Indian J. Pure Appl. Math., **51**(4): 1893-1901, December 2020

DOI: 10.1007/s13226-020-0502-9

CLASSIFYING HEPTAVALENT SYMMETRIC GRAPHS OF ORDER $40p$

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(*Received* 22 *July* 2018, *accepted* 3 *January* 2020)

A graph is symmetric if its automorphism group acts transitively on the set of arcs of the graph. Let p be a prime. In this paper, we proved that there is only one connected heptavalent symmetric graphs of order $40p$, it is a vertex primitive graph of order $40{\cdot}3 = 120$ admitting S_7 as its full automorphism group.

Key words : Symmetric graph; s-transitive graph; coset graph; orbital graph.

2010 Mathematics Subject Classification : 05C25, 20B25.

1. INTRODUCTION

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [19, 21] or [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An s-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X is simply called

s-arc-transitive, *s-regular* or *s-transitive* if it is $(\text{Aut}(X), s)$-arc-transitive, $(\text{Aut}(X), s)$-regular or $(\text{Aut}(X), s)$-transitive, respectively.

As we all known that the structure of the vertex stabilizers of symmetric graphs is very useful to classify such graphs, and this structure of the cubic or tetravalent case was given by Miller [16] and Potočnik [18]. Thus, classifying symmetric graphs with valency 3 or 4 has received considerable attention and a lot of results have been achieved, see [5, 24, 25]. Guo [9] determined the exact structure of pentavalent case. Following this structure, a series of pentavalent symmetric graphs was classified in [14, 22, 23]. Recently, Guo [10] gave the exact structure of heptavalent case. Then a series of results about classifying heptavalent symmetric graphs have been achieved, see for example [12, 17]. Note that pentavalent symmetric graphs of order $40p$ with p a prime were classified in [13]. Thus, in this paper, we classify connected heptavalent symmetric graphs of order $40p$ for each prime p.

2. Preliminary Results and Graph Constructions

Let X be a connected G-symmetric graph with $G \leq \text{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [15, Theorem 9], we have the following:

Proposition 2.1 — Let X be a connected heptavalent G-symmetric graph with $G \leq \text{Aut}(X)$, and let N be a normal subgroup of G. Then one of the following holds:

(1) N is transitive on $V(X)$;
(2) X is bipartite and N is transitive on each part of the bipartition;
(3) N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, the quotient graph X_N is a connected heptavalent G/N-symmetric graph.

The following proposition characterizes the vertex stabilizers of connected heptavalent s-transitive graphs (see [10, Theorem 1.1]).

Proposition 2.2 — Let X be a connected heptavalent (G, s)-transitive graph for some $G \leq \text{Aut}(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 3$ and one of the following holds:

(1) For $s = 1$, $G_v \cong \mathbb{Z}_7$, D_{14}, F_{21}, D_{28}, $F_{21} \times \mathbb{Z}_3$;
(2) For $s = 2$, $G_v \cong F_{42}$, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$, $\text{PSL}(3,2)$, A_7, S_7, $\mathbb{Z}_2^3 \rtimes \text{SL}(3,2)$ or $\mathbb{Z}_2^4 \rtimes \text{SL}(3,2)$;
(3) For $s = 3$, $G_v \cong F_{42} \times \mathbb{Z}_6$, $\text{PSL}(3,2) \times \text{S}_4$, $\text{A}_7 \times \text{A}_6$, $\text{S}_7 \times \text{S}_6$, $(\text{A}_7 \times \text{A}_6) \rtimes \mathbb{Z}_2$, $\mathbb{Z}_2^6 \rtimes (\text{SL}(2,2) \times \text{SL}(3,2))$ or $[2^{20}] \rtimes (\text{SL}(2,2) \times \text{SL}(3,2))$.

