Indian J. Pure Appl. Math., **51**(4): 1919-1930, December 2020
 DOI: 10.1007/s13226-020-0504-7

PENTAVALENT 1-TRANSITIVE DIGRAPHS WITH NON-SOLVABLE AUTOMORPHISM GROUPS

Masoumeh Akbarizadeh*, Mehdi Alaeiyan* and Raffaele Scapellato**

Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16844, Iran

***Dipartimento di Matematica, Politecnico di Milano, Milano, Italy*

e-mails: masoumeh.akbarizadeh@gmail.com; alaeiyan@iust.ac.ir

king_raphael@yahoo.it

(*Received* 12 *August* 2018; *accepted* 9 *January* 2020)

A digraph $\overrightarrow{\Gamma}$ is said to be 1-transitive if its automorphism group acts transitively on the 1-arcs but not on the 2-arcs of $\overrightarrow{\Gamma}$. We give a tentatively complete classification of pentavalent strongly connected 1-transitive digraphs of order $2^a p^b q$, where p and q are two distinct odd primes, $a \in \{3,\ldots,8\}, b \in \{1,\ldots,4\}$, whose automorphism groups are non-solvable. It is shown that such digraphs exist if and only if $q = 3$ or 13 and $p \in \{7, 11, 17, 19, 31, 41\}$.

Key words : Non-solvable group; arc transitive; pentavalent digraph; coset digraph.

2010 Mathematics Subject Classification : 05C25, 20B25.

1. Introduction

By a *digraph* $\overrightarrow{\Gamma}$ we shall mean an ordered pair (V, A), where $V = V(\overrightarrow{\Gamma})$ is a finite non-empty set and $A = A(\overrightarrow{\Gamma}) \subseteq V \times V$ is an anti-reflexive binary relation on V. We shall refer to V and A in a graph theoretical fashion as the *vertex-set* and the *arc-set*, and call their elements *vertices* and *arcs* of $\overrightarrow{\Gamma}$, respectively. A digraph $\overrightarrow{\Gamma}$ is called a *graph* when the relation $A(\overrightarrow{\Gamma})$ is symmetric and is said to be *asymmetric* when the relation $A(\overrightarrow{\Gamma})$ is asymmetric. So all digraphs considered in this paper will be *simple* (without multiple edges or loops).

For a positive integer s, an *s-arc* of $\overrightarrow{\Gamma}$ is a sequence $(v_0, \ldots, v_s)$ of $s+1$ vertices of $\overrightarrow{\Gamma}$ such that each three consecutive vertices in the sequence are pairwise distinct and such that (v_{i-1}, v_i) is an arc of $\overrightarrow{\Gamma}$ for every $i \in \{1, \ldots, s\}$. A 1-arc is also simply called an *arc*. An *automorphism* of $\overrightarrow{\Gamma}$ is a permutation of V which preserves the relation A. Let G be a subgroup of the full automorphism group Aut($\overrightarrow{\Gamma}$) of $\overrightarrow{\Gamma}$. We say that $\overrightarrow{\Gamma}$ is *G-vertex-transitive* or *G-arc transitive* provided that G acts transitively on V or A, respectively. Similarly, we say that $\overrightarrow{\Gamma}$ is (G,s)*-arc transitive* if G acts transitively on the set of s-arcs of $\overrightarrow{\Gamma}$. An G-arc transitive digraph is said to be $(G,1)$-transitive if the graph is not $(G,2)$-arc transitive. When $G = Aut(\overrightarrow{\Gamma})$, the prefix G in the above notation is usually omitted.

If v is the tail and u the head of some arc $x \in A$, then we say that u is an *out-neighbour* of v and v an *in-neighbour* of u. Let $v \in V$ and $\Gamma^+(v) = \{x \in A : tail(x) = v\}$ and $\Gamma^-(v) = \{x \in A : head(x) = v\}$, we call the sizes of these two sets the *out-valency* and the *in-valency* of $v \in V$, respectively. A digraph is called *k-valent* if both its out-valency and its in-valency are equal to k.

Classification of arc transitive graphs and digraphs with given valency and with restricted order has received considerable attention in the literature. For the case of pentavalent graphs, some such classifications were presented in [1, 6, 7, 9-11, 16]. Recently, 2-arc transitive two-valent digraphs of certain orders were classified [12]. In this paper, we give a tentatively complete classification of strongly connected $(G,1)$-transitive 5-valent digraphs of order $2^a p^b q$, where p and q are distinct odd primes, $a \in \{3,\dots,8\}, b \in \{1,\dots,4\}$, whose automorphism groups are non-solvable.

