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ON CUMULATIVE TSALLIS ENTROPY AND ITS DYNAMIC PAST VERSION

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Cumulative Tsallis entropy and its residual have developed by many others, and they proposed alternative (second) forms of them. In this article, we illustrate the differences and the relation between the proposed model and its alternative form. Furthermore, we study some properties and features for the concept of cumulative Tsallis entropy and its residual. In addition, we propose a dynamic past form of cumulative Tsallis entropy and obtain some of its properties.

Key words : Cumulative residual Tsallis entropy; dynamic cumulative entropy; mean inactivity time; stochastic orders; Tsallis entropy.

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1. Introduction

A classical measure of uncertainty is the Shannon entropy (Shannon [19]) for the non-negative continuous random variable X, with probability density function (pdf) $f(.)$, defined as

$$H(X) = -E(\ln f(X)) = -\int_0^\infty f(x)\ln f(x)dx, \tag{1}$$

which has been widely used in many areas such communication theory, computer science, physical and biological sciences and fuzzy sets. With introducing some additional parameters many generalizations of Shannon entropy are obtained in literature, which make these entropies sensitive to different shapes of probability distributions. A well known generalization of Shannons entropy is Tsallis entropy of order α that was first introduced by Havrda and Charvat [8] in the status of cybernetics concept. Then, Tsallis [21] exploited its non-extensive features and placed it in a physical context. This measure is defined for the continuous random variable X as

$$T_\alpha(X) = \frac{1}{\alpha-1}E(1-(f(X))^{\alpha-1}) = \frac{1}{\alpha-1}\left(1-\int_0^\infty (f(x))^\alpha dx\right), \tag{2}$$

where $0 < \alpha \neq 1$. Clearly, when $\alpha \to 1$, $T_\alpha(X) \to H(X)$. Rao *et al.* [15] (see also, Wang *et al.* [23]) defined a new measure of uncertainty based on the survival function of X, and called it cumulative residual entropy (CRE), obtained by replacing the $pdf\ f(.)$ in (1.1) by the survival function $\bar{F}(.)$, as follows

$$\varepsilon(X) = -\int_0^\infty \bar{F}(x)\ln\bar{F}(x)dx, \tag{3}$$

CRE measure is always non-negative and can be applied in reliability and image alignment. It is based on cdf rather than pdf, which is generally more stable since the distribution function is more regular being defined in an integral form unlike the density function, which is defined as the derivative of the distribution. Rao [16] presented some more mathematical properties of CRE and gave an alternative formula for it. For more recent works on CRE one can refer to Asadi and Zohrevand [1], Navarro *et al.* [13], Baratpour [2], Sunoj and Linu [20], and Zardasht *et al.* [24].

Recently, Sati and Gupta [17] proposed the cumulative residual Tsallis entropy (CRTE) of order α, which defined as

$$\eta_\alpha(X) = \frac{1}{\alpha-1}\left(1-\int_0^\infty(\bar{F}(x))^\alpha dx\right). \tag{4}$$

An information measure similar to (1.3) is the cumulative entropy, defined as (see, Di Crescenzo and Longobardi [6])

$$c\varepsilon(X) = -\int_0^\infty F(x)\ln F(x)dx, \tag{5}$$

where $F(.)$ is the cdf of a random variable X. However, since the argument of the logarithm is a probability, we have $0 \leq c\varepsilon(X) < \infty$, whereas $H(X)$ may be negative in the absolutely continuous case. Moreover, $c\varepsilon(X) = 0$ if and only if X is a constant.

Rajesh and Sunoj [14] introduced an alternate measure of CRTE of order α as

$$\xi_\alpha(X) = \frac{1}{\alpha-1}\left(\int_0^\infty(\bar{F}(x)-(\bar{F}(x))^\alpha)dx\right), \alpha \neq 1, \alpha > 0. \tag{6}$$

Cali *et al.* [4] proposed the cumulative Tsallis entropy (CTE) based on definition of (1.5) as

$$c\xi_\alpha(X) = \frac{1}{\alpha-1}\left(\int_0^\infty(F(x)-(F(x))^\alpha)dx\right), \alpha \neq 1, \alpha > 0. \tag{7}$$

Kumar [11] studied some properties and characterization results for dynamic cumulative residual Tsallis entropy (DCRTE). Also, developed some characterization results of Tsallis survival entropy of the first order statistic. Khammar and Jahanshahi [10] proposed generalizations of CRTE and DCRTE called weighted cumulative residual Tsallis entropy (WCRTE) and dynamic weighted cumulative residual Tsallis entropy (DWCRTE), respectively.