From [6, pp.12-14], [20, Theorem 2] and [11, Theorem A], we may obtain the following proposition by checking the orders of non-abelian simple groups:

Proposition 2.3 — Let p be a prime, and let G be a non-abelian simple group of order $|G| \,\big|\, (2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}p)$. Then G has 3-prime factor, 4-prime factor or 5-prime factor, and is one of the following groups:

Table 1: **Non-abelian simple** $\{2,3,5,7,p\}-$**groups**

3-prime factor					
G	Order	G	Order	G	Order
A_5	$2^2{\cdot}3{\cdot}5$	$\mathrm{PSL}(2,8)$	$2^3{\cdot}3^2{\cdot}7$	$\mathrm{PSU}(4,2)$	$2^6{\cdot}3^4{\cdot}5$
A_6	$2^3{\cdot}3^2{\cdot}5$	$\mathrm{PSL}(2,17)$	$2^4{\cdot}3^2{\cdot}17$	$\mathrm{PSU}(3,3)$	$2^5{\cdot}3^3{\cdot}7$
$\mathrm{PSL}(2,7)$	$2^3{\cdot}3{\cdot}7$	$\mathrm{PSL}(3,3)$	$2^4{\cdot}3^3{\cdot}13$		
4-prime factor					
G	Order	G	Order	G	Order
A_7	$2^3{\cdot}3^2{\cdot}5{\cdot}7$	$\mathrm{PSL}(2,27)$	$2^2{\cdot}3^3{\cdot}7{\cdot}13$	$\mathrm{PSU}(5,2)$	$2^{10}{\cdot}3^5{\cdot}5{\cdot}11$
A_8	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	$\mathrm{PSL}(2,31)$	$2^5{\cdot}3{\cdot}5{\cdot}31$	$\mathrm{PSp}(4,4)$	$2^8{\cdot}3^2{\cdot}5^2{\cdot}17$
A_9	$2^6{\cdot}3^4{\cdot}5{\cdot}7$	$\mathrm{PSL}(2,49)$	$2^4{\cdot}3{\cdot}5^2{\cdot}7^2$	$\mathrm{PSp}(6,2)$	$2^9{\cdot}3^4{\cdot}5{\cdot}7$
A_{10}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7$	$\mathrm{PSL}(2,81)$	$2^4{\cdot}3^4{\cdot}5{\cdot}41$	M_{11}	$2^4{\cdot}3^2{\cdot}5{\cdot}11$
$\mathrm{PSL}(2,11)$	$2^2{\cdot}3{\cdot}5{\cdot}11$	$\mathrm{PSL}(2,127)$	$2^7{\cdot}3^2{\cdot}7{\cdot}127$	M_{12}	$2^6{\cdot}3^3{\cdot}5{\cdot}11$
$\mathrm{PSL}(2,13)$	$2^2{\cdot}3{\cdot}7{\cdot}13$	$\mathrm{PSL}(3,4)$	$2^6{\cdot}3^2{\cdot}5{\cdot}7$	J_2	$2^7{\cdot}3^3{\cdot}5^2{\cdot}7$
$\mathrm{PSL}(2,16)$	$2^4{\cdot}3{\cdot}5{\cdot}17$	$\mathrm{PSU}(3,4)$	$2^6{\cdot}3{\cdot}5^2{\cdot}13$	$\mathrm{P\Omega}^+(8,2)$	$2^{12}{\cdot}3^5{\cdot}5^2{\cdot}7$
$\mathrm{PSL}(2,19)$	$2^2{\cdot}3^2{\cdot}5{\cdot}19$	$\mathrm{PSU}(3,5)$	$2^4{\cdot}3^2{\cdot}5^3{\cdot}7$	$\mathrm{Sz}(8)$	$2^6{\cdot}5{\cdot}7{\cdot}13$
$\mathrm{PSL}(2,25)$	$2^3{\cdot}3{\cdot}5^2{\cdot}13$	$\mathrm{PSU}(3,8)$	$2^9{\cdot}3^4{\cdot}7{\cdot}19$	${}^2F_4(2)'$	$2^{11}{\cdot}3^3{\cdot}5^2{\cdot}13$
5-prime factor					
G	Order	G	Order	G	Order
A_{11}	$2^7{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}11$	$\mathrm{PSL}(2,449)$	$2^6{\cdot}3^2{\cdot}5^2{\cdot}7{\cdot}449$	$\mathrm{PSp}(8,2)$	$2^{16}{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}17$
A_{12}	$2^9{\cdot}3^5{\cdot}5^2{\cdot}7{\cdot}11$	$\mathrm{PSL}(2,2^6)$	$2^6{\cdot}3^2{\cdot}5{\cdot}7{\cdot}13$	M_{22}	$2^7{\cdot}3^2{\cdot}5{\cdot}7{\cdot}11$
$\mathrm{PSL}(2,29)$	$2^2{\cdot}3{\cdot}5{\cdot}7{\cdot}29$	$\mathrm{PSL}(2,251)$	$2^2{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}251$	$\mathrm{P\Omega}^-(8,2)$	$2^{12}{\cdot}3^4{\cdot}5{\cdot}7{\cdot}17$
$\mathrm{PSL}(2,41)$	$2^3{\cdot}3{\cdot}5{\cdot}7{\cdot}41$	$\mathrm{PSL}(4,4)$	$2^{12}{\cdot}3^4{\cdot}5^2{\cdot}7{\cdot}17$	$G_2(4)$	$2^{12}{\cdot}3^3{\cdot}5^2{\cdot}7{\cdot}13$
$\mathrm{PSL}(2,71)$	$2^3{\cdot}3^2{\cdot}5{\cdot}7{\cdot}71$	$\mathrm{PSL}(5,2)$	$2^{10}{\cdot}3^2{\cdot}5{\cdot}7{\cdot}31$		

