

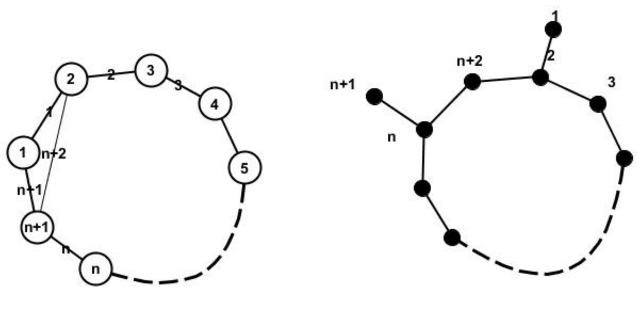
***Theorem* 2.2** — *[3]. Let $G.H$ be the coalescence in which the vertex u of G is identified with the vertex v of H.*

$\phi(G.H,x) = \phi(G,x)\phi(H-v,x) + \phi(G-u,x)\phi(H,x) - x\phi(G-u,x)\phi(H-v,x)$

In this section we give some analytic expression for the characteristic polynomial of some Gallai graphs.

2.1 *Cycle containing a* K_3

Consider a cycle C_{n+1} with an edge connecting any two vertices at a distance 2 in C_{n+1}(or a cycle C_n sharing a common edge with a triangle).



G

Gamma(G)

$H = \Gamma(C_{n+1}+e)$ is a cycle C_n with two pendant edges attached to two alternative vertices. By using Theorem 2.1 we have,

$$\begin{aligned}\Phi(H,x) &= x^2\Phi(C_n,x) - 2x\Phi(P_{n-1},x) - \Phi(P_{n-3}\cup K_1,x)\\ &= x^2\prod_{i=0}^{n-1}\left(x-2cos\frac{2\pi i}{n}\right) - 2x\prod_{i=1}^{n-1}\left(x-2cos\frac{\pi i}{n}\right) - x\prod_{i=1}^{n-3}\left(x-2cos\frac{\pi i}{n-2}\right)\end{aligned}$$

2.2 *Path sharing a common edge with a triangle*

H= Path P_{n+1} sharing a common edge with a triangle.

Case 1 : (common edge is the pendant edge of P_{n+1})

$G = \Gamma(H)$= P_{n+1} together with a pendant edge attached to the second vertex $\cup K_1$.

$$\phi(G,x) = x\left\{x\left(\prod_{i=1}^{n}\left(x-2cos\frac{\pi i}{n+1}\right)\right) - x\left(\prod_{i=1}^{n-2}\left(x-2cos\frac{\pi i}{n-1}\right)\right)\right\}.$$

Case 2 : Let the common edge be the m^{th} edge. Then, $G = \Gamma(H)$= P_n together with one pendant edge attached to the $m-1^{th}$ vertex and one pendant edge at the $m+1^{th}$ vertex. So, by the same method we have, $\phi(G,x)$= $x^2\phi(p_n,x) - x\phi(p_{m-2},x)\phi(P_{n-m+1},x) - [x\phi(P_m,x) - x\phi(P_{m-2},x)]\phi(P_{n-m-1},x)$.

2.3 *Path sharing a common vertex with a triangle*

H= Path P_{n+1} sharing a common vertex with a triangle.

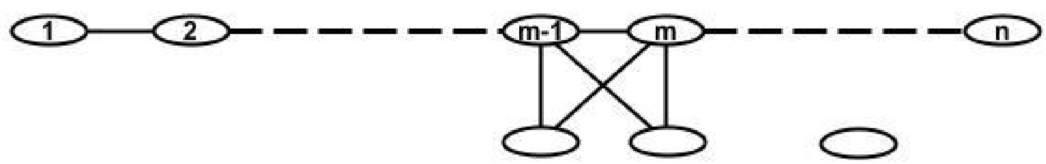
Case 1 : The common vertex is the pendant vertex.

$G = \Gamma(H)$= path P_{n+1} together with a pendant edge attached to the second vertex $\cup K_1$.