2. Constructions as Coset Digraphs

In this section, we shall construct several pentavalent arc transitive digraphs. Let G be a group generated by a core-free subgroup H and an element g with $g^{-1} \notin HgH$. One can construct the coset digraph, denoted $Cos(G,H,g)$, whose vertex-set is the set G/H of right cosets of H in G, and where (Hx,Hy) is an arc if and only if $yx^{-1} \in HgH$. Note that the condition $g^{-1} \notin HgH$ guarantees that the arc-set is an asymmetric relation. Moreover, since $G = < H, g >$, the digraph $Cos(G,H,g)$ is strongly connected. The digraph $Cos(G,H,g)$ is G-arc transitive (with G acting upon G/H by right multiplication) [13].

Obviously that every arc transitive digraph without vertices of out-valency 0 (in particular, every arc transitive digraph) is vertex-transitive. By [3, Proposition 2.1], every vertex-transitive digraphs can be represented as a coset digraph.

Example 2.1 : Let $G = PSL(2,11)$. Then G contains the following elements:

$a = (1\ 7\ 4\ 2\ 9)(3\ 12\ 6\ 10\ 5)$, $\quad g = (1\ 12\ 2\ 4\ 3)(5\ 8\ 11\ 6\ 10)$,

$g_1 = (1\ 3\ 11\ 9\ 2)(3\ 9\ 8\ 6\ 4)$,

$g_2 = (1\ 10\ 2\ 11\ 12)(4\ 12\ 11\ 10\ 6)$,

$g_3 = (1\ 8\ 10)(2\ 7\ 4)(3\ 5\ 12)(6\ 11\ 9)$,

$g_4 = (1\ 3\ 9)(2\ 8\ 10)(4\ 12\ 5)(6\ 11\ 7)$.

Let $H = < a >$. By Magma [2], $G = < H, g > = < H, g_i >$ for each integer $1 \leqslant i \leqslant 4$. Define

$\mathcal{D}_{PSL(2,11)} := Cos(G, H, g)$ and $\mathcal{D}^i_{PSL(2,11)} := Cos(G, H, g_i)$ $(1 \leq i \leq 4)$.

Again by Magma [2], the digraphs $\mathcal{D}^i_{PSL(2,11)}$ $(1 \leq i \leq 4)$ and $\mathcal{D}_{PSL(2,11)}$ are pairwise non-isomorphic strongly connected pentavalent arc transitive of order $2^2 \cdot 3 \cdot 11$ with Aut$(\mathcal{D}_{PSL(2,11)}) = PSL(2, 11)$ and Aut$(\mathcal{D}^i_{PSL(2,11)}) = PGL(2, 11)$, where $1 \leq i \leq 4$.

Example 2.2 : Let $G = PGL(2, 11)$. Then G contains the following elements:

$a = (1\ 5\ 8\ 6\ 9)(2\ 11\ 7\ 3\ 12)$, $g = (1\ 7\ 11\ 8\ 3\ 12\ 4\ 10\ 6\ 9\ 2\ 5)$,

$g_1 = (1\ 11\ 9\ 6\ 8\ 12\ 5\ 10\ 2\ 4)$, $g_2 = (1\ 10\ 7\ 9\ 12\ 5\ 4\ 3\ 8\ 11)$,

$g_3 = (1\ 5\ 10\ 3\ 7\ 9\ 6\ 4\ 2\ 11)$, $g_4 = (1\ 8\ 4\ 2\ 5\ 7\ 9\ 12\ 10\ 6)$.

Let $H = < a >$. By Magma [2], $G = < H, g > = < H, g_i >$ for each integer $1 \leqslant i \leqslant 4$. Define

$\mathcal{D}_{PGL(2,11)} := Cos(G, H, g)$ and $\mathcal{D}^i_{PGL(2,11)} := Cos(G, H, g_i)$, $(1 \leq i \leq 4)$.

Again by Magma [2], the digraphs $\mathcal{D}^i_{PGL(2,11)}$ $(1 \leq i \leq 4)$ and $\mathcal{D}_{PGL(2,11)}$ are pairwise non-isomorphic strongly connected pentavalent arc transitive of order $2^3 \cdot 3 \cdot 11$ with Aut$(\mathcal{D}_{PGL(2,11)}) = PGL(2, 11)$, Aut$(\mathcal{D}^i_{PGL(2,11)}) \cong PGL(2, 11) \times C_2$ where $1 \leq i \leq 3$ and $\mid Aut(\mathcal{D}^4_{PGL(2,11)}) \mid = 4\times \mid PGL(2, 11) \mid$.