In view of [12, Theorem 1.1] and [17, Theorem 1.1], the classifications of connected heptavalent symmetric graphs of order kp are given, where $k = 8$, 10 or 20, and p is a prime. Thus, we have that following characterization of these graphs:

Proposition 2.4 — Let X be connected heptavalent symmetric graph of order kp with p a prime and $k = 8$, 10 or 20. Let $v \in V(X)$ and s a positive integer. If $k = 8$ or 10, then the descriptions of $|V(X)|$, Aut(X), s-transitivity and Aut$(X)_v$ are as follows. If $k = 20$, then there is no such graph.

Table 2: **Heptavalent s-transitive graphs of order kp**

$k{\cdot}p = \lvert V(X)\rvert$	s-transitive	Aut(X)	Aut$(X)_v$
$8{\cdot}2 = 16$	2-transitive	$\mathrm{S}_8 \times \mathbb{Z}_2$	S_7
$8{\cdot}3 = 24$	1-transitive	$\mathrm{PGL}(2,7)$	D_{14}
$10{\cdot}31 = 310$	3-transitive	$\mathrm{Aut}(\mathrm{PSL}(5,2))$	$\mathbb{Z}_2^6 \rtimes (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$

The following graph is a vertex-primitive 2-transitive graph of order 120, defined on the group S_7.

Construction 2.5 — Let $G = \mathrm{S}_7$. Then G has a maximal subgroup $H \cong F_{42}$ and hence G has a primitive permutation representation on 120 points. By Magma [3], this representation has one self-paired suborbit of length 7, and denoted the corresponding orbital graph by $\mathcal{SG}_{120}$. Moreover, Aut$(\mathcal{SG}_{120}) \cong G$.

3. Classification

This section is devoted to classifying connected heptavalent symmetric graphs of order $40p$ for each prime p.

***Theorem* 3.1** — *Let X be a connected heptavalent graph of order $40p$ with p a prime. Then X is symmetric if and only if $p = 3$ and $X \cong \mathcal{SG}_{120}$ with* Aut$(X) \cong \mathrm{S}_7$.

Proof : Let $A = \mathrm{Aut}(X)$ and $v \in V(X)$. If $p = 2$, then $|V(X)| = 80$. By [8, Theorem 1.1], there is no such graph of order 80. If $p = 3$, then $|V(X)| = 120$. By [7, Theorem 1.1], $X \cong \mathcal{SG}_{120}$ and $A \cong \mathrm{S}_7$. Thus, in what follows, we may assume that $p \geq 5$. Take $v \in V(X)$. Then by Proposition 2.2, $|A_v| \mid 2^{24}{\cdot}3^4{\cdot}5^2{\cdot}7$ and hence $|A| \mid 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}p$. We separate the proof into two cases: A has a solvable minimal normal subgroup; A has no solvable normal subgroup.