$\phi(G,x) = x\{x(\prod_{i=1}^{n+1}(x - 2cos\dfrac{\pi i}{n+2})) - x(\prod_{i=1}^{n-1}(x - 2cos\dfrac{\pi i}{n}))\}$

Case 2 : The common vertex is the m^{th} vertex.

$G = \Gamma(H)$ is



$\phi(G,x)$ is obtained by considering the theorem (Let $G.H$ be the coalescence in which the vertex u of G is identified with the vertex v of H).

$\phi(G.H,x) = \phi(G,x)\phi(H-v,x) + \phi(G-u,x)\phi(H,x) - x\phi(G-u,x)\phi(H-v,x)$.

Here $G = (P_{m-1}.K_4 - e).P_{n-m+1}$ together with an isolated vertex.

Let $G' = P_{m-1}.K_4 - e$

$\phi(G',x) = \phi(P_{m-1},x)\phi(K_{1,2},x) + \phi(P_{m-2},x)\phi(K_4 - e,x) - x\phi(P_{m-2},x)\phi(K_{1,2},x)$

$\phi(G,x) = [\phi(G',x)\phi(P_{n-m},x) + \phi(T',x)\phi(P_{n-m+1},x) - x\phi(T',x)\phi(P_{n-m},x)]x$ (where T' is P_m together with a pendant edge attached to the second vertex).

Then $\phi(T',x) = x\phi(P_m,x) - \phi(P_{m-2} \cup K_1,x)$

We have,

$\phi(G,x) = \{[\phi(P_{m-1},x)\phi(K_{1,2},x) + \phi(P_{m-2},x)\phi(K_4 - e,x) - x\phi(P_{m-2},x)\phi(K_{1,2},x)]$
$\phi(P_{n-m},x) + [x\phi(P_m,x) - x\phi(P_{m-2},x)]\phi(P_{n-m+1},x) -$

$$x[x\phi(P_m,x) - x\phi(P_{m-2},x)]\phi(P_{n-m},x)\ \}\ x.$$

2.4 *Star graph sharing a common edge with a traingle*

$G = \Gamma(K_{1,n} + e)$, i.e., $K_{1,n-1}$ sharing a common edge with a triangle.

Here $G = (K_n - e) \cup K_1$

$Spec(G) = (-1)^{n-3}, 0^2, \dfrac{(n-3)+\sqrt{(n+1)^2-8}}{2}, \dfrac{(n-3)-\sqrt{(n+1)^2-8}}{2}$

***Theorem* 2.3** — $Spec(\Gamma(G \vee K_n))$= $Spec(\Gamma(G))$, $Spec(G^c)$ *(n times)*, $0^{\frac{n(n-1)}{2}}$

PROOF : We make the following observations, regarding the vertices corresponding to the edges of $G \vee K_n$.

(1) The vertices corresponding to the edges of G are adjacent in $\Gamma(G \vee K_n)$ if and only if they are adjacent in $\Gamma(G)$.

(2) The vertices corresponding to the edges of K_n are not adjacent in $\Gamma(G \vee K_n)$ since they are belong to a K_3 in $G \vee K_n$.

(3) The vertices corresponding to the edges drawn from two different vertices of K_n in $\Gamma(G \vee K_n)$ are not adjacent since they are belong to a K_3 in $G \vee K_n$.

(4) The vertices corresponding to the edges drawn from the same vertex of K_n to G is adjacent in $\Gamma(G \vee K_n)$ if and only if the end vertices are non-adjacent in G.

Let M denotes the adjacency matrix of $\Gamma(G \vee K_n)$. Then M is of order $q + \dfrac{n(n-1)}{2} + np$. From the above explanation M is,

$$M = \begin{bmatrix} [\Gamma(G)] & 0 & 0 & 0 & . & . & . & 0 \\ 0 & [G^c] & 0 & 0 & . & . & . & 0 \\ 0 & 0 & [G^c] & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where G^c occurs n times and zero $\dfrac{n(n-1)}{2}$ times. □

Corollary 2.4 — $Spec(\Gamma(G \vee K_1))$= $Spec(\Gamma(G))$, $Spec(G^c)$.