Example 2.3 : Let $G = PSL(2, 19)$. Then G contains the following elements:

$a = (1\ 6\ 15\ 3\ 14)(2\ 17\ 12\ 18\ 16)(4\ 7\ 10\ 9\ 5)(8\ 13\ 20\ 11\ 19)$,

$g_1 = (1\ 18\ 20\ 17\ 2\ 8\ 6\ 14\ 15\ 11\ 5\ 9\ 13\ 7\ 3\ 4\ 12\ 10\ 16)$,

$g_2 = (1\ 3\ 10\ 13\ 8\ 17\ 7\ 16\ 11\ 14)(2\ 19\ 15\ 20\ 18\ 12\ 6\ 4\ 9\ 5)$,

$g_3 = (1\ 4\ 8\ 14\ 9\ 10\ 2\ 15\ 5)(6\ 19\ 11\ 12\ 7\ 13\ 17\ 20\ 16)$,

$g_4 = (2\ 15\ 10\ 17\ 4\ 19\ 12\ 11\ 3)(5\ 6\ 20\ 16\ 18\ 14\ 8\ 9\ 7)$,

$g_5 = (2\ 11\ 17\ 13\ 6\ 9\ 20\ 10\ 7)(3\ 18\ 15\ 5\ 16\ 19\ 12\ 8\ 14)$,

$g_6 = (1\ 3\ 2\ 6\ 7\ 16\ 4\ 18\ 15\ 5)(8\ 9\ 13\ 12\ 14\ 10\ 20\ 17\ 11\ 19)$,

$g_7 = (1\ 3\ 9)(2\ 10\ 5)(6\ 20\ 19)(7\ 15\ 12)(8\ 14\ 16)(11\ 18\ 17)$,

$g_8 = (1\ 4\ 20\ 3\ 13\ 19\ 18\ 16\ 9\ 10\ 7\ 2\ 17\ 14\ 15\ 8\ 6\ 5\ 11)$,

$g_9 = (1\ 14\ 3\ 18\ 4\ 9\ 19\ 17\ 16\ 7)(2\ 13\ 12\ 10\ 20\ 5\ 11\ 6\ 15\ 8)$,

$g_{10} = (1\ 7\ 2)(3\ 4\ 14)(5\ 11\ 10)(6\ 17\ 15)(8\ 9\ 18)(12\ 6\ 20)$,

$g_{11} = (2\ 15\ 19\ 9\ 4\ 5\ 10\ 6\ 8\ 14\ 16\ 12\ 17\ 18\ 13\ 3\ 7\ 20\ 11)$,

$g_{12} = (1\ 12\ 14\ 2\ 20)(3\ 5\ 16\ 17\ 15)(4\ 18\ 8\ 7\ 11)(6\ 10\ 9\ 19\ 13)$,

$g_{13} = (1\ 17\ 19\ 3\ 10\ 4\ 13\ 8\ 20)(2\ 11\ 5\ 12\ 16\ 18\ 14\ 15\ 7)$,
$g_{14} = (1\ 18\ 11\ 9\ 2\ 19\ 4\ 15\ 5\ 16)(3\ 7\ 10\ 13\ 17\ 12\ 14\ 20\ 6\ 8)$,
$g_{15} = (1\ 12\ 19\ 8\ 20\ 2\ 14\ 3\ 10)(4\ 5\ 11\ 17\ 18\ 16\ 13\ 9\ 6)$,
$g_{16} = (1\ 5\ 16\ 19\ 4\ 2\ 3\ 20\ 14\ 13)(6\ 10\ 18\ 17\ 11\ 8\ 9\ 7\ 12\ 15)$,
$g_{17} = (1\ 19\ 18\ 11\ 2\ 17\ 5\ 10\ 12\ 9)(3\ 14\ 7\ 6\ 4\ 16\ 13\ 15\ 20\ 8)$.

Let $H = < a >$. By Magma [2], $G =< H, g_i >$ for each $1 \leq i \leq 17$. Define

$\mathcal{D}^{(i)}_{PSL(2,19)} := Cos(G, H, g_i)$ $(1 \leq i \leq 17)$.

By Magma [2], the coset digraphs $\mathcal{D}^{(i)}_{PSL(2,19)}$ $(1 \leq i \leq 17)$, are pairwise non-isomorphic pentavalent arc transitive digraphs of order $2^2 \cdot 3^2 \cdot 19$ and Aut$(\mathcal{D}^{(i)}_{PSL(2,19)}) = PSL(2, 19)$ for $i = 1, \cdots 9$ and Aut$(\mathcal{D}^{(i)}_{PSL(2,19)}) = PGL(2, 19)$ where $10 \leq i \leq 17$.