In the present paper, we introduce the CRTE of order α which has been introduced by Sati and Gupta [17] and alternative CTE of order α with its dynamic past version that has not been introduced before. The proposed measures have some additional features and simple relationships with other important information and reliability measures.

2. CUMULATIVE TSALLIS AND ITS RESIDUAL

In this section, we will give some properties and relationships about the cumulative Tsallis entropy and its residual, some of them illustrate misconceptions and corrections of such measures. According to Rajesh and Sunoj [14] and Cali *et al*. [4], from (3), (5), (6) and (7), the following statements hold

$$\begin{aligned} \xi_\alpha(X) &\underset{\alpha\to1}{\longrightarrow} \varepsilon(X), \\ c\xi_\alpha(X) &\underset{\alpha\to1}{\longrightarrow} c\varepsilon(X). \end{aligned} \tag{8}$$

But form Sati and Gupta [17], they mentioned that $\eta_\alpha(X)$ tends to $\varepsilon(X)$ when $\alpha \to 1$, which can't be hold. Similarly, let X be a random variable with support in $[a,b]$, $a,b \in \mathbb{R}^+$. Thus, the alternative form of the CTE of order α, which defined as

$$c\eta_\alpha(X) = \frac{1}{\alpha-1}\left(1 - \int_a^b (F(x))^\alpha dx\right). \tag{9}$$

doesn't tends to $c\varepsilon(X)$ when $\alpha \to 1$.

Example 2.1 : With the cdf of power function distribution:

$$F(x) = \left(\frac{x}{c}\right)^a, 0 < x < c, a > 0. \tag{10}$$

Take $a = 2, c = 1$, then we have

1. $c\eta_\alpha(X) = \frac{1}{\alpha-1}(1 - \frac{1}{1+2\alpha})$.
2. $\eta_\alpha(X) = \frac{1}{\alpha-1}(1 - \frac{\sqrt{\pi}\Gamma(1+\alpha)}{2\Gamma(\frac{3}{2}+\alpha)})$.
3. $c\xi_\alpha(X) = \frac{1}{\alpha-1}(\frac{1}{3} - \frac{1}{1+2\alpha})$.
4. $\xi_\alpha(X) = \frac{1}{\alpha-1}(\frac{2}{3} - \frac{\sqrt{\pi}\Gamma(1+\alpha)}{2\Gamma(\frac{3}{2}+\alpha)})$.
5. $\lim\limits_{\alpha\to1} c\eta_\alpha(X) = \infty$.
6. $\lim\limits_{\alpha\to1} \eta_\alpha(X) = \infty$.

7. $c\varepsilon(X) = \frac{2}{9} = \lim_{\alpha \to 1} c\xi_\alpha(X)$.

8. $\varepsilon(X) = \frac{2}{9}(5 - \ln(64)) = \lim_{\alpha \to 1} \xi_\alpha(X)$.

Thus, from the pervious example, we conclude that $c\eta_\alpha(X) \neq c\xi_\alpha(X)$, $c\eta_\alpha(X) \underset{\alpha \to 1}{\nrightarrow} c\varepsilon_\alpha(X)$ and $\eta_\alpha(X) \neq \xi_\alpha(X)$, $\eta_\alpha(X) \underset{\alpha \to 1}{\nrightarrow} \varepsilon_\alpha(X)$.

For some families and distributions which its range is infinite such as Pareto, exponential and Weibull distributions. We can easily find that the CTE $c\eta_\alpha(X)$ doesn't exist. But for some distributions like power or uniform distributions which its range has an upper and lower limit, we can see that the CTE $c\eta_\alpha(X)$ exists, i.e. the random variable X defined with support in $[a, b]$, $a, b \in \mathbb{R}^+$. Meanwhile, the CRTE $\eta_\alpha(X)$ exists, where X is defined in its support. Based on this notation, we provide the following applications and features for those measures.

Remark 2.1 : If the random variable X follows uniform distribution $U(a, b)$ with CDF

$$F(x) = \left(\frac{x - a}{b - a}\right), a < x < b, a, b > 0. \tag{11}$$

We conclude that

1. $c\xi_\alpha(X) = \xi_\alpha(X)$, $c\eta_\alpha(X) = \eta_\alpha(X)$ and $\xi_\alpha(X) = \frac{1}{2(\alpha-1)}(b - a) - \eta_\alpha(X)$.

