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# ON EIGHT COLOUR PARTITIONS

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*Dedicated to Prof. Chandrashekar Adiga on the occasion of his 62$^{nd}$ birthday*

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In 2013, Baruah and Sarmah, and Xia and Yao independently obtained generating function for the sequences $p_{-8}(2n+1)$ and $p_{-8}(4n+3)$, where $p_{-8}(n)$ counts the number of partitions of $n$ in eight colours. In this article, we generalize the identities and as a consequence, establish several Ramanujan type congruences modulo higher powers of 2.

**Key words** : Colour partitions; congruences.

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## 1. Introduction and Preliminaries

For any nonnegative integer $n$, let $p(n)$ denote the number of partitions of $n$. The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1}.$$

Here and throughout this paper, we set

$$f_k := \prod_{j=1}^{\infty}(1-q^{kj}), \quad |q| < 1$$

for any positive integer $k$. By Euler's pentagonal number theorem [2, pp. 10-12], we have

$$f_1 = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{k(3k-1)}{2}}. \tag{1.1}$$

A partition is said to be $k-$coloured if each part appears as $k$ different colours. Let $p_{-k}(n)$ count the number of $k-$coloured partitions of $n$. Then

$$\sum_{n=0}^{\infty} p_{-k}(n) q^n = \frac{1}{f_1^k}.$$

Note that $p_{-1}(n) = p(n)$.

In 1919, Ramanujan [12] obtained the generating function for $p(5n+4)$ as

$$\sum_{n=0}^{\infty} p(5n+4) q^n = 5\frac{f_5^5}{f_1^6}. \tag{1.2}$$

It follows from (1.2) that $p(5n+4) \equiv 0 \pmod 5$. Also, he conjectured that for any integer $k \geq 1$,

$$p(5^k n + \delta_k) \equiv 0 \pmod{5^k},$$

where $\delta_k$ is the reciprocal modulo $5^k$ of 24, and outlined the proof in [13, pp. 156-177], (also see [6]). In 1938, Watson [14] proved the above conjecture and in 1981, Hirschhorn and Hunt [10] gave a simple proof of the conjecture. Also, many congruences modulo powers of 7, 11 and 13 enjoyed by $p(n)$ have been found.

In the recent past, a number of congruences satisfied by $p_{-k}(n)$ have been found (see, for example, [3, 7, 8, 11]). In partitcular, Atkin [3] proved the following congruences modulo powers of prime.

***Theorem* 1.1** — [3, Theorem 1.1] — *Suppose* $k > 0$ *and* $q = 2, 3, 5, 7$ *or* $13$. *If* $24n \equiv k \pmod{q^r}$, *then* $p_{-k}(n) \equiv 0 \pmod{q^{\frac{1}{2}\alpha r + \epsilon}}$, *where* $\epsilon = \epsilon(q,k) = O(\log k)$, *and where* $\alpha$ *depends on* $q$ *and the residue of* $k$ *modulo* $24$ *according to a certain table.*

Recently, with the aid of Jacobi's identity, Hirschhorn [9] obtained several congruences for $p_{-3}(n)$ modulo high powers of 3, namely, for all $\alpha, n \geq 0$,

$$p_{-3}\left(3^{2\alpha+1} n + \frac{5 \cdot 3^{2\alpha+1} + 1}{8}\right) \equiv 0 \pmod{3^{2\alpha+2}},$$

which are analogous to Ramanujan's congruences for the partition function.

For $k = 8$, Baruah and Sarmah [4, Theorem 5.1] by obtaining the generating functions

$$\sum_{n=0}^{\infty} p_{-8}(2n+1)q^n = 8\frac{f_2^{16}}{f_1^{24}} \tag{1.3}$$

and

$$\sum_{n=0}^{\infty} p_{-8}(4n+3)q^n = 192\frac{f_2^{24}}{f_1^{32}} + 16384q\frac{f_2^{48}}{f_1^{56}}, \tag{1.4}$$

showed that

$$p_{-8}(2n+1) \equiv 0 \pmod{2^3}, \tag{1.5}$$

$$p_{-8}(4n+3) \equiv 0 \pmod{2^6} \tag{1.6}$$

and

$$p_{-8}(8n+7) \equiv 0 \pmod{2^9}. \tag{1.7}$$

In the same year, utilizing addition formula, Xia and Yao [15, Theorem 2.1] also obtained (1.3) and slightly different version of (1.4).