Example 2.4 : Let $G = PGL(2, 19)$. Then G contains the following elements:
$a = (1\ 15\ 3\ 13\ 14)(2\ 5\ 16\ 9\ 6)(4\ 17\ 8\ 11\ 7)(10\ 18\ 12\ 19\ 20)$,
$g_1 = (1\ 16\ 10\ 18\ 17\ 13\ 5\ 9\ 2\ 15\ 19\ 11\ 7\ 6\ 14\ 8\ 3\ 12)$,
$g_2 = (1\ 10\ 12\ 14\ 3\ 16\ 15\ 19\ 13\ 20\ 6\ 7\ 2\ 17\ 18\ 4\ 11\ 5\ 9\ 8)$,
$g_3 = (1\ 20\ 19\ 5\ 13\ 2\ 16\ 9\ 12\ 17\ 14\ 10\ 6\ 3\ 8\ 11\ 4\ 18\ 7\ 15)$,
$g_4 = (1\ 17\ 3\ 15\ 20\ 7\ 12\ 4\ 10\ 6\ 9\ 11\ 2\ 19\ 13\ 14\ 8\ 5\ 16\ 18)$,
$g_5 = (1\ 2\ 5\ 6\ 19\ 18\ 12\ 17\ 14\ 20\ 4\ 16\ 11\ 3\ 7\ 13\ 10\ 15\ 9\ 8)$,
$g_6 = (1\ 4\ 15\ 11\ 13\ 2\ 9\ 19\ 17\ 5\ 20\ 8\ 6\ 16\ 3\ 12\ 14\ 10)$,
$g_7 = (1\ 13\ 5\ 2\ 8\ 15\ 10\ 6\ 9\ 14\ 3\ 12\ 17\ 20\ 16\ 11\ 18\ 4)$,
$g_8 = (1\ 11\ 12\ 2\ 7\ 3\ 9\ 6\ 5\ 13\ 4\ 19\ 10\ 18\ 17\ 14\ 20\ 16)$,
$g_9 = (1\ 8\ 3\ 15\ 4\ 16\ 11\ 18\ 20\ 6\ 5\ 9\ 17\ 7\ 12\ 2\ 10\ 14\ 13\ 19)$,
$g_{10} = (1\ 8\ 7\ 3\ 18\ 10\ 20\ 6\ 17\ 11\ 5\ 16\ 2\ 12\ 4\ 19\ 15\ 14)$,
$g_{11} = (1\ 5\ 13\ 9\ 16\ 3\ 19\ 7\ 11\ 12\ 4\ 10\ 15\ 6\ 17\ 2\ 8\ 20)$,
$g_{12} = (1\ 2\ 18\ 14\ 15\ 12\ 20\ 3\ 7\ 19\ 6\ 8\ 10\ 17\ 9\ 13\ 16\ 4)$,
$g_{13} = (1\ 15\ 9\ 10\ 7\ 13\ 11\ 6\ 8\ 18\ 14\ 5\ 16\ 12\ 2\ 4\ 19\ 17\ 3\ 20)$,
$g_{14} = (1\ 7\ 4\ 12)(2\ 11\ 19\ 16)(3\ 15\ 9\ 13)(5\ 6\ 17\ 18)(8\ 20\ 10\ 14)$,
$g_{15} = (1\ 4\ 10\ 6\ 8\ 16)(2\ 14\ 7\ 19\ 3\ 18)(5\ 13\ 15\ 11\ 17\ 20)$.

Let $H = < a >$. Define

$\mathcal{D}^{(i)}_{PGL(2,19)} := Cos(G, H, g_i)$, $(1 \leq i \leq 15)$.

By Magma [2], $G = < H, g_i >$ for each $1 \leq i \leq 15$ and the coset digraphs $\mathcal{D}^{(i)}_{PGL(2,19)}$ are pairwise non-isomorphic pentavalent arc transitive of order $2^3 \cdot 3^2 \cdot 19$. Again by Magma [2], Aut$(\mathcal{D}^{(i)}_{PGL(2,19)}) = PGL(2,19)$, $(1 \leq i \leq 8)$ and $\mid Aut(\mathcal{D}^{(i)}_{PGL(2,19)}) \mid = 2\times \mid PGL(2,19) \mid$ for $(9 \leq i \leq 14)$ and $\mid Aut(\mathcal{D}^{(15)}_{PGL(2,19)}) \mid = 4\times \mid PGL(2,19) \mid$.

Example 2.5 : Let $a = (1\ 5\ 12\ 17\ 16)(3\ 13\ 6\ 15\ 11)(4\ 8\ 10\ 9\ 14)$. Then $H = < a >$ is a subgroup of $G = PSL(2, 2^4)$ and G contains the following elements:

$g_1 = (1\ 14\ 17\ 10\ 11)(2\ 3\ 13\ 16\ 12)(4\ 9\ 6\ 15\ 7)$,

$g_2 = (1\ 16\ 9\ 5\ 10\ 12\ 17\ 13\ 6\ 4\ 15\ 14\ 3\ 11\ 2\ 8\ 7)$,

$g_3 = (1\ 5\ 12\ 6\ 2\ 4\ 16\ 9\ 8\ 11\ 10\ 3\ 15\ 17\ 13\ 7\ 14)$,

$g_4 = (1\ 9\ 3\ 15\ 17)(2\ 13\ 4\ 5\ 7)(6\ 11\ 16\ 12\ 10)$,

$g_5 = (1\ 17\ 5\ 3\ 14\ 15\ 9\ 7\ 12\ 11\ 8\ 16\ 2\ 6\ 10\ 13\ 4)$,

$g_6 = (1\ 6\ 12\ 13\ 11\ 8\ 3\ 5\ 7\ 2\ 16\ 14\ 15\ 4\ 9\ 17\ 10)$.