2. If $b = a + 2$, then $c\xi_\alpha(X) = \xi_\alpha(X) = c\eta_\alpha(X) = c\eta_\alpha(X)$.

Remark 2.2 : Rajesh and Sunoj [14] mentioned that $\xi_\alpha(X) \neq \eta_\alpha(X)$ as $\xi_\alpha(X) = \eta_\alpha(X) + \frac{\mu - 1}{\alpha - 1}$, $\mu = \mathbb{E}(X)$. Meanwhile, they also mentioned that $\xi_\alpha(X) \geq 0$ (≤ 0) for all x according as $\alpha > 1 (0 < \alpha < 1)$, but we can easily show that it can't be hold, see Figer (1).

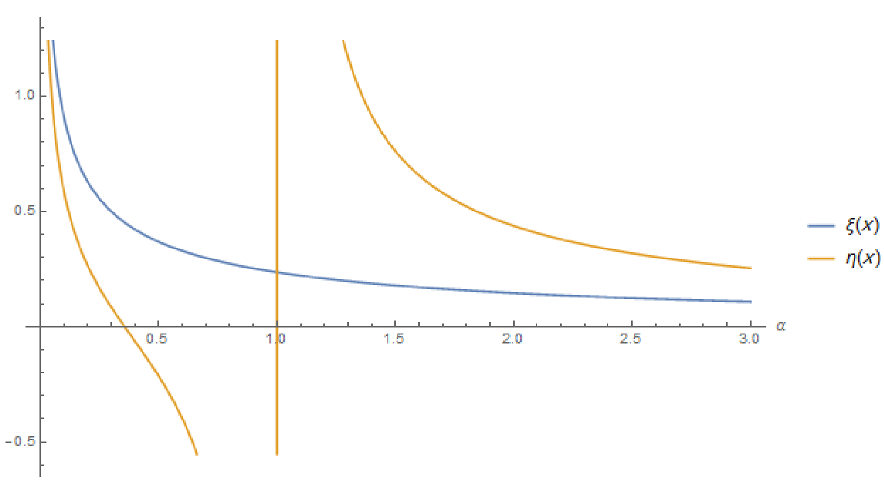

Figure 1: Weibull distribution with CDF $F(x) = 1 - e^{-ax^b}$, $a = 2, b = 3$.

Remark 2.3 : (1) If X is a non-negative and absolutely continuous random variable which defined with support in $[a, b]$, then, from (7) and (9), $c\xi_\alpha(X) = c\eta_\alpha(X) + \frac{b-\mu-1}{\alpha-1}$.

(2) Let a random variable X have a pdf $f(.)$. Then the pdf of the equilibrium random variable X_E corresponding to X is given by $g(x) = \frac{\bar{F}(x)}{E(X)}$. Using (2), we have $(\alpha - 1)\eta_\alpha(X) = \mu^\alpha[1 - (\alpha - 1)T_\alpha(X_E)]$.

Remark 2.4 : If X is a random variable defined in its support, then

1. $c\eta_\alpha(X) = \frac{1}{\alpha-1}$ if and only if $F(x) = 0$.
2. $\eta_\alpha(X) = \frac{1}{\alpha-1}$ if and only if $F(x) = 1$.

In the following lemma, we show that $\eta_\alpha(X)$ and $c\eta_\alpha(X)$ are not invariant under non-singular transformation.

Lemma 2.1 — Let X be a non-negative continuous random variable with CDF $F(.)$ and $\phi(X)$ is a strictly increasing differentiable function. Then, from (4), (9), we have

1. $\eta_\alpha(\phi(X)) = \frac{1}{\alpha-1}(1 - \int_{\phi^{-1}(0)}^{\infty} \bar{F}(x)\phi'(x)dx)$.
2. $c\eta_\alpha(\phi(X)) = \frac{1}{\alpha-1}(1 - \int_{\phi^{-1}(a)}^{\phi^{-1}(b)} F(x)\phi'(x)dx)$.

Corollary 2.1 — If $\phi(x) = a_1x + b_1$, $a_1 > 0$ and $b_1 \geq 0$, then

1. $\eta_\alpha(\phi(X)) = \frac{1-a_1}{\alpha-1} + a_1\eta_\alpha(X)$.
2. $c\eta_\alpha(\phi(X)) = \frac{1-a_1}{\alpha-1} + a_1c\eta_\alpha(X)$.

Definition 2.1 — X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$ if and only if $\bar{F}(t) \leq \bar{G}(t)$, for all $t \in \mathbb{R}$. For more details, see Shaked and Shanthikumar [18].