In this paper, using analysis of huffing operator acting on specified terms and induction on the matrix recurrence, we generalize the generating functions (1.3) and (1.4) and deduce several Ramanujan-type infinite families of congruences modulo high powers of 2. The main results of this paper are as follow.

***Theorem* 1.2** — *For all nonnegative integers $n$ and $\alpha \geq 1$, we have*

$$p_{-8}\left(2^{2\alpha-1}n + \frac{2^{2\alpha-1}+1}{3}\right) \equiv 0 \pmod{2^{3\alpha}}, \tag{1.8}$$

$$p_{-8}\left(2^{2\alpha+1}n + \frac{5\cdot 2^{2\alpha}+1}{3}\right) \equiv 0 \pmod{2^{3\alpha+8}}, \tag{1.9}$$

$$p_{-8}\left(2^{2\alpha+1}n + \frac{7\cdot 2^{2\alpha-1}+1}{3}\right) \equiv 0 \pmod{2^{3\alpha+2}}, \tag{1.10}$$

$$p_{-8}\left(2^{2\alpha+2}n + \frac{13\cdot 2^{2\alpha-1}+1}{3}\right) \equiv 0 \pmod{2^{3\alpha+1}}, \tag{1.11}$$

$$p_{-8}\left(2^{2\alpha+2}n + \frac{19\cdot 2^{2\alpha-1}+1}{3}\right) \equiv 0 \pmod{2^{3\alpha+3}}, \tag{1.12}$$

$$p_{-8}\left(2^{2\alpha+2}n + \frac{11\cdot 2^{2\alpha}+1}{3}\right) \equiv 0 \pmod{2^{3\alpha+10}}, \tag{1.13}$$

$$p_{-8}\left(2^{2\alpha+3}n + \frac{17\cdot 2^{2\alpha}+1}{3}\right) \equiv 0 \pmod{2^{3\alpha+9}} \tag{1.14}$$

*and if $n$ is not a generalised pentagonal number, then*

$$p_{-8}\left(2^{2\alpha+2}n+\frac{2^{2\alpha-1}+1}{3}\right)\equiv 0 \pmod{2^{3\alpha+1}}. \tag{1.15}$$

Note that congruences (1.5)-(1.7) are special cases of (1.8) and (1.9). In the next section, we provide a set of preliminary results. Establish intended generalized generating functions for $p_{-8}(n)$ in Section 3 and prove Theorem 1.2 in the last section.

## 2. PRELIMINARIES

In this section, we present some preliminary results which are used in proving our main results.

*Lemma* 2.1 — We have

$$f_1^4=\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2} \tag{2.1}$$

and

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}. \tag{2.2}$$

PROOF : Lemma 2.1 is an immediate consequence of dissection formulas of Ramanujan, collected in Berndts book [5, Entry 25, p. 40]. □

*Lemma* 2.2 — Let

$$S=\frac{qf_4^8}{f_1^8},\ T=\frac{q^2f_4^{24}}{f_2^{24}}.$$

Then, for any $j\geq 1$

$$S^j=T(S^{j-2}+16S^{j-1}).$$

PROOF : Squaring (2.1) on both sides, we deduce that

$$f_1^8=\frac{f_4^{20}}{f_2^4f_8^8}+16q^2\frac{f_2^4f_8^8}{f_4^4}-8qf_4^8. \tag{2.3}$$

Setting $-q$ for $q$ in (2.3) and using the fact that $\prod_{j=0}^{\infty}(1+q^j)=\frac{f_2^3}{f_1f_4}$, we obtain

$$\frac{f_2^{24}}{f_1^8f_4^8}=\frac{f_4^{20}}{f_2^4f_8^8}+16q^2\frac{f_2^4f_8^8}{f_4^4}+8qf_4^8. \tag{2.4}$$