PROOF : It follows directly from the above theorem.

Corollary 2.5 — If G is triangle free then the $Spec(\Gamma(G \vee K_1))$= $Spec(L(G))$, $Spec(G^c)$

Corollary 2.6 — If G is triangle free r-regular graph with p vertices and q edges then,

$$\phi_{\Gamma(G\vee K_1)}(x)) = (x+2)^{q-p}\phi_G(x-r+2).(-1)^p.\frac{x-p+r+1}{x+r+1}\phi_G(x+1).$$

***Theorem* 2.7** — *Let G be a semi-regular bipartite graph with n_1 independent vertices of degree r_1 and n_2 independent vertices of degree r_2 where $n_1 \geq n_2$ Then* $\phi_{L(G)}(x) = (x+2)^{\beta}\phi_G(\sqrt{\alpha_1\alpha_2})$ $\sqrt{\frac{\alpha_1}{\alpha_2}}^{n_1-n_2}$ where $\alpha_i = x - r_i + 2$ and $\beta = n_1r_1 - n_1 - n_2$

Corollary 2.8 — If G is a semi-regular bipartite graph we can calculate the spectrum of $\Gamma(G \vee K_1)$

PROOF : For a semi-regular bipartite graph $\Gamma(G) \cong L(G)$. Hence, by Corollary 2.5 and Theorem 2.7 the result follows. □

By using Theorem 2.7 we have the following results.

3.1 *Wheel graph*

Gallai of Wheel graph W_n

We know $W_n = C_{n-1} \vee K_1$

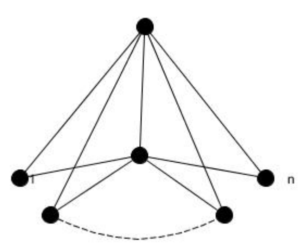
So, by the above result,

$Spec(\Gamma(W_n))$ = $Spec(\Gamma(C_{n-1}))$, $Spec(C^c_{n-1})$.

Therefore,

$$Spec(\Gamma(w_n)) = \{-2cos\frac{2\pi j}{n-1} - 1, j = 1, ..., n-2, n-4, 2cos\frac{2\pi j}{n-1}, j = 0, 1, ..., n-2\}$$

3.2 *Star-Pyramid*



$H = K_{1,n} \vee K_1$

$Spec(\Gamma(H))$ = $Spec(\Gamma(K_{1,n}))$, $Spec(K^c_{1,n})$

= $Spec(K_n)$, $Spec(K_n)$, $Spec(K_1)$

= $(n-1)^2, (-1)^{2n-2}, 0$

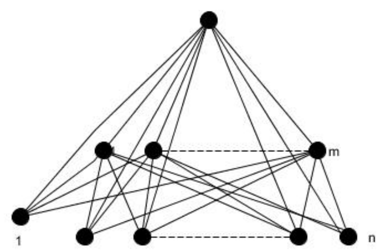
3.3 **$\mathbf{K}_{m,n}$**-*Pyramid*

$H = K_{m,n} \vee K_1$

$Spec(\Gamma(H))$ = $Spec(\Gamma(K_{m,n}))$, $Spec(K^c_{m,n})$

= $Spec(L(K_{m,n}))$, $Spec(K_n)$, $Spec(K_m)$

= $(m+n-2),(n-2)^{(m-1)},(m-2)^{(n-1)},-2^{(m-1)(n-1)},$
$n-1,m-1,-1^{m+n-2}$



4.1 $(\mathbf{K}_n \cup \mathbf{K}_m)$ -*Pyramid*

$H = \Gamma[(K_n \cup K_m) \vee K_1]$

$Spec(\Gamma(H))$ = $Spec(\dfrac{n(n-1)+m(m-1)}{2}K_1)$, $spec(K_{mn})$

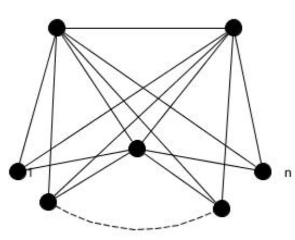
= $0^{\frac{n(n+1)+m(m+1)-4}{2}}, -\sqrt{mn}, \sqrt{mn}$

Similarly we can find the spectrum of $\Gamma(P_n \vee K_1)$.