By Magma [2], $G = < H, g_i >$ for each integer $1 \leq i \leq 6$. Define

$\mathcal{D}^{(i)}_{PSL(2,2^4)} := Cos(G, H, g_i)$, $i = 1, 2, \ldots, 6$.

Again by Magma [2], the coset digraphs $\mathcal{D}^{(i)}_{PSL(2,2^4)}$ $(1 \leq i \leq 6)$ are pairwise non-isomorphic pentavalent arc transitive of order $2^4 \cdot 3 \cdot 17$ with Aut$(\mathcal{D}^{(i)}_{PSL(2,2^4)}) = PGL(2, 2^4)$ for $i = 1, 2, 3$ and $\mid Aut(\mathcal{D}^{(4)}_{PSL(2,2^4)}) \mid = \mid Aut(\mathcal{D}^{(5)}_{PSL(2,2^4)}) \mid = 2\times \mid G \mid$ and $\mid Aut(\mathcal{D}^{(6)}_{PSL(2,2^4)}) \mid = 4\times \mid G \mid$.

Example 2.6 : Let $H = < (1\ 7\ 4\ 6\ 5) >$. Then H is a subgroup of $G = A_7$ and G contains the following elements:

$g_1 = (1\ 4\ 7)(2\ 3\ 5)$, $g_2 = (1\ 2\ 6\ 3\ 7\ 4\ 5)$, $g_3 = (1\ 7\ 2\ 6\ 5\ 4\ 3)$,

$g_4 = (1\ 2\ 4\ 6\ 3\ 5\ 7)$, $g_5 = (1\ 5\ 6\ 7\ 2\ 4\ 3)$, $g_6 = (1\ 4\ 5\ 3\ 2\ 6\ 7)$,

$g_7 = (1\ 5\ 3\ 2\ 4\ 7\ 6)$, $g_8 = (2\ 4\ 7)(3\ 6\ 5)$, $g_9 = (1\ 7)(2\ 4\ 5)(3\ 6)$,

$g_{10} = (1\ 4\ 7\ 2)(3\ 5)$, $g_{11} = (1\ 2\ 3\ 6\ 7\ 5\ 4)$, $g_{12} = (1\ 2\ 3\ 7)(4\ 5)$.

By Magma [2], $G = < H, g_i >$ for each integer $1 \leq i \leq 12$. Define

$\mathcal{D}^{(i)}_{A_7} := Cos(G, H, g_i)$, $(1 \leq i \leq 12)$.

Again by Magma [2], $\mathcal{D}^{(i)}_{A_7}$ $(1 \leq i \leq 12)$ are pairwise non-isomorphic pentavalent arc transitive digraphs of order $2^3 \cdot 3^2 \cdot 7$ with Aut$(\mathcal{D}^{(i)}_{A_7}) = A_7$ $(1 \leq i \leq 6)$, Aut$(\mathcal{D}^{(i)}_{A_7}) = S_7 (7 \leq i \leq 10)$ and $|$Aut$(\mathcal{D}^{(i)}_{A_7}) \mid = 4\times \mid A_7 \mid$ for $i = 11, 12$.

Example 2.7 : Let $G = S_7$. Then G contains the following elements:

$a = (2\ 3\ 4\ 5\ 7)$, $g_1 = (1\ 3)(2\ 4\ 5\ 6\ 7)$, $g_2 = (1\ 4\ 2)(6\ 7)$,

$g_3 = (1\ 6\ 5)(2\ 3\ 7\ 4)$, $g_4 = (1\ 4\ 6\ 5\ 7)(2\ 3)$, $g_5 = (1\ 6\ 2\ 3\ 7)(4\ 5)$,

$g_6 = (1\ 3\ 5)(2\ 4\ 7\ 6)$, $g_7 = (1\ 4)(2\ 3\ 7\ 6\ 5)$, $g_8 = (1\ 2\ 3\ 6\ 5\ 4)$,

$g_9 = (1\ 6\ 7)(2\ 4\ 5\ 3), \quad g_{10} = (1\ 6\ 7\ 5)(2\ 3\ 4)$.

Let $H =< a >$. By Magma [2], $G =< H, g_i >$ for each integer $1 \leq i \leq 10$. Define

$\mathcal{D}_{S_7}^{(i)} := Cos(G, H, g_i)$, $(1 \leq i \leq 10)$.

Again by Magma [2], $\mathcal{D}_{S_7}^{(i)}$ $(1 \leq i \leq 10)$ are pairwise non-isomorphic pentavalent arc transitive digraphs of order $2^4 \cdot 3^2 \cdot 7$ and Aut$(\mathcal{D}_{S_7}^{(i)}) = S_7$ $(1 \leq i \leq 5)$, $|$Aut$(\mathcal{D}_{S_7}^{(i)}) \mid = 2\times \mid S_7 \mid$ $(i = 6, 7, 8)$ and $|$Aut$(\mathcal{D}_{S_7}^{(i)}) \mid = 4\times \mid S_7 \mid$ for $i = 9, 10$.