We can rewrite (2.3) as

$$\frac{1}{S}=\frac{f_4^{12}}{qf_2^4f_8^8}+16q\frac{f_2^4f_8^8}{f_4^{12}}-8 \tag{2.5}$$

and then (2.4) as

$$\frac{1}{S}+16=\frac{f_2^{24}}{qf_4^{16}f_1^{8}}.$$

If follows from the definition of $S$ and last equation that

$$\frac{1}{S}\left(\frac{1}{S}+16\right)=\frac{f_2^{24}}{q^2f_4^{24}}=\frac{1}{T},$$

which yields

$$S=T\left(\frac{1}{S}+16\right),$$

thus for any $j\geq 1$,

$$S^j=T\left(S^{j-2}+16S^{j-1}\right).$$

This completes the proof of Lemma 2.1. $\square$

Let us define an operator $H$ as follows:

$$H\left(\sum_{n=0}^{\infty}a_nq^n\right)=\sum_{n=0}^{\infty}a_{2n}q^{2n}$$

and $H(1)=1$.

In view of (2.5), we see that

$$H\left(\frac{1}{S}\right)=-8.$$

Also, from Lemma 2.2

$$H\left(S^j\right)=T\left(H\left(S^{j-2}\right)+16H\left(S^{j-1}\right)\right). \tag{2.6}$$

In particular,

$$H(S)=T\left(H\left(\frac{1}{S}\right)+16H(1)\right)=8T, \tag{2.7}$$

$$H(S^2)=T\left(H(1)+16H(S)\right)=T+128T^2 \tag{2.8}$$

and

$$H(S^3)=T\left(H(S)+16H(S^2)\right)=24T^2+2048T^3. \tag{2.9}$$

We define an infinite matrix $M=(m_{j,\,k})_{j,\,k\geq 1}$ as follows:

1. $m_{1\,,1}=8,\ m_{1,\,k}=0$ for all $k\geq 2$.

2. $m_{2,\,1} = 1,\ m_{2,\,2} = 128,\ m_{2,\,k} = 0$, for all $k \geq 3$.

3. $m_{j,\,1} = 0$ for all $j \geq 3$.

4. $m_{j,\,k} = 16m_{j-1,\,k-1} + m_{j-2,\,k-1}$ for all $j \geq 3,\ k \geq 2$.

Using the definition of $M$ and induction, it is easy to prove that

5. $m_{j\,,k} = 0$ for all $k > j$.

6. $m_{j,\,k} = 0$ for all $j > 2k$.

7. $m_{2j,\,j} = 1$ for all $j \geq 1$.

8. $m_{j,\,j} = 2^{4j-1}$ for all $j \geq 1$.

We omit the details of the proof. The first eight rows of $M$ are given by

$$\begin{pmatrix} 2^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2^7 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2^3\times 3 & 2^{11} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 2^9 & 2^{15} & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2^3\times 5 & 2^{11}\times 5 & 2^{19} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 2^7\times 3^2 & 2^{16}\times 3 & 2^{23} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2^3\times 7 & 2^{12}\times 7 & 2^{19}\times 7 & 2^{27} & 0 & \cdots \\ 0 & 0 & 0 & 1 & 2^{11} & 2^{17}\times 5 & 2^{26} & 2^{31} & \cdots \end{pmatrix}.$$

*Lemma* 2.3 — For any positive integer $j$, we have

$$H\left(S^j\right) = \sum_{k=1}^{j} m_{j,\,k} T^k = \sum_{k=\lfloor \frac{j+1}{2} \rfloor}^{j} m_{j,\,k} T^k. \tag{2.10}$$

PROOF : The second equality follows from (6). We use mathematical induction to prove the first equality. From (2.7) and (2.8), we see that the above identity holds for $j = 1, 2$. Suppose that (2.10) holds for some integer $j > 2$. From (2.6), we see that

$$\begin{aligned} H\left(S^{j+1}\right) &= T\left(\sum_{k=1}^{j-1} m_{j-1,\,k} T^k + 16 \sum_{k=1}^{j} m_{j,\,k} T^k\right) \\ &= 16 m_{j,\,j} T^{j+1} + \sum_{k=2}^{j} \left(16 m_{j,\,k-1} + m_{j-1,\,k-1}\right) T^k. \end{aligned}$$

Using (3),(4) and (8) in the above equation, we arrive at

$$H\left(S^{j+1}\right)=\sum_{k=1}^{j+1} m_{j+1,\,k} T^k.$$

This completes the induction.