Corollary 2.9 — $Spec(\Gamma(G \vee K_2))$= $Spec(\Gamma(G))$, $Spec(G^c)$, $Spec(G^c)$,0

PROOF : It follows directly from the above theorem.

$\mathbf{K}_{1,n}$ $\vee K_2$



Here $H = \Gamma(K_{1,n} \vee K_2)$

$Spec(\Gamma(K_{1,n}))$, $Spec(K^c_{1,n})^2$' to '$Spec(H)$ = $Spec(\Gamma(K_{1,n}))$, $Spec(K^c_{1,n})^2, 0$.

= $(n-1)^3, (-1)^{3n-3}, 0^3$

Similarly, for all the graphs G in the above examples we can find the spectrum of Gallai graph of $G \vee K_2$ easily by Corollary 2.9.

Result 1

The matrix of $\Gamma(G \vee G)$ is

$$\begin{bmatrix}
[\Gamma(G)] & [0]_{q*p} & . & ... & [0]_{q*p} & [0]_{q*p} & [0]_q \\
[0]_{p*q} & [G^c] & I_p or [0]_p & ... & I_p or [0]_p & I_p or [0]_p & [0]_{p*q} \\
[0]_{p*q} & I_p or [0]_p & [G^c] & ... & I_p or [0]_p & I_p or [0]_p & [0]_{p*q} \\
. & . & . & ... & . & . & . \\
. & . & . & ... & . & . & . \\
. & . & . & ... & . & . & . \\
[0]_{p*q} & I_p or [0]_p & . & ... & [G^c] & I_p or [0]_p & [0]_{p*q} \\
[0]_{p*q} & I_p or [0]_p & . & ... & I_p or [0]_p & [G^c] & [0]_{p*q} \\
[0]_q & [0]_{q*p} & . & ... & [0]_{q*p} & [0]_{q*p} & [\Gamma(G)]
\end{bmatrix}$$

Let $1, 2, ..., p$ and $e_1, e_2, ..., e_q$ denote the vertices and edges of G. In $G \vee G$, let $e_{i1}, e_{i2}, ...e_{ip}$ denote the edges drawn from vertex i (in the second copy of G) to the vertices of G. Then the matrix of $\Gamma(G \vee G)$ is as given above. Where $I_p, 0_p$ denote the identity matrix and zero matrix respectively. I_p occurs if vertex i is not adjacent to vertex j in G

Result 2

The matrix of $\Gamma(G \Box K_2)$ is

$$M = \begin{bmatrix} [\Gamma(G)] & [0]_q & B^T \\ [0]_q & [\Gamma(G)] & B^T \\ B & B & [0]_p \end{bmatrix}$$

(B denotes the incidence matrix of G)

3. Some Characterizations Regarding Spectrum of $\Gamma(G)$

In this section we give some characterization for the graphs regarding the spectrum of Gallai graphs. When we consider the class of Gallai graph of all graphs, we cannot characterize a graph belongs to this class by using the spectrum. For example, consider $\Gamma(C_6)$ it cannot be characterized by the spectrum in the class of Gallai Graphs. Since $\Gamma(K_3 \odot K_1)$ (where $\odot$ denotes the *corona*) has the same spectrum as that of $\Gamma(C_6)$. Also it is proved that there exist infinitely many pairs of non-isomorphic graphs of the same order with the same Gallaigraphs [1]. If $\Gamma(G) \cong H$ then G is called a root-graph of H.