Case 1 : A has a solvable minimal normal subgroup.

Let N be a solvable minimal normal subgroup of A. Then $|N| \mid 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}p$, and N is elementary abelian. Thus, $N \cong \mathbb{Z}_q^k$ with $q = 2, 3, 5, 7$ or p and k a positive integer. Note that $|V(X)| = 2^3{\cdot}5{\cdot}p$. By Proposition 2.1, N is semiregular and X_N is also a connected heptavalent A/N-symmetric graph. It follows that $|N| \mid 40p$ and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_5$ or $\mathbb{Z}_p$. Since there is no connected heptavalent regular graph of odd order, we have that $N \ncong \mathbb{Z}_2^3$. Note that $|X_N| = 40p/|N|$ and $p \geq 5$. By Proposition 2.4, $N \ncong \mathbb{Z}_2$, $\mathbb{Z}_5$ or $\mathbb{Z}_p$. Thus, $N \cong \mathbb{Z}_2^2$ and $|X_N| = 10p$.

Again by Proposition 2.4, $p = 31$ and $A/N \lesssim \mathrm{PSL}(5,2) \rtimes \mathbb{Z}_2$. By Magma [3], $\mathrm{PSL}(5,2) \rtimes \mathbb{Z}_2$ is the only minimal arc-transitive subgroup and hence $A/N \cong \mathrm{PSL}(5,2) \rtimes \mathbb{Z}_2$. It follows that A/N has a normal subgroup $M/N \cong \mathrm{PSL}(5,2)$, which has two orbits on $V(X_N)$. By Atlas [4], $\mathrm{Mult}(\mathrm{PSL}(5,2)) = 1$. Thus, we have that $M \cong \mathrm{PSL}(5,2) \times \mathbb{Z}_2^2$ and $A \cong (\mathrm{PSL}(5,2) \times \mathbb{Z}_2^2).\mathbb{Z}_2$. It forces that A has a normal subgroup $H \cong \mathbb{Z}_2$ and X_H is a heptavalent symmetric graph of order $20{\cdot}31$. However, by Proposition 2.4, there is no such graph of order $20{\cdot}31$, a contradiction.

Case 2 : A has no solvable normal subgroup.

For convenience, we still use N to denote a minimal normal subgroup of A. Then N is non-solvable. Let $N = T^k$ with T a non-abelian simple group and k a positive integer. Then T has at least 3-prime factors. Since $|T| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}p$, we have that T is one of the simple groups listed in Proposition 2.3. By Proposition 2.1, N has at most two orbits on $V(X)$, and hence $|N| = 40p|N_v|$ or $20p|N_v|$.

Assume that $k \geq 2$. Since T is a non-abelian simple group, we have that $3 \,\big|\, |T|$ and $T_v \neq 1$. If $p > 7$, then $p \nmid |T|$ because $p^2 \nmid |N| = |T^k|$. It follows that p divides the order of X_N. By Proposition 2.1, $N = T^k$ is semiregular and hence $T_v = 1$, a contradiction. Thus, $p \leq 7$. Recall that $p \geq 5$, $|T^k| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7{\cdot}p$ and $20{\cdot}p \,\big|\, |T^k|$.

Let $p = 5$. Then $7^2 \nmid |T^k|$ and T is a simple $\{2,3,5\}$-group. By Proposition 2.3, $N \cong \mathrm{A}_5^2$, A_5^3, A_5^4 or A_6^2. The normality of N in A implies that $N_v \trianglelefteq A_v$. If $k \geq 4$, then $5^2 \,\big|\, |N_v|$ and hence N_v has a subgroup isomorphic to A_7 by Proposition 2.2. It follows that $7 \,\big|\, |T|$, a contradiction. Thus, $N \ncong \mathrm{A}_5^4$. Suppose that $N \cong \mathrm{A}_5^2$. Then $|N_v| = 36$ or 18. By Atlas [3], $N_v \cong \mathrm{A}_4 \times \mathbb{Z}_3$ or $\mathrm{S}_3 \times \mathrm{S}_3$ for $|N_v| = 36$, and $N_v \cong \mathrm{S}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$ for $|N_v| = 18$. However, A_v has no such normal subgroup by Proposition 2.3, a contradiction. Suppose that $N \cong \mathrm{A}_5^3$. Then $|N_v| = 2160$ or 1080. By Magma [3], N_v has a characteristic subgroup isomorphic to A_5, which is normal in A_v. However, A_v has no normal subgroup isomorphic to A_5, a contradiction. Suppose that $N \cong \mathrm{A}_6^2$. Then $|N_v| = 1296$ or 648. By Magma [3], A_6^2 has no subgroups of such orders, also a contradiction. Thus, $p \neq 5$.