It is worth to note that even the expression from Adiga and the second author [1, p. 3144, Eq. 2.9], can be used to prove our results.

## 3. Generating Functions

In this section, we obtain generating functions for the sequences in Theorem 1.2.

Let us define another infinite matrix $(x_{\alpha,\,i})_{\alpha,j\geq 1}$ by

9. $x_{1,\,1}=8,\ x_{1,\,k}=0$ for all $k\geq 2$.

10.

$$x_{\alpha+1,\,j}=\begin{cases}\displaystyle\sum_{i=1}^{\infty} x_{\alpha,\,i}\, m_{3i,\,i+j} & \text{if } \alpha \text{ is odd,}\\ \displaystyle\sum_{i=1}^{\infty} x_{\alpha,\,i}\, m_{3i+1,\,i+j} & \text{if } \alpha \text{ is even.}\end{cases}$$

Using the properties of the numbers $m_{i,\,j}$ and induction, we can easily see that

11.

$$x_{\alpha,\,j}=0 \ \text{ if } \begin{cases} j>\frac{2^{\alpha+1}-2}{3} & \text{and } \alpha \text{ is even,}\\ j>\frac{2^{\alpha+1}-1}{3} & \text{and } \alpha \text{ is odd;}\end{cases}$$

and

12.

$$x_{\alpha,\,\frac{2^{\alpha+1}-2}{3}}=2^{8(2^{\alpha}-1)-5\alpha} \ \text{ if } \alpha \text{ is even,}$$

$$x_{\alpha,\,\frac{2^{\alpha+1}-1}{3}}=2^{8(2^{\alpha}-1)-5\alpha} \ \text{ if } \alpha \text{ is odd.}$$

The first four rows of $x$ are given by

$$\begin{pmatrix} 2^3 & 0 & 0 & 0 & 0 & \cdots\\ 2^6\times 3 & 2^{14} & 0 & 0 & 0 & \cdots\\ 2^6\times 3 & 2^{15}\times 31 & 2^{21}\times 227 & 2^{33}\times 7 & 2^{41} & \cdots\\ 2^9\times 1993 & 2^{17}\times 729187 & 2^{31}\times 265617 & 2^{38}\times 3070947 & \cdots & \cdots \end{pmatrix}. \tag{3.1}$$

***Theorem* 3.1** — *For any positive integer* $\alpha$,

$$\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha-1}n + \frac{2^{2\alpha-1}+1}{3}\right) q^{n+1} = \frac{1}{f_2^8} \sum_{j=1}^{\frac{1}{3}(4^{\alpha}-1)} x_{2\alpha-1,\,j}\, q^j \frac{f_2^{24j}}{f_1^{24j}} \tag{3.2}$$

*and*

$$\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha}n + \frac{2^{2\alpha+1}+1}{3}\right) q^{n+1} = \frac{1}{f_1^8} \sum_{j=1}^{\frac{2}{3}(4^{\alpha}-1)} x_{2\alpha,\,j}\, q^j \frac{f_2^{24j}}{f_1^{24j}}. \tag{3.3}$$

PROOF : We prove (3.2) and (3.3) by mathematical induction on $\alpha$. From (1.3) and (1.4), it follows that (3.2) and (3.3) hold for $\alpha = 1$.

Suppose that (3.2) holds for some positive integer $\alpha > 1$. Applying the operator $H$ to both sides gives

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha-1}(2n+1) + \frac{2^{2\alpha-1}+1}{3}\right) q^{2n+2}\\ &= \frac{1}{f_2^8} \sum_{j=1}^{\frac{1}{3}(4^{\alpha}-1)} x_{2\alpha-1,j} H\left(q^j \frac{f_2^{24j}}{f_1^{24j}}\right).\end{aligned} \tag{3.4}$$