***Theorem* 3.1** — *[5]. The number of closed walks of length k in a graph G is equal to S_k, where*

$$S_k = \Sigma_{i=1}^{p} {\lambda_i}^k$$

***Theorem* 3.2** — *Let G be a graph with p vertices and q edges, with eigen values $\lambda_1, \lambda_2, ..., \lambda_p$. Let $\mu_1, \mu_2, ...\mu_q$ denote the eigen values of $\Gamma(G)$ then, $\Sigma_{i=1}^{q}|\mu_i| \leq \sqrt{2q(\frac{\Sigma_{i=1}^{p} d_i^2}{2} - q - \frac{\Sigma_{i=1}^{p}\lambda_i^3}{2})}$ where $\{d_1, d_2, ...d_p\}$ denote the degree sequence of G.*

PROOF : Let G be a graph with p vertices and q edges, with eigen values $\lambda_1, \lambda_2, ..., \lambda_p$. From Theorem 3.1 we have,

$$\Sigma_{i=1}^{p} \frac{\lambda_i^2}{2} = q$$

$$\Sigma_{i=1}^{p} \frac{\lambda_i^3}{6} = \text{Number of } K_3\text{'s}$$

In $L(G)$, number of of vertices = q.

Number of edges of edges = $\Sigma_{i=1}^{p} \frac{d_i^2}{2} - q$, where $d_1, d_2, ...d_p$ denote the degree sequence of G.

Now for $\Gamma(G)$, number of vertices = q.

Number of edges = $\Sigma_{i=1}^{p} \frac{d_i^2}{2} - q$ - 3 times the number of triangles in G

Let $\mu_1, \mu_2, ..., \mu_q$ denote the eigen values of $\Gamma(G)$, by Holder's inequality we have,

$$\begin{aligned}\Sigma_{i=1}^{q}|\mu_i| &= \Sigma_{i=1}^{q}|\mu_i.1| \\ &\leq (\Sigma_{i=1}^{q}|\mu_i^2|)^{1/2}(\Sigma 1^2)^{1/2} \\ &\leq (2(\frac{\Sigma_{i=1}^{p} d_i^2}{2} - q - \frac{\Sigma_{i=1}^{p}\lambda_i^3}{2}))^{1/2} q^{1/2} \\ &\leq \sqrt{2q(\frac{\Sigma_{i=1}^{p} d_i^2}{2} - q - \frac{\Sigma_{i=1}^{p}\lambda_i^3}{2})}\end{aligned}$$

Also $\lambda_1 \geq \frac{2 * no.of edges}{no.of vertices}$

Therefore $\mu_1 \geq (\frac{\Sigma_{i=1}^{p} d_i^2}{q} - 2 - \frac{\Sigma_{i=1}^{p}\lambda_i^3}{q})$. □

***Theorem* 3.3** — *Disjoint union of K_n's are the only graphs whose Gallai graphs has only one eigen value.*

PROOF : Since $\Gamma(G)$ has only one eigen value the eigen value is zero. Hence, $\Gamma(G)$ is the union of isolated vertices. Hence, the root graph is the disjoint union of K_n's. □

***Theorem* 3.4** — *The following are the only graphs whose Gallai graphs have only two distinct* *eigen values.*

(1) Disjoint union of stars.

(2) Disjoint union of P_3.

PROOF : Let A denote the adjacency matrix of $\Gamma(G)$. And λ_1 and λ_2 are the two distinct eigen values with multiplicity m_1 and m_2 respectively. The following cases arise

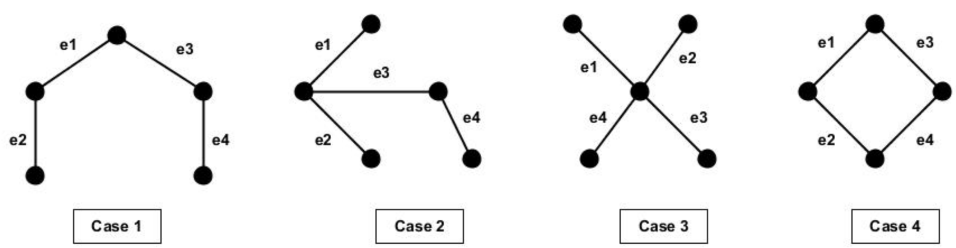
Case 1 : m_1=m_2

Since $\Sigma\lambda_i = 0$ in this case $\lambda_1 = -\lambda_2$. Then the minimal polynomial of A is