Let $p = 7$. Then $|T^k| \,\big|\, 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7^2$ and $20{\cdot}7 \,\big|\, |T^k|$. It follows that $7 \,\big|\, |T|$ and $k = 2$. By Proposition 2.3, T is isomorphic to A_7, A_8 or $\mathrm{PSL}(3,4)$. Suppose that $N \cong \mathrm{A}_7^2$. Then $|N_v| = 2^4{\cdot}3^4{\cdot}5{\cdot}7$ or $2^3{\cdot}3^4{\cdot}5{\cdot}7$. By Magma [3], $N_v \cong \mathrm{A}_7 \times (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2)$ or $\mathrm{A}_7 \times \mathbb{Z}_3^2$. However, A_v has no such normal subgroups, a contradiction. Suppose that $N \cong \mathrm{A}_8^2$. Then $|N_v| = 2^{10}{\cdot}3^4{\cdot}5{\cdot}7$ or $2^9{\cdot}3^4{\cdot}5{\cdot}7$. By Magma [3], N_v has a normal subgroup isomorphic to A_8 or $N_v \cong \mathrm{A}_7 \times (\mathbb{Z}_2^4 \rtimes \mathrm{S}_3^2)$. For the former, A_8 cannot be as a permutation of degree 7, a contradiction. For the latter, A_v has no such normal subgroup, also a contradiction. Suppose that $N \cong \mathrm{PSL}(3,4)^2$. Then $|N_v| = 2^{10}{\cdot}3^4{\cdot}5{\cdot}7$ or $2^9{\cdot}3^4{\cdot}5{\cdot}7$. Then by Atlas [4], $N_v \cong \mathrm{PSL}(3,4) \times (\mathbb{Z}_3^2 \rtimes Q_8)$. However, A_v has no such normal subgroups, a contradiction.

Thus, $k = 1$ and $N = T$ is a non-abelian simple group listed in Proposition 2.3. It follows that $N_v \neq 1$ and by Proposition 2.1, N has at most two orbits on $V(X)$ and $|N_v| = |N|/20p$ or $|N|/40p$.

Subcase 2.1 : Suppose that $p = 5$. Then $|N| \bigm| 2^{27}{\cdot}3^4{\cdot}5^4{\cdot}7$. By Proposition 2.3, N is isomorphic to $\mathrm{PSU}(3,5)$, J_2 or $\mathrm{P}\Omega^+(8,2)$. Note that $|N_v| = |N|/20{\cdot}5$ or $|N|/40{\cdot}5$.

Let $N \cong \mathrm{PSU}(3,5)$ or $\mathrm{P}\Omega^+(8,2)$. Then $|N_v| = 2^2{\cdot}3^2{\cdot}5{\cdot}7$ or $2{\cdot}3^2{\cdot}5{\cdot}7$ for $N \cong \mathrm{PSU}(2,5)$, $|N_v| = 2^9{\cdot}3^5{\cdot}7$ or $2^{10}{\cdot}3^3{\cdot}7$ for $N \cong \mathrm{P}\Omega^+(8,2)$. By Magma [3], N has no subgroups of such orders, a contradiction.

Let $N \cong \mathrm{J}_2$. Then $|N_v| = 2^4{\cdot}3^3{\cdot}7$ or $2^5{\cdot}3^3{\cdot}7$. By Atlas [4], $|N_v| = 2^5{\cdot}3^3{\cdot}7$ and $N_v \cong \mathrm{PSU}(3,3)$. Clearly, $\mathrm{PSU}(3,3)$ cannot be as a permutation group of degree 7, a contradiction.