From Lemma 2.3, we have

$$\begin{aligned} H\left(q^j \frac{f_2^{24j}}{f_1^{24j}}\right) &= \frac{f_2^{24j}}{q^{2j} f_4^{24j}} H\left(q^{3j} \frac{f_4^{24j}}{f_1^{24j}}\right)\\ &= \frac{f_2^{24j}}{q^{2j} f_4^{24j}} \sum_{k=\lfloor \frac{3j+1}{2} \rfloor}^{3j} m_{3j,\,k}\, q^{2k} \frac{f_4^{24k}}{f_2^{24k}}\\ &= \sum_{k=\lfloor \frac{j+1}{2} \rfloor}^{2j} m_{3j,\,k+j}\, q^{2k} \frac{f_4^{24k}}{f_2^{24k}}. \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha-1}(2n+1) + \frac{2^{2\alpha-1}+1}{3}\right) q^{2n+2}\\ &= \frac{1}{f_2^8} \sum_{j=1}^{\frac{1}{3}(4^{\alpha}-1)} \sum_{k=\lfloor \frac{j+1}{2} \rfloor}^{2j} x_{2\alpha-1,\,j}\, m_{3j,\,k+j}\, q^{2k} \frac{f_4^{24k}}{f_2^{24k}}.\end{aligned}$$

Interchanging the order of summantion and using properties (5), (6) and (11) to extend the sums to all positive integers, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha}n + \frac{2^{2\alpha+1}+1}{3}\right) q^{n+1}\\
&= \frac{1}{f_1^8} \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} x_{2\alpha-1,\,j}\, m_{3j,\,k+j} \right) q^{2k} \frac{f_2^{24k}}{f_1^{24k}}\\
&= \frac{1}{f_1^8} \sum_{k=1}^{\frac{2}{3}(4^\alpha-1)} x_{2\alpha,\,k}\, q^k \frac{f_2^{24k}}{f_1^{24k}}
\end{aligned}$$

which is (3.3). Here last equality follows from properties (10) and (11).

Suppose (3.3) holds for some positive integer $\alpha$. Multiplying the above equation by $q$ and applying the operator $H$ to the resulting expression, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha}(2n) + \frac{2^{2\alpha+1}+1}{3}\right) q^{2n+2}\\
&= \sum_{j=1}^{\frac{2}{3}(4^\alpha-1)} x_{2\alpha,\,j} H\left( q^{j+1} \frac{f_2^{24j}}{f_1^{24j+8}} \right). \qquad (3.6)
\end{aligned}$$

From Lemma 2.3, we have

$$\begin{aligned}
H\left( q^{j+1} \frac{f_2^{24j}}{f_1^{24j+8}} \right) &= \frac{f_2^{24j}}{q^{2j} f_4^{24j+8}} H\left( q^{3j+1} \frac{f_4^{24j}}{f_1^{24j+8}} \right)\\
&= \frac{f_2^{24j}}{q^{2j} f_4^{24j+8}} \sum_{k=\lfloor \frac{3j+2}{2} \rfloor}^{3j+1} m_{3j+1,\,k}\, q^{2k} \frac{f_4^{24k}}{f_2^{24k}}\\
&= \sum_{k=\lfloor \frac{j+2}{2} \rfloor}^{2j+1} m_{3j+1,\,k+j}\, q^{2k} \frac{f_4^{24k-8}}{f_2^{24k}}. \qquad (3.7)
\end{aligned}$$

Combine (3.6) and (3.7), interchange the order of the summation and extend sums to all positive integers, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha+1}n+\frac{2^{2\alpha+1}+1}{3}\right)q^{n+1}\\
&=\frac{1}{f_2^8}\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} x_{2\alpha,\,j}\, m_{3j+1,\,k+j}\right)q^k\frac{f_2^{24k}}{f_1^{24k}}\\
&=\frac{1}{f_2^8}\sum_{k=1}^{\frac{1}{3}(4^{\alpha+1}-1)} x_{2\alpha+1,\,k}\, q^k\frac{f_2^{24k}}{f_1^{24k}}
\end{aligned}$$

which is (3.2) with $\alpha$ replaced by $\alpha+1$. This completes the proof of (3.2) and (3.3) by induction. □

## 4. CONGRUENCES

For a positive integer $n$, let $\vartheta_2(n)$ be the highest power of 2 that divides $n$ and $\vartheta_2(0) = +\infty$ by convention.