Lemma 2.2 — Let X and Y be two non-negative continuous random variables with CDF's $F(.)$ and $G(.)$ and finite mean $\mathbb{E}(X)$ and $\mathbb{E}(Y)$, respectively. If $X \leq_{st} Y$, then, for $1 < \alpha \in \mathbb{N}$,

1. $\eta_\alpha(X) \geq \eta_\alpha(Y)$ and $\eta_\alpha(X) - \eta_\alpha(Y) \leq \mathbb{E}(Y) - \mathbb{E}(X)$.
2. $c\eta_\alpha(X) \leq c\eta_\alpha(Y)$ and $c\eta_\alpha(X) - c\eta_\alpha(Y) \geq \mathbb{E}(X) - \mathbb{E}(Y)$.

PROOF : Suppose $1 < \alpha \in \mathbb{N}$ and $X \leq_{st} Y$, from (4) and (9), we have

$$\begin{aligned}
\eta_\alpha(X) - \eta_\alpha(Y) &= \frac{1}{\alpha-1}\int_0^\infty [\bar{G}^\alpha(t) - \bar{F}^\alpha(t)]dt \ (\geq 0) \\
&\leq \frac{1}{\alpha-1}\int_0^\infty [\bar{G}(t) - \bar{F}(t)]dt \\
&\leq \mathbb{E}(Y) - \mathbb{E}(X).
\end{aligned} \tag{12}$$

Meanwhile, we can write

$$\begin{aligned}
c\eta_\alpha(X) - c\eta_\alpha(Y) &= \frac{1}{\alpha-1}\int_a^b [G^\alpha(t) - F^\alpha(t)]dt \\
&\leq \frac{1}{\alpha-1}\int_a^b [G(t) - F(t)][\sum_{i=1}^{\alpha} G^{i-1}(t)F^{\alpha-i}(t)]dt \\
&\leq \frac{1}{\alpha-1}\int_a^b [G(t) - F(t)] = \frac{1}{\alpha-1}\int_a^b [\bar{F}(t) - \bar{G}(t)] \ (\leq 0) \\
&\geq \mathbb{E}(X) - \mathbb{E}(Y).
\end{aligned} \tag{13}$$

The next example is it ensures the previous theorem.

Example 2.2 : Let X and Y be two random variables of power function distribution with CDF's $F(x) = (\frac{x}{c})^a, 0 < x < c, a > 0$ and $G(y) = (\frac{y}{c})^b, 0 < y < c, b > 0$, respectively. From (9), we have

$$\begin{aligned}
c\eta_\alpha(X) - c\eta_\alpha(Y) &= \frac{1-c+a\alpha}{(\alpha-1)(a\alpha+1)} - \frac{1-c+b\alpha}{(\alpha-1)(b\alpha+1)} \\
&= \frac{c(a\alpha - b\alpha)}{(\alpha-1)(a\alpha+1)(b\alpha+1)},
\end{aligned} \tag{14}$$

$$\mathbb{E}(X) - \mathbb{E}(Y) = \frac{c(a-b)}{(a+1)(b+1)}. \tag{15}$$

For $c = 1$, $a = 1$, $b = 2$, we have $c\eta_\alpha(X) \leq c\eta_\alpha(Y)$ and $c\eta_\alpha(X) - c\eta_\alpha(Y) > \mathbb{E}(X) - \mathbb{E}(Y)$, whenever $\alpha > 1$, while clearly $X \leq_{st} Y$. From (4), it is to be noted that for $c = 1$, $a = 1$, $b = 2$, $\alpha = 2$, $\eta_\alpha(X) - \eta_\alpha(Y) = \frac{1}{5}$ and $\mathbb{E}(Y) - \mathbb{E}(X) = \frac{1}{6}$, but $\eta_\alpha(X) - \eta_\alpha(Y) < \mathbb{E}(Y) - \mathbb{E}(X)$, whenever $\alpha > 2.217$.

Example 2.3 : Let X and Y be two random variables of Pareto distribution with CDF's $F(x) = 1 - x^{-a}, 1 < x < \infty, a > 0$ and $G(y) = 1 - y^{-b}, 1 < y < \infty, b > 0$, respectively. From (4), we have

$$\begin{aligned}
\eta_\alpha(X) - \eta_\alpha(Y) &= \frac{a\alpha - 2}{(\alpha-1)(a\alpha-1)} - \frac{b\alpha - 2}{(\alpha-1)(b\alpha-1)} \\
&= \frac{a\alpha - b\alpha}{(\alpha-1)(a\alpha-1)(b\alpha-1)},
\end{aligned} \tag{16}$$

$$\mathbb{E}(X) - \mathbb{E}(Y) = \frac{a-b}{(a-1)(b-1)}. \tag{17}$$

For $a = 4$, $b = 2$, we have $\eta_\alpha(X) \geq \eta_\alpha(Y)$ and $\eta_\alpha(X) - \eta_\alpha(Y) \leq \mathbb{E}(Y) - \mathbb{E}(X)$, whenever $\alpha > 1$, while clearly $X \leq_{st} Y$.