$(A - \lambda_1 I)(A + \lambda_1 I) = 0$

$A^2 = \lambda_1^2 I$

Which means that $[\Gamma(G)]^2$ is a λ_1^2-regular graph and in $\Gamma(G)$ there does not exist walk of length 2 connecting any two vertices. That is $\Gamma(G)$ is either K_2 or disjoint union of K_2's. Let $\Gamma(G) = \cup_i K_2$, with vertex set $S = \{v_1, v_2, ..., v_i, ...\}$ corresponding to the edges $e_1, e_2, ...$ in G so that e_{2n} and e_{2n-1} form an induced P_3 in G. Clearly if G is the disjoint union of P_3's then $\Gamma(G) = \cup_i K_2$. So, the next possibility is P_3's sharing common vertices. Then the following cases arise,



Consider the first case. In this case e_1 and e_3 are incident, so in $\Gamma(G)$ there is an edge e_1e_3. In order to overcome this add an additional edge such that e_1, e_3 together with that edge induce a K_3.

Since the eigen values are distinct, zero cannot be an eigen value, $\Gamma(G)$ does not contains isolated vertices. Therefore we cannot add edges outside the set $\{e_1, e_2, ...\}$. Hence, the next possibility is add induced P_3's i.e., e_{2n} and e_{2n-1} sharing common vertex. Then again we have edges sharing common vertices but, the vertices corresponding to them are not adjacent in $\Gamma(G)$. So, we have to repeat the same procedure. But, in all the above cases the addition of such edges effect the adjacency of any of the vertices e_{2n} and e_{2n-1} which are already adjacent in $\Gamma(G)$. This leads to a contradiction.

Case 2 : $m_1 \neq m_2$

In this case the minimal polynomial of A is

$(A - \lambda_1 I)(A - \lambda_2 I) = 0$

$A^2 = (\lambda_1 + \lambda_2)A - \lambda_1\lambda_2 I$

From this equation it is clear that $[\Gamma(G)]^2$ is $-\lambda_1\lambda_2$-regular and in $\Gamma(G)$, for any two distinct vertices e_i and e_j there exist a walk of length two if and only if there exists an edge between them. So, when we consider the connected component of $\Gamma(G)$ all the vertices are adjacent to each other. That is the connected component is K_n. $\Gamma(G)$ is disjoint union of K_n's means G is disjoint union of $K_{1,n}$'s.

***Theorem* 3.5** — *$\Gamma(H)$ where, H is a graph obtained by making a vertex of K_n adjacent with a vertex of K_m, can be characterized by the spectrum in the class of Gallai graphs of all graphs.*

PROOF : Consider the spectrum of $\Gamma(H)$ which consists of the numbers $\sqrt{m+n-2}, 0^{\frac{m^2-m+n^2-n-2}{2}}$, $-\sqrt{m+n-2}$. Consider a graph G with spectrum same as that of $\Gamma(H)$ in the class of Gallai graphs. From the spectrum it follows that the number of vertices in G is $\dfrac{m^2-m+n^2-n+2}{2}$. Therefore, the number of edges in the root graph is $\dfrac{m^2-m+n^2-n+2}{2}$. Since, sum of the cubes of the spectral values is zero, there is no K_3 in G. So, no two vertices have the same neighbors in the root-graph. Hence, in the root graph, there is no induced K_{13}. Since G has only three distinct eigen values, diameter of $G \leq 2$. Hence, there is no P_5 in the root graph. Also the number of edges in G is $m+n-2$ implies the root graph contains $m+n-2$ induced P_3's. So, by the above explanation the root graph contains $m+n-2$ induced P_3 with $\dfrac{m^2-m+n^2-n+2}{2}$ edges with out K_{13}. So, in the root graph the following cases arise.