In this section, we consider the powers of 2 that divide the numbers $m_{j,\,k}$ and $x_{j,\,k}$. Using this information we prove Theorem 1.2.

*Lemma* 4.1 — For any integers $j, k \geq 1$ and $k \leq j \leq 2k$, we have

$$\vartheta_2(m_{j,\,k}) \geq 4(2k-j)-1. \tag{4.1}$$

PROOF : The proof easily follows from the definition of $m_{j,\,k}$ and properties (5)-(8). □

*Lemma* 4.2 — For all positive integers $j$ and $k$, we have

$$\vartheta_2(x_{2j-1,\,k}) \geq 3j+7(k-1) \tag{4.2}$$

and

$$\vartheta_2(x_{2j,\,k}) \geq 3(j+1)+8(k-1). \tag{4.3}$$

In both cases, equality holds for $k=1$.

PROOF : We use mathematical induction to prove these results, passing alternatively between (4.2) and (4.3). From (3.1) it follows that (4.2) and (4.3) hold for $j=1,2$ and equality hold in each case for $k=1$.

Assume that (4.2) is true for some positive integer $j$. Consider the case $k=1$ of (4.3).

$$\begin{aligned}
x_{2j,\,1} &= \sum_{i=1}^{\infty} x_{2j-1,\,i}\, m_{3i,\,i+1} = x_{2j-1,\,1}\, m_{3,2} + x_{2j-1,\,2} m_{6,\,3}\\
&= 3\cdot 2^3 x_{2j-1,\,1} + x_{2j-1,\,2}.
\end{aligned} \tag{4.4}$$

By our assumption, $\vartheta_2(x_{2j-1,\,1}) = 3j$ and $\vartheta_2(x_{2j-1,\,2}) \geq 3j + 7$. Hence it follows that $\vartheta_2(x_{2j,\,1}) = 3j+3$. For $k \geq 2$, using the definition of $x_{\alpha,\,j}$ and properties (5) and (6) of the numbers $m_{j,\,k}$, we get

$$\vartheta_2(x_{2j,\,k}) = \vartheta_2\left(\sum_{\frac{k}{2}\leq i\leq 2k} x_{2j-1,\,i}\, m_{3i,\,i+k}\right) \geq \min_{\frac{k}{2}\leq i\leq 2k} \vartheta_2(x_{2j-1,\,i}\, m_{3i,\,i+k}). \tag{4.5}$$

By Lemma 4.1, $\vartheta_2(m_{3i,\,i+k}) \geq 4(2k-i)-1$, whenever $i \leq 2k$. Using this fact and the induction hypothesis,

$$\begin{aligned}\min_{\frac{k}{2}\leq i\leq 2k} \vartheta_2(x_{2j-1,\,i}\, m_{3i,\,i+k}) &\geq \min_{\frac{k}{2}\leq i\leq 2k} 3j + 7(i-1) + 4(2k-i) - 1\\ &= \min_{\frac{k}{2}\leq i\leq 2k} 3j + 3i + 8(k-1)\\ &\geq 3j + 3 + 8(k-1).\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$\vartheta_2(x_{2j,\,k}) \geq 3(j+1) + 8(k-1). \tag{4.7}$$

From (4.7), we conclude that if (4.2) is true for some positive integer $j$ then (4.3) is also true for that $j$.

Suppose (4.3) is true for some positive integer $j$. Consider

$$x_{2j+1,\,1} = \sum_{i=1}^{\infty} x_{2j,\,i}\, m_{3i+1,\,i+1} = x_{2j,\,1}\, m_{4,\,2} = x_{2j,\,1}.$$