Let X and Y be two non-negative continuous random variables with CDF's $F(.)$ and $G(.)$, such that

$$G(t) = (F(t))^{\theta}, \text{ for all } t \in S_X, \tag{18}$$
$$\bar{G}(t) = (\bar{F}(t))^{\theta}, \text{ for all } t \in S_X, \tag{19}$$

where S_X is the support of X and $\theta > 0$. The identity (18) is known as the proportional reversed hazard rate (PRHR) model. Meanwhile, (19) is known as the Cox proportional hazard (CPH) model.

Lemma 2.3 — Let X and Y be two non-negative continuous random variables with CDF's $F(.)$ and $G(.)$,

1. If $F(.)$ and $G(.)$ satisfy the PRHR model, then

$$(\alpha - 1)c\eta_{\alpha}(y) = (\alpha\theta - 1)c\eta_{\alpha\theta}(y) \tag{20}$$

2. If $F(.)$ and $G(.)$ satisfy the CPH model, then

$$(\alpha - 1)\eta_{\alpha}(y) = (\alpha\theta - 1)\eta_{\alpha\theta}(y). \tag{21}$$

It is obvious that if X and Y have the same distribution then $\eta_{\alpha}(y) = \eta_{\alpha}(x)$ ($c\eta_{\alpha}(y) = c\eta_{\alpha}(x)$). However, in the following example we will show that the converse is not hold.

Example 2.4 : Suppose the random variable X has uniform distribution in $(0, b)$ with $b > 0$, i.e., $F(x) = \frac{x}{b}, 0 < x < b$, and X and Y satisfy the CPH model, then $\eta_{\alpha\theta}(x) = \frac{1+\alpha-b}{(\alpha-1)(\alpha+1)}$ and from (21), we have

$$\begin{aligned}(\alpha - 1)\eta_{\alpha}(y) &= (\alpha\theta - 1)\eta_{\alpha\theta}(y)\\ &= \frac{1 + \alpha\theta - b}{\alpha\theta + 1}.\end{aligned} \tag{22}$$

If $\theta = \frac{1}{\alpha^2}$, then $\eta_{\alpha}(y) = \eta_{\alpha}(x)$, which implies that $\eta_{\alpha}(x)$ does not uniquely characterize the distribution of X. Similarly for $c\eta_{\alpha}(x)$.

Lemma 2.4 — Let $X_1, ..., X_n$ be i.i.d non-negative continuous random variables with common CDF $F(.)$. Then, for $1 < \alpha \in \mathbb{N}$

1. $c\eta_{\alpha}(X_{1:n}) \leq 1 + \alpha\mathbb{E}(X)$,
2. $c\eta_{\alpha}(X_{n:n}) \leq 1 + n\alpha\mathbb{E}(X)$,

where $X_{1:n} = \min\{X_1, ..., X_n\}$ and $X_{n:n} = \max\{X_1, ..., X_n\}$.

PROOF : 1. For non-negative random variable X and using Bernoulli's inequality, we have

$$\begin{aligned}
c\eta_\alpha(X) &= \frac{1}{\alpha-1}\left(1-\int_a^b (1-\bar{F}^n(x))^\alpha dx\right) \\
&\leq \frac{1}{\alpha-1}\left[1-\int_a^b (1-\alpha\bar{F}^n(x))dx\right] \\
&= \frac{1}{\alpha-1}\left[1-(b-a)+\alpha\int_a^b \bar{F}^n(x)dx\right] \\
&\leq \frac{1}{\alpha-1}\left[1+\alpha\int_a^b \bar{F}^n(x)dx\right] \\
&\leq \frac{1}{\alpha-1}\left[1+\alpha\int_a^b \bar{F}(x)dx\right] \\
&\leq \frac{1}{\alpha-1}[1+\alpha\mathbb{E}(X)] \\
&\leq 1+\alpha\mathbb{E}(X).
\end{aligned} \tag{23}$$

2. Similarly by using Bernoulli's inequality, for non-negative random variable X,

$$\begin{aligned}
c\eta_\alpha(X) &= \frac{1}{\alpha-1}\left(1-\int_a^b (F(x))^{n\alpha} dx\right) \\
&\leq \frac{1}{\alpha-1}\left[1-\int_a^b (1-n\alpha\bar{F}(x))dx\right] \\
&= \frac{1}{\alpha-1}\left[1-(b-a)-n\alpha a+n\alpha\mathbb{E}(X)\right] \\
&\leq \frac{1}{\alpha-1}[1+n\alpha\mathbb{E}(X)] \\
&\leq 1+n\alpha\mathbb{E}(X).
\end{aligned} \tag{24}$$

Suppose i.i.d lifetimes $X_1,...,X_n$ be a system consisting of n components. Then, the lifetimes of the system $X_{1:n}$ and $X_{n:n}$ are known for series and parallel system, respectively. Thus, Lemma (2.4) provides upper bounds for CTE of series and parallel systems based on the mean lifetime of their components.