Example 2.8 : Let $G = PSL(2, 31)$. Then G contains the following elements:

$a = (1\ 11\ 8\ 3\ 16)(4\ 12\ 10\ 15\ 13)(5\ 17\ 14\ 7\ 9)$,

$g_1 = (1\ 14\ 17\ 10\ 11)(2\ 3\ 13\ 16\ 12)(4\ 9\ 6\ 15\ 7)$,

$g_2 = (1\ 9\ 3\ 15\ 17)(2\ 13\ 4\ 5\ 7)(6\ 11\ 16\ 12\ 10)$,

$g_3 = (1\ 12\ 9)(2\ 17\ 15)(3\ 11\ 4)(5\ 16\ 8)(6\ 14\ 13)$,

$g_4 = (1\ 7\ 4\ 6\ 2\ 17\ 15\ 11\ 13\ 10\ 16\ 3\ 8\ 9\ 14)$,

$g_5 = (1\ 7\ 2\ 10\ 8\ 9\ 12\ 6\ 3\ 17\ 11\ 14\ 15\ 13\ 4\ 16\ 5)$,

$g_6 = (1\ 6\ 2\ 16\ 7\ 4\ 17\ 5\ 11\ 14\ 8\ 10\ 3\ 13\ 15\ 9\ 12)$.

Let $H =< a >$. Define

$\mathcal{D}_{PSL(2,31)}^{(i)} := Cos(G, H, g_i)$, $(1 \leq i \leq 6)$.

By Magma [2], $G = < H, g_i >$ for each $1 \leq i \leq 6$ and the coset digraphs $\mathcal{D}_{PSL(2,31)}^{(i)}$ are pairwise non-isomorphic pentavalent arc transitive of order $2^5 \cdot 3 \cdot 31$ such that Aut$(\mathcal{D}_{PSL(2,31)}^{(i)}) = PSL(2, 31)$ $(i = 1, 2, 3)$, Aut$(\mathcal{D}_{PSL(2,31)}^{(i)}) \cong PGL(2, 31)$ for $i = 4, 5$ and $|$Aut$(\mathcal{D}_{PSL(2,31)}^{(6)}) \mid = 4\times \mid PSL(2, 31) \mid$.

Example 2.9 : Let $G = M_{11}$. Then G contains the following elements:

$a = (1\ 11\ 6\ 7\ 3)(4\ 8\ 10\ 5\ 9)$,

$g_1 = (1\ 2\ 11\ 10)(3\ 7\ 8\ 9)$,

$g_2 = (2\ 9\ 10\ 3\ 4)(5\ 11\ 8\ 6\ 7)$,

$g_3 = (1\ 2\ 11\ 3\ 9)(5\ 7\ 10\ 8\ 6)$,

$g_4 = (1\ 3\ 6\ 5\ 9\ 8\ 11\ 7\ 4\ 10\ 2)$,

$g_5 = (1\ 10\ 2\ 9)(3\ 7\ 6\ 11)$,

$g_6 = (1\ 3\ 7\ 5\ 6\ 8\ 11\ 10\ 2\ 4\ 9)$,

$g_7 = (1\ 8\ 3\ 9\ 6\ 2\ 10\ 5\ 7\ 4\ 11)$,

$g_8 = (1\ 7\ 10\ 6\ 2\ 11\ 9\ 3)(4\ 8)$,

$g_9 = (1\ 7\ 9\ 11\ 2\ 5\ 3\ 4)(6\ 10)$,

$g_{10} = (1\ 4\ 11\ 7\ 6\ 10\ 3\ 2)(5\ 9)$,

$g_{11} = (1\ 2\ 8\ 7\ 9\ 10\ 6\ 3\ 4\ 11\ 5)$,

$g_{12} = (1\ 6\ 10\ 5\ 7\ 4\ 9\ 11\ 2\ 3\ 8)$,

$g_{13} = (1\ 8\ 3\ 10\ 2\ 7\ 4\ 5)(9\ 11)$,

$g_{14} = (1\ 7\ 6\ 8\ 5\ 4\ 10\ 3\ 9\ 11\ 2)$,

$g_{15} = (1\ 6\ 10\ 5)(2\ 7\ 11\ 9)$,

$g_{16} = (1\ 7\ 9)(2\ 4)(3\ 10\ 11\ 6\ 8\ 5)$,

$g_{17} = (2\ 6)(3\ 7\ 10\ 11\ 4\ 5\ 8\ 9)$,

$g_{18} = (2\ 7\ 10\ 6\ 5\ 4\ 9\ 8)(3\ 11)$,

$g_{19} = (1\ 11\ 6\ 2\ 10\ 8\ 9\ 3)(5\ 7)$,

$g_{20} = (1\ 7\ 5\ 9\ 2\ 11\ 8\ 4\ 10\ 3\ 6)$,
$g_{21} = (1\ 3\ 5\ 2\ 8\ 6)(4\ 7\ 9)(10\ 11)$,
$g_{22} = (1\ 2\ 8\ 9\ 5\ 11)(3\ 7)(4\ 10\ 6)$,
$g_{23} = (1\ 8\ 11\ 4\ 10\ 2\ 7\ 5\ 3\ 9\ 6)$,
$g_{24} = (1\ 9\ 5\ 2\ 8\ 10\ 7\ 3\ 4\ 11\ 6)$,
$g_{25} = (1\ 7\ 10\ 5)(2\ 11\ 8\ 3)$,
$g_{26} = (1\ 9\ 10\ 5\ 4\ 2\ 8\ 6)(7\ 11)$,
$g_{27} = (1\ 11\ 2\ 7\ 5\ 8\ 10\ 9\ 6\ 4\ 3)$,
$g_{28} = (1\ 4\ 7\ 3\ 9\ 6\ 11\ 8)(2\ 10)$,
$g_{29} = (2\ 10\ 9\ 4)(5\ 6\ 7\ 11)$,
$g_{30} = (1\ 2\ 4\ 8\ 10\ 9)(3\ 11\ 6)(5\ 7)$,
$g_{31} = (2\ 6\ 4\ 9)(3\ 10\ 7\ 11)$,
$g_{32} = (1\ 2\ 8\ 6\ 11\ 4)(3\ 7\ 9)(5\ 10)$,
$g_{33} = (1\ 8\ 3\ 2\ 10\ 9\ 5\ 4)(7\ 11)$,
$g_{34} = (1\ 4\ 8\ 3\ 5\ 10\ 6\ 2)(7\ 11)$,
$g_{35} = (2\ 11\ 6\ 9)(3\ 4\ 8\ 10)$,
$g_{36} = (1\ 9\ 6\ 11\ 2)(4\ 10\ 5\ 8\ 7)$,
$g_{37} = (1\ 10\ 4\ 3\ 8)(2\ 9\ 7\ 5\ 11)$,
$g_{38} = (1\ 6\ 11\ 4\ 7\ 5\ 2\ 8\ 3\ 9\ 10)$,
$g_{39} = (1\ 4\ 3\ 6\ 10\ 11)(2\ 7)(5\ 8\ 9)$,
$g_{40} = (1\ 6\ 4\ 2\ 3)(5\ 10\ 9\ 8\ 11)$,
$g_{41} = (1\ 5\ 9\ 7)(2\ 10\ 3\ 6)$,
$g_{42} = (1\ 2\ 8)(3\ 6\ 11\ 9\ 10\ 7)(4\ 5)$,
$g_{43} = (1\ 4\ 3)(2\ 11\ 8\ 6\ 7\ 10)(5\ 9)$,
$g_{44} = (1\ 2\ 10\ 4\ 5\ 7\ 3\ 9\ 8\ 11\ 6)$,
$g_{45} = (1\ 5\ 10\ 3\ 7\ 6\ 9\ 4)(2\ 11)$,
$g_{46} = (1\ 11\ 5\ 9\ 3)(2\ 7\ 8\ 4\ 10)$,
$g_{47} = (1\ 3\ 7\ 4\ 9\ 2\ 10\ 6\ 5\ 11\ 8)$,
$g_{48} = (1\ 3\ 8\ 11\ 9\ 7)(2\ 10\ 4)(5\ 6)$,
$g_{49} = (1\ 8\ 5)(2\ 7\ 6\ 4\ 3\ 9)(10\ 11)$,
$g_{50} = (1\ 5)(2\ 3\ 10\ 7\ 9\ 8)(4\ 11\ 6)$,
$g_{51} = (1\ 11\ 7\ 2\ 9\ 8\ 3\ 5)(4\ 10)$,
$g_{52} = (1\ 3\ 9)(2\ 5\ 11\ 8\ 4\ 10)(6\ 7)$,
$g_{53} = (1\ 4)(2\ 6\ 11\ 10\ 9\ 8\ 3\ 5)$,
$g_{54} = (1\ 4\ 9\ 11\ 8)(2\ 5\ 3\ 6\ 7)$,
$g_{55} = (1\ 11\ 7\ 10\ 6)(2\ 8\ 3\ 4\ 9)$,
$g_{56} = (1\ 11\ 3\ 5\ 4\ 9\ 2\ 10)(7\ 8)$,
$g_{57} = (1\ 6\ 9)(2\ 10\ 3)(4\ 7\ 11)$,
$g_{58} = (1\ 10)(2\ 6\ 5\ 8\ 3\ 11\ 4\ 9)$,
$g_{59} = (1\ 10\ 6\ 5\ 11\ 4\ 3\ 9)(2\ 7)$,
$g_{60} = (1\ 5\ 3\ 9\ 11\ 10\ 6\ 2\ 7\ 4\ 8)$,
$g_{61} = (1\ 4)(2\ 3\ 5\ 7\ 8\ 11)(6\ 9\ 10)$,
$g_{62} = (1\ 2\ 3\ 8\ 7\ 4)(5\ 10\ 11)(6\ 9)$,
$g_{63} = (1\ 5\ 11\ 7\ 2\ 8\ 9\ 10)(3\ 4)$.