Subcase 2.2 : Suppose that $p = 7$. Then $|N| \bigm| 2^{27}{\cdot}3^4{\cdot}5^3{\cdot}7^2$ and $20{\cdot}7 \bigm| |N|$ or $40{\cdot}7 \bigm| |N|$. In particular, $7^2 \bigm| |A|$. By Proposition 2.3, N is isomorphic to one of the following simple groups:

$$\mathrm{A}_7,\ \mathrm{A}_8,\ \mathrm{A}_9,\ \mathrm{A}_{10},\ \mathrm{PSL}(2,49)$$
$$\mathrm{PSL}(3,4),\ \mathrm{PSp}(6,2),\ \mathrm{J}_2,\ \mathrm{P}\Omega^+(8,2).$$

Let $N \cong \mathrm{A}_7$. Then $|N_v| = 3^2$ or $2{\cdot}3^2$. Then by Magma [3], $N_v \cong \mathbb{Z}_3^2$ or $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$. It forces that A_v has a normal subgroup isomorphic to $\mathbb{Z}_3^2$, and by Proposition 2.2, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{A}_8$. Then $|N_v| = 2^3{\cdot}3^2$ or $2^4{\cdot}3^2$. By Magma [3], $N_v \cong \mathbb{Z}_3^2 \rtimes D_8$, $(\mathbb{Z}_2^2 \rtimes \mathbb{Z}_3^2) \rtimes \mathbb{Z}_2$ or $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_3^2$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{A}_9$. Then $|N_v| = 2^3{\cdot}3^4$ or $2^4{\cdot}3^4$. By Atlas [4], $|N_v| = 2^3{\cdot}3^4$ and $N_v \cong \mathbb{Z}_3^3 \rtimes \mathrm{S}_4$. However, A_v has no such normal subgroups, a contradiction.

Let $N \cong \mathrm{A}_{10}$, $\mathrm{PSL}(2,49)$, $\mathrm{PSp}(6,2)$, J_2 or $\mathrm{P}\Omega^+(8,2)$. Then $|N_v| = 2^4{\cdot}3^4{\cdot}5$ or $2^5{\cdot}3^4{\cdot}5$ for $N \cong \mathrm{A}_{10}$, $|N_v| = 2{\cdot}3{\cdot}5{\cdot}7$ or $2^2{\cdot}3{\cdot}5{\cdot}7$ for $N \cong \mathrm{PSL}(2,49)$, $|N_v| = 2^6{\cdot}3^4$ or $2^7{\cdot}3^4$ for $N \cong \mathrm{PSp}(6,2)$, $|N_v| = 2^4{\cdot}3^4{\cdot}5$ or $2^4{\cdot}3^4{\cdot}5$ for $N \cong \mathrm{J}_2$, or $|N_v| = 2^9{\cdot}3^5{\cdot}5$ or $2^{10}{\cdot}3^5{\cdot}5$ for $N \cong \mathrm{P}\Omega^+(8,2)$. By Magma [3], N has no subgroups of such orders, a contradiction.

Let $N \cong \mathrm{PSL}(3,4)$. Then $|N_v| = 2^3{\cdot}3^2$ or $2^4{\cdot}3^2$. By Magma [3], $|N_v| = 2^3{\cdot}3^2$ and $N_v \cong \mathbb{Z}_3^2 \rtimes Q_8$. However, A_v has no such normal subgroups, a contradiction.

Subcase 2.3 : Suppose that $p > 7$. Then $|N| \bigm| 2^{27}{\cdot}3^4{\cdot}5^5{\cdot}7{\cdot}p$ and $20{\cdot}p \bigm| |N|$ or $40{\cdot}p \bigm| |N|$. By Proposition 2.2, N is isomorphic to one of the following simple groups:

$$\begin{aligned}&\mathrm{PSL}(2,11),\ \mathrm{PSL}(2,16),\ \mathrm{PSL}(2,19),\ \mathrm{PSL}(2,25),\ \mathrm{PSL}(2,31)\\&\mathrm{PSL}(2,81),\ \mathrm{PSU}(3,4),\ \mathrm{PSU}(5,2),\ \mathrm{PSp}(4,4),\ \mathrm{M}_{11},\ \mathrm{M}_{12}\\&\mathrm{Sz}(8),\ {}^2F_4(2)',\ A_{11},\ \mathrm{PSL}(2,29),\ \mathrm{PSL}(2,41),\ \mathrm{PSL}(2,71),\ \mathrm{PSL}(2,449)\\&\mathrm{PSL}(2,251),\ \mathrm{PSL}(4,4),\ \mathrm{PSL}(5,2),\ \mathrm{M}_{22},\ \mathrm{P}\Omega^-(8,2),\ G_2(4)\end{aligned}$$

Note that $p > 7$. Since $|N_v| = |N|/20p$ or $|N|/40p$, we have that a Sylow 3-subgroup of N_v is also a Sylow 3-subgroup of N. By Proposition 2.2, a Sylow 3-subgroup of A_v is elementary abelian, and so is that of N_v. With the calculation of Magma [3], we have that $N \ncong \mathrm{M}_{12}$, ${}^2F_4(2)'$, A_{11}, $\mathrm{PSL}(4,4)$, $\mathrm{P}\Omega^-(8,2)$ or $G_2(4)$.

Let $N \cong \mathrm{PSL}(2,11)$. Then $|N_v| = 3$ and $N_v \cong \mathbb{Z}_3$. Since $N_v \trianglelefteq A_v$, we have that $A_v \cong F_{21} \times \mathbb{Z}_3$, $F_{42} \times \mathbb{Z}_3$ or $F_{42} \times \mathbb{Z}_6$ by Proposition 2.2. Assume that $C_A(N) = 1$. Then by "N/C-Theorem", $A \cong A/C_A(N) \lesssim \mathrm{Aut}(\mathrm{PSL}(2,11))$. This is impossible because $\mathrm{Aut}(\mathrm{PSL}(2,11)) \cong \mathrm{PGL}(2,11)$ and $7 \bigm| |A/N|$. Thus, $C_A(N) \neq 1$, and by our assumption, $C_A(N)$ is non-solvable. Note that $C_A(N) \cong C_A(N)N/N \trianglelefteq A/N$. This is also impossible because A/N is solvable.

Let $N \cong \mathrm{PSL}(2,16)$. Then $|N_v| = 6$ or 12. By Magma [3], $N_v \cong \mathrm{S}_3$ or A_4. By the normality of N_v in A_v and Proposition 2.2, $A_v \cong \mathrm{PSL}(3,2) \times \mathrm{S}_4$ and $N_v \cong \mathrm{A}_4$. A similar argument as above, $C_A(N)$ is non-solvable. Since $C_A(N) \lesssim A/N$, $A_vN/N \cong \mathrm{PSL}(3,2) \times \mathbb{Z}_2$ and $|A : A_vN| = 2$, we have that $C_A(N)$ has a subgroup $H \cong \mathrm{PSL}(3,2)$, which is normal in A. It follows that H acting on $V(X)$ has at least 5 orbits because $5 \nmid |H|$. By Proposition 2.1, $H \bigm| 40p$ and hence H is solvable, a contradiction.

Let $N \cong \mathrm{PSL}(2,19)$. Then $|N_v| = 9$ and $N_v \cong \mathbb{Z}_9$. However, by Proposition 2.2, A_v has no subgroup isomorphic to $\mathbb{Z}_9$, a contradiction.