The lifetime of the system is $Y = \min\{X_1,...,X_n\}$, with $pdf\ g(y) = n(\bar{F}(x))^{n-1}f(x)$ and survival function $\bar{G}(y) = (\bar{F}(x))^n$. Clearly, X_i's and Y satisfy CPH model, $i = 1,2,...,n$. Therefore,

$$\eta_n(nX) = -1 + n\ \eta_n(X), \tag{25}$$

$$\eta_{n\alpha}(nX) = \frac{1-n}{n\alpha-1} + n\ \eta_{n\alpha}(X), \tag{26}$$

from (21) and (26) we obtain a lower bound to the CRTE for the series system in terms of CRTE of its component, as follows

$$\begin{aligned} n(\alpha-1)\eta_\alpha(Y) &= n(n\alpha-1)\eta_{n\alpha}(X) \\ &= (n\alpha-1)\eta_{n\alpha}(nX)+n-1 \\ &\geq (n\alpha-1)\eta_{n\alpha}(nX). \end{aligned} \tag{27}$$

3. Dynamic Past Cumulative Tsallis Entropy

Let X represents the random lifetime of an item survived up to time t. Therefore, $X_t = [X - t|X > t]$ is the residual lifetime of the system at age t, which its mean residual life is denoted by $m(t) = \mathbb{E}(X_t)$. Di Crescenzo and Longobardi [5] defined the uncertainty of $X_{(t)} = [X|X \leq t]$ and called past entropy, which describes the past lifetime of the system at age t, with mean past lifetime $\mu(t) = \mathbb{E}[X_{(t)}]$. The inactivity time is the duration of the time between an inspection time t and the failure time X, given that at time t the system has been found failed which is represented by the random variable $[t - X|X \leq t]$, $t > 0$, with mean inactivity time

$$\tilde{\mu}(t) = \mathbb{E}[t - X|X \leq t] = \frac{1}{F(t)}\int_0^t F(x)dx. \tag{28}$$

The derivative of $\tilde{\mu}(t)$ is given by

$$\tilde{\mu}'(t) = 1 - \tau_X(t)\tilde{\mu}(t), t > 0, F(t) > 0, \tag{29}$$

which expressed in term of the reversed hazard rate (if existing) $\tau(t) = \frac{f(t)}{F(t)}$. For further results about these concepts in reliability theory see Ebrahimi [], Kayid and Ahmad [9] and Misra *et al.* [12].

The dynamic cumulative residual Tsallis entropy (DCRTE) is proposed by Sati and Gupta [17] as follows

$$\phi_\alpha(X;t) = \frac{1}{\alpha-1}[1 - \int_t^\infty \bar{F}^\alpha_{X_t}(x)dx], \tag{30}$$

Rajesh and Sunoj [14] introduced the second (alternative) form of the DCRTE as

$$\psi_\alpha(X;t) = \frac{1}{\alpha-1}[\int_t^\infty (\bar{F}_{X_t}(x) - \bar{F}^\alpha_{X_t}(x))dx], \tag{31}$$

where $\bar{F}_{X_t}(x)$ is the survival function of X_t. Cali *et al.* [4] derived the dynamic past cumulative Tsallis entropy (DPCTE) of a non-negative absolutely continuous random variable X as

$$c\xi_\alpha(X;t) = \frac{1}{\alpha-1}[\int_0^t (F_{X_{(t)}}(x) - F^\alpha_{X_{(t)}}(x))dx]. \tag{32}$$

We propose the original (alternative) form of the DPCTE as

$$c\eta_\alpha(X;t) = \frac{1}{\alpha-1}[1 - \int_0^t F^\alpha_{X_{(t)}}(x)dx], \tag{33}$$

where $X(t) = [X|X \leq t]$ is the random variable that describes the past lifetime of system at age t and $F_{X(t)} = \frac{F(x)}{F(t)}$ is the distribution function of $X_{(t)}$.

The following theorem shows that the DPCTE characterizes the distribution.