Case 1 : Two P_3's joined to form P_4's.

Case 2 : P_3's joined together to form stars. Since $K_{1,3}$ is forbidden the pendent edges excluding one edge should form a complete graph.

Case 3 : Two complete graphs K_p and K_q joined with an edge.

Case 4 : Disjoint union of the above cases (same or different cases).

When we consider the graph as the disjoint union in Case 1, the corresponding Gallai graph is the disjoint union of P_3. Clearly the spectrum is different from that of G.

Again if the graph is the disjoint union of the above graphs, then the corresponding Gallai graph is the disjoint union of stars with spectrum different from that of G. So, Case 4 also not occur.

So, the only possibilities are Case 2 and 3. That is P_3's joined at the middle vertex, forming

K_{m+n-1} with a pendant edge. In this case the corresponding Gallai graph has spectrum $\sqrt{m+n-2}, 0^{\frac{m^2+n^2+2mn-3m-3n}{2}}, -\sqrt{m+n-2}$. This is same as the spectrum of G only when

$$m^2+n^2+2mn-3m-3n = m^2-m+n^2-n+2$$

Solving this equation we get $m = 1$ which is same as the graph H (when $m = 1$).

Consider Case 3 i.e., two complete graphs K_p and K_q joined with an edge we prove that $p = m, q = n$. When we consider the Gallai of the graph the spectum is $\sqrt{p+q-2}, 0^{\frac{p^2-p+q^2-q-2}{2}}$, $-\sqrt{p+q-2}$. This implies,

$$m+n-2 = p+q-2$$

$$m^2-m+n^2-n-2 = p^2-p+q^2-q-2$$

Solving these two equations we get $p = m, q = n$. Hence, the theorem.

***Theorem* 3.6** — $\Gamma(C_5)$ *can be characterized by the spectum in the class of Gallai graph of all graphs.*

PROOF : Consider the spectrum of $\Gamma(C_5)$ which consists of the numbers $-1.61803^2, .61803^2, 2$. Consider a graph G with spectrum same as that of $\Gamma(C_5)$. From the spectrum it is clear that the number of vertices in G is 5 and the number of edges is 5. Also G is 2-regular since, $5\lambda_1 = \lambda_1^2 + \lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2$ [5]. So, clearly G is either a cycle or disjoint union of cycles. Since G is K_3 free G is C_5. Now consider the root graphs of C_5. Let $v_1, v_2, ...v_5$ denote the vertices of C_5. Corresponding to the vertex v_1 there exists an edge e_1 in the root graph such that e_1 is incident to atleast two edges out of which e_1 not forms K_3 with edges e_2 and e_5. So, there is two possibilities either e_5, e_1 and e_2 form a $K_{1,3}$ or P_3. For the first case, since G is K_3-free root graph is $K_{1,3}$-free. This force to triangulate e_2 and e_5 by adding a new edge. Since there is no isolated vertex we can triangulate only by adding the edges e_3 or e_4. But, it will effect the adjacency between e_2 and e_3, e_5 and e_4. So, this case is not possible. For the second case, e_5, e_1, e_2 form a P_3, since e_3 is adjacent to e_2 we can attach e_3 to the pendant vertex of e_2 or to the other vertex. If e_3 is attached to the pendant vertex, e_4 (since e_4 is adjacent to both e_5 and e_3) can attach in such a way that all the edges together form a C_5. If e_3 is attached to the vertex of e_2 then e_1 and e_3 become incident. This force e_4 to be attached so that e_1, e_3, e_4 form a K_3. Then it will effect the adjacency between e_3 and e_4. So, this is also not possible. Hence, the only root graph is C_5.

4.1 *We have the following theorem, proof of which is on similar lines*

***Theorem* 3.7** — $\Gamma(C_7)$ *and* $\Gamma(C_9)$ *can be characterized by the spectum in the class of Gallai graph of all graphs.*

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