Let $N \cong \mathrm{PSL}(2,25)$, $\mathrm{PSU}(3,4)$, $\mathrm{PSU}(5,2)$, $\mathrm{PSp}(4,4)$, $\mathrm{PSL}(2,29)$, $\mathrm{PSL}(2,71)$, $\mathrm{PSL}(2,449)$, $\mathrm{PSL}(2,251)$ or M_{22}. Then $|N_v| = 15$ or 30 for $N \cong \mathrm{PSL}(2,25)$, $|N_v| = 2^3{\cdot}3{\cdot}5$ or $2^4{\cdot}3{\cdot}5$ for $N \cong \mathrm{PSU}(3,4)$ $|N_v| = 2^7{\cdot}3^5$ or $2^8{\cdot}3^5$ for $N \cong \mathrm{PSU}(5,2)$, $|N_v| = 2^5{\cdot}3^2{\cdot}5$ or $2^6{\cdot}3^2{\cdot}5$ for $N \cong \mathrm{PSp}(4,4)$, $|N_v| = 21$ for $N \cong \mathrm{PSL}(2,29)$, $|N_v| = 2{\cdot}3^2{\cdot}7$ or $3^2{\cdot}7$ for $N \cong \mathrm{PSL}(2,71)$, $|N_v| = 2^3{\cdot}3^2{\cdot}5{\cdot}7$ or $2^4{\cdot}3^2{\cdot}5{\cdot}7$ for $N \cong \mathrm{PSL}(2,449)$, $|N_v| = 3^2{\cdot}5^2{\cdot}7$ for $N \cong \mathrm{PSL}(2,251)$, or $|N_v| = 2^4{\cdot}3^2{\cdot}7$ or $2^5{\cdot}3^2{\cdot}7$ for $N \cong \mathrm{M}_{22}$. By Magma [3], N has no subgroups of such orders, a contradiction.

Let $N \cong \mathrm{PSL}(2,31)$. Then $|N_v| = 12$ or 24. By Magma [3], $N_v \cong \mathrm{A}_4$ or S_4. By Proposition 2.2, $A_v \cong \mathrm{PSL}(3,2) \times \mathrm{S}_4$ because $N_v \trianglelefteq A_v$. Similar, $C_A(N) \neq 1$ and A has a normal subgroup $H \cong \mathrm{PSL}(3,2)$ and $H \leq C_A(N)$. Since $5 \nmid |H|$, we have that H is semiregular on $V(X)$ by Proposition 2.1. It then forces that H is solvable, a contradiction.

Let $N \cong \mathrm{PSL}(2,81)$. Then $|N_v| = 2{\cdot}3^4$ or $2^2{\cdot}3^4$. By Magma [3], N_v has a characteristic subgroup of order 3^4. This implies that A_v has a normal subgroup of order 3^4, which is contrary to the fact of Proposition 2.2.

Let $N \cong \mathrm{M}_{11}$. Then $|N_v| = 2{\cdot}3^2$ or $2^2{\cdot}3^2$. By Magma [3], N_v has a normal Sylow 3-subgroup $P \cong \mathbb{Z}_3^2$. It follows that $P \trianglelefteq A_v$. However, A_v has no normal subgroup isomorphic $\mathbb{Z}_3^2$, a contradiction.

Let $N \cong \mathrm{Sz}(8)$. Then $|N_v| = 2^3{\cdot}7$ or $2^4{\cdot}7$. By Magma [3], $|N_v| = 2^3{\cdot}7$ and $N_v \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$. However, A_v has no such normal subgroup, a contradiction.

Let $N \cong \mathrm{PSL}(2,41)$. Then $|N_v| = 21$ or 42. By Magma [3], $N_v \cong \mathbb{Z}_{21}$ or D_{42}. However, A_v has no such normal subgroup, a contradiction.

Let $N \cong \mathrm{PSL}(2,2^6)$. Then $|N_v| = 2^3{\cdot}3^2{\cdot}7$ or $2^4{\cdot}3^2{\cdot}7$. By Magma [3], $|N_v| = 2^3{\cdot}3^2{\cdot}7$ and $N_v \cong \mathrm{PSL}(2,8)$. However, $\mathrm{PSL}(2,8)$ cannot be as a permutation group of degree 7, a contradiction.

Let $N \cong \mathrm{PSL}(5,2)$. Then $|N_v| = 2^7{\cdot}3^2{\cdot}7$ or $2^8{\cdot}3^2{\cdot}7$. By Magma [3] and Atlas [4], $|N_v| = 2^7{\cdot}3^2{\cdot}7$ and $N_v \cong \mathbb{Z}_2^6 \rtimes (\mathrm{S}_3 \times F_{21})$. However, A_v has no such normal subgroup, a contradiction. $\square$

Acknowledgement

This work was supported by the National Natural Science Foundation of China (11301154).

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