***Theorem* 3.1** — *Let X is a non-negative random variable having the pdf, CDF and reversed hazard rate $f(.)$, $F(.)$, and $\tau(x)$, respectively. Assume that the DPCTE $c\eta_\alpha(X;t) < \infty$, $t \geq 0$, $\alpha > 0 (\neq 1)$. Then $c\eta_\alpha(X;t)$, (whereas $c\eta'_\alpha(X;t)$ exist) uniquely determines the CDF $F(.)$.*

PROOF : From (33), the DPCTE can be written as

$$(\alpha-1)c\eta_\alpha(X;t) = 1 - \frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}, \tag{34}$$

Differentiating (34) with respect to t, we have

$$\begin{aligned}(\alpha-1)c\eta'_\alpha(X;t) &= \alpha\frac{f(t)}{F(t)}\frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)} - 1 \\ &= \alpha\tau(t)(1-(\alpha-1)c\eta_\alpha(X;t)) - 1,\end{aligned} \tag{35}$$

where $\tau(t) = \frac{f(t)}{F(t)}$ is the reversed hazard rate function of random variable X. Consider two CDF's $F_1(t)$ and $F_2(t)$ having DPCTE $c\eta_\alpha(X_1;t)$ and $c\eta_\alpha(X_2;t)$ and reversed hazard rate functions $\tau_1(t)$ and $\tau_2(t)$, respectively. Let $c\eta_\alpha(X_1;t) = c\eta_\alpha(X_2;t)$ which implies that $(\alpha-1)c\eta'_\alpha(X_1;t) = (\alpha-1)c\eta'_\alpha(X_2;t)$, then

$$\tau_1(t)(1-(\alpha-1)c\eta_\alpha(X_1;t)) = \tau_2(t)(1-(\alpha-1)c\eta_\alpha(X_2;t)). \tag{36}$$

Now as $c\eta_\alpha(X_1;t) = c\eta_\alpha(X_2;t)$, therefore, (36) reduces to $\tau_1(t) = \tau_2(t)$ or equivalently $F_1(t) = F_2(t)$.

***Theorem* 3.2** — *Let X be a random variable with support in $[0,b]$, with b finite. For all $t \in [0,b]$ and for $\alpha > 1$ we have*

1. $(\alpha-1)c\eta_\alpha(X;t) = q\tilde{\mu}(t) + 1$ *if and only if* $F(t) = (\frac{t}{b})^{\frac{k}{1-k}}$, $k = \frac{1+q}{1-\alpha}$,
2. $(\alpha-1)c\eta_\alpha(X;t) = q\mu(t) - t + 1$ *if and only if* $F(t) = (\frac{t}{b})^{\frac{1-k}{k}}$, $k = \frac{\alpha(q-1)}{q(\alpha-1)}$,

where q is a constant.

PROOF : 1. Let $(\alpha-1)c\eta_\alpha(X;t)=q\tilde{\mu}(t)+1$ for all $t\in[0,b]$. Differentiating both side with respect to t and from (29) and (35) we have

$$
\begin{aligned}
&(\alpha-1)c\eta'_\alpha(X;t)=q\tilde{\mu}'(t)\\
&\Rightarrow \alpha\tau(t)(1-(\alpha-1)c\eta_\alpha(X;t))-1=q[1-\tau(t)\tilde{\mu}(t)]\\
&\Rightarrow \alpha\tau(t)(1-q\tilde{\mu}(t)-1)-1=q-q\tau(t)\tilde{\mu}(t)\\
&\Rightarrow \tau(t)\tilde{\mu}(t)=\frac{q+1}{1-\alpha}\\
&\Rightarrow \tau(t)\tilde{\mu}(t)=k,
\end{aligned}
$$

where $k=\frac{1+q}{1-\alpha}$, q is a constant. Note that (29) gives

$$\tau(t)=\frac{1-\tilde{\mu}'(t)}{\tilde{\mu}(t)},$$

then we have

$$\tilde{\mu}'(t)=1-k.$$

This differential equation yields $\tilde{\mu}(t)=(1-k)t$, where $\tilde{\mu}(0)=0$. Finally, we obtain

$$\tau(t)=\frac{k}{1-k}\frac{1}{t}.$$

This implies that

$$F(t)=\left(\frac{t}{b}\right)^{\frac{k}{1-k}},0\leq t\leq b.$$

2. Recall that the mean inactivity time can be expressed as

$$\tilde{\mu}(t)=t-\mu(t). \tag{37}$$

Let $(\alpha-1)c\eta_\alpha(X;t)=q\mu(t)-t+1$ for all $t\in[0,b]$. Differentiating both side with respect to t and from (29) and (35) we have