By our assumption, $\vartheta_2(x_{2j,1}) = 3(j+1)$ and hence $\vartheta_2(x_{2j+1,1}) = 3(j+1)$. Let $k \geq 2$, by the definition of $x_{\alpha,\,j}$ and properties (5) and (6), we see that

$$\begin{aligned}\vartheta_2(x_{2j+1,\,k}) &= \vartheta_2\left(\sum_{\frac{k-1}{2}\leq i\leq 2k-1} x_{2j,\,i}\, m_{3i+1,\,i+k}\right)\\ &\geq \min_{\frac{k-1}{2}\leq i\leq 2k-1} \vartheta_2(x_{2j,\,i}\, m_{3i+1,\,i+k})\\ &\geq \min_{\frac{k-1}{2}\leq i\leq 2k-1} 3(j+1) + 8(i-1) + 4(2k-i-1) - 1\\ &= \min_{\frac{k-1}{2}\leq i\leq 2k-1} 3(j+1) + 7(k-1) + (k+4i-6)\\ &\geq 3(j+1) + 7(k-1).\end{aligned}$$

which is (4.2) with $j$ replaced by $j+1$. Here second inequality of the last expression follows from the Lemma 4.1. $\square$

PROOF OF THEOREM 1.2 : By the binomial theorem, we can see that for all positive integers $k$ and $m$

$$f_k^{2^m} \equiv f_{2k}^{2^{m-1}} \pmod{2^m}. \tag{4.8}$$

In view of (4.2) and (4.8), we see that

$$x_{2\alpha-1,\,k} \equiv 0 \pmod{2^{3\alpha+7(k-1)}} \tag{4.9}$$

and

$$\frac{f_2^{16}}{f_1^{24}} \equiv f_2^4 \pmod{2^3}. \tag{4.10}$$

It follows from (3.2), (4.9) and (4.10) that,

$$\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha-1}n + \frac{2^{2\alpha-1}+1}{3}\right) q^n \equiv x_{2\alpha-1,\,1}\, f_2^4 \pmod{2^{3\alpha+3}}, \tag{4.11}$$

which implies that

$$\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha}n + \frac{2^{2\alpha-1}+1}{3}\right) q^n \equiv x_{2\alpha-1,\,1}\, f_1^4 \pmod{2^{3\alpha+3}}. \tag{4.12}$$

Congruence (1.8) follows from (4.9) and (4.11). By substituting (2.1) in (4.12) and extracting the terms involving odd and even powers of $q$,

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha+1}n + \frac{7\cdot 2^{2\alpha-1}+1}{3}\right) q^n &\equiv -4x_{2\alpha-1,\,1}\, \frac{f_1^2 f_4^4}{f_2^2} \\ &\equiv -4x_{2\alpha-1,\,1}\, \frac{f_4^4}{f_2} \pmod{2^{3\alpha+3}}\end{aligned} \tag{4.13}$$

and

$$\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha+1}n + \frac{2^{2\alpha-1}+1}{3}\right) q^n \equiv x_{2\alpha-1,\,1}\, \frac{f_2^{10}}{f_1^2 f_4^4} \equiv x_{2\alpha-1,\,1}\, f_2 \pmod{2^{3\alpha+1}}. \tag{4.14}$$

Congruences (1.10) and (1.12) follow from (4.13). Congruences (1.11) and (1.15) follow from (4.14).

From (4.3), we have

$$x_{2\alpha,\,k} \equiv 0 \pmod{2^{3\alpha+3+8(k-1)}}. \tag{4.15}$$

In view of (3.3) and (4.15),

$$\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha}n+\frac{2^{2\alpha+1}+1}{3}\right)q^n \equiv x_{2\alpha,1}\,\frac{f_2^{24}}{f_1^{32}} \pmod{2^{3\alpha+11}}. \tag{4.16}$$

Substituting (2.2) in (4.16) and extracting the terms involving odd powers of $q$, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha+1}n+\frac{5\cdot 2^{2\alpha}+1}{3}\right)q^n &\equiv 2^5 x_{2\alpha,1}\,\frac{f_2^{100}}{f_1^{84}f_4^{24}} \\ &\equiv 2^5 x_{2\alpha,1}\,f_2^{10} \pmod{2^{3\alpha+10}},\end{aligned} \tag{4.17}$$

which implies that

$$\sum_{n=0}^{\infty} p_{-8}\left(2^{2\alpha+1}n+\frac{5\cdot 2^{2\alpha}+1}{3}\right)q^n \equiv 2^5 x_{2\alpha,1}\,f_4^{5} \pmod{2^{3\alpha+9}}. \tag{4.18}$$

Congruences (1.9) and (1.13) follow from (4.17). Congruence (1.14) follows from (4.18).

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