$$
\begin{aligned}
&(\alpha-1)c\eta'_\alpha(X;t)=q\mu'(t)-1\\
&\Rightarrow \alpha\tau(t)(1-(\alpha-1)c\eta_\alpha(X;t))-1=q[1-\tilde{\mu}'(t)]-1\\
&\Rightarrow \alpha\tau(t)(1-q(t-\tilde{\mu}(t))-1)-1=q[1-1+\tau(t)\tilde{\mu}(t)]-1\\
&\Rightarrow \tilde{\mu}(t)=\frac{\alpha(q-1)}{q(\alpha-1)}t\\
&\Rightarrow \tilde{\mu}(t)=kt,
\end{aligned}
$$

where $k=\frac{\alpha(q-1)}{q(\alpha-1)}$, q is a constant. So, we have

$$\tau(t)=\frac{1-k}{k}\frac{1}{t},$$

which implies that

$$F(t)=\left(\frac{t}{b}\right)^{\frac{1-k}{k}}, 0\leq t\leq b.$$

The converse for both (1.) and (2.) is quite straightforward.

The effect of increasing transformation on the DPCTE will be discussed in the following.

Lemma 3.1 — Let $Y=\varphi(X)$ an increasing differentiable function. Then the DPCTE of order α for the random variable Y is given by

$$c\eta_\alpha(Y;t)=\frac{1}{\alpha-1}\left[1-\frac{\int_0^{\varphi^{-1}(t)}F^\alpha(x)\varphi'(x)dx}{F^\alpha(\varphi^{-1}(t))}\right]. \tag{38}$$

Corollary 3.1 — If $Y=aX+b$, with $a>0$ and $b\geq 0$, then

$$\begin{aligned} c\eta_\alpha(Y;t) &= \frac{1}{\alpha-1}\left[1-\frac{\int_0^{\varphi^{-1}(t)}F^\alpha(x)\varphi'(x)dx}{F^\alpha(\varphi^{-1}(t))}\right] \\ &= \frac{1}{\alpha-1}\left[1-a\,\frac{\int_0^{\frac{t-b}{a}}F^\alpha(x)dx}{F^\alpha(\frac{t-b}{a})}\right] \\ &= \frac{1-a}{\alpha-1}+a\;c\eta_\alpha\left(X;\frac{t-b}{a}\right). \end{aligned} \tag{39}$$

***Theorem* 3.3** — *Let X be a random variable with support in $[0,b]$ and symmetric about $\frac{b}{2}$, i.e. $F(x)=\bar{F}(b-x)$, $0\leq x\leq b$. Therefore, from (30) and (33), we have*

$$c\eta_\alpha(X;t)=\phi_\alpha(X;b-t). \tag{40}$$

PROOF : From (30) and (33), we have

$$\begin{aligned} c\eta_\alpha(X;t) &= \frac{1}{\alpha-1}\left[1-\frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}\right] \\ &= \frac{1}{\alpha-1}\left[1-\frac{\int_0^t(1-F(b-x))^\alpha dx}{(1-F(b-t))^\alpha}\right] \\ &= \frac{1}{\alpha-1}\left[1+\frac{\int_b^{b-t}(\bar{F}(y))^\alpha dy}{(\bar{F}(b-t))^\alpha}\right] \\ &= \frac{1}{\alpha-1}\left[1-\frac{\int_{b-t}^{b}(\bar{F}(y))^\alpha dy}{(\bar{F}(b-t))^\alpha}\right] \\ &= \phi_\alpha(X;b-t). \end{aligned}$$

Example 3.1 : If the random variable X has uniform distribution $U[0,b]$, $0 \le t \le b$, we have

$$\begin{aligned} c\eta_\alpha(X;t) &= \frac{1}{\alpha-1}\left[1-\frac{\int_0^t F^\alpha(x)dx}{F^\alpha(t)}\right] \\ &= \frac{1+\alpha-t}{(\alpha+1)(\alpha-1)}, \end{aligned}$$

$$\begin{aligned} \phi_\alpha(X;b-t) &= \frac{1}{\alpha-1}\left[1-\frac{\int_t^b \bar{F}^\alpha(x)dx}{\bar{F}^\alpha(t)}\right] \\ &= \frac{1-b+\alpha+t}{(\alpha+1)(\alpha-1)}, \end{aligned}$$

which is in agreement with the previous theorem.

4. CONCLUSION

We have dealt with CRTE and CTE with its alternative forms and interpreted the relation and difference between them. Moreover, we have mentioned the adequate family and distributions that the CRTE and CTE defined in its support. Besides, developing some properties. In the last part, we have given an extension of CTE called DPCTE which characterize the distribution. Based on the proposed DPCTE, some relations and properties have introduced.

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