Let $H = < a >$. Define

$\mathcal{D}_{M_{11}}^{(i)} := Cos(G, H, g_i)$ ($1 \leq i \leq 63$).

By Magma [2], $G = < H, g_i >$ for each $1 \leq i \leq 63$ and the coset digraphs $\mathcal{D}_{M_{11}}^{(i)}$ are pairwise non-isomorphic pentavalent arc transitive of order $2^4 \cdot 3^2 \cdot 11$ such that Aut($\mathcal{D}_{M_{11}}^{(i)}$) $= M_{11}$ ($1 \leq i \leq 52$), |Aut($\mathcal{D}_{M_{11}}^{(i)}$) $| = 2\times \mid M_{11} \mid$ for $53 \leq i \leq 58$, |Aut($\mathcal{D}_{M_{11}}^{(i)}$) $\mid = 4\times \mid M_{11} \mid$ where $i = 59, 60, 61, 62$ and |Aut($\mathcal{D}_{M_{11}}^{63}$) $\mid = 12\times \mid M_{11} \mid$.

Example 2.10 : Let $G = PGL(2, 31)$. Then G contains the following elements:

$a = (1\ 10\ 23\ 20\ 19)(2\ 27\ 13\ 32\ 7)(4\ 29\ 24\ 31\ 18)(5\ 16\ 22\ 25\ 17)$
$(6\ 9\ 15\ 26\ 14)(8\ 21\ 30\ 12\ 11)$,

$g = (1\ 28\ 8\ 11\ 5\ 27\ 9\ 32\ 10\ 17)(2\ 29\ 13\ 20\ 30\ 21\ 3\ 25\ 19\ 22)$

$(4\ 23\ 12\ 16\ 15\ 14\ 18\ 7\ 26\ 31)$.

Let $H = < a >$. By Magma [2], the coset digraph $\mathcal{D}_{PGL(2,31)} := Cos(G, H, g)$ is pentavalent arc transitive of order $2^6 \cdot 3 \cdot 31$ and Aut$(\mathcal{D}_{PGL(2,31)}) = PGL(2, 31)$.

Example 2.11 : Let $G = A_8$. Then $H = < (2\ \ 3\ \ 8\ \ 6\ \ 7) >$ is a subgroup of G. Take $g = (1\ 6\ 8)(2\ 4)(3\ 5) \in G$ and define $\mathcal{D}_{A_8} := Cos(G, H, g)$. By Magma [2], $< H, g > = G$ and $\mathcal{D}_{A_8}$ is a pentavalent arc transitive digraph of order $2^6 \cdot 3^2 \cdot 7$ with Aut$(\mathcal{D}_{A_8}) = A_8$.

Example 2.12 : Let $G = S_8$. Then $H = < (4\ \ 6\ \ 5\ \ 7\ \ 8) >$ is a subgroup of G. Take $g = (1\ 8\ 2\ 5\ 3\ 4) \in G$ and define $\mathcal{D}_{S_8} := Cos(G, H, g)$. By Magma [2], $< H, g > = G$ and $\mathcal{D}_{S_8}$ is a pentavalent arc transitive digraph of order $2^7 \cdot 3^2 \cdot 7$ with Aut$(\mathcal{D}_{S_8}) = S_8$.

Example 2.13 : Let $G = A_9$. Then $H = < (1\ \ 9\ \ 6\ \ 3\ \ 5) >$ is a subgroup of G. Take $g = (1\ 6\ 4)(2\ 8\ 5\ 9\ 7) \in G$ and define $\mathcal{D}_{A_9} := Cos(G, H, g)$. By Magma [2], $< H, g > = G$ and $\mathcal{D}_{A_9}$ is a pentavalent arc transitive digraph of order $2^6 \cdot 3^4 \cdot 7$ with Aut$(\mathcal{D}_{A_9}) = A_9$.

Example 2.14 : Let $G = PSL(3, 4)$. Then $H = < (1\ \ 4\ \ 12\ \ 18\ \ 7)(3\ \ 8\ \ 5\ \ 13\ \ 19)(6\ \ 21\ \ 15\ \ 10\ \ 14)(9\ \ 20\ \ 16\ \ 11\ \ 17) >$ is a subgroup of G. Take $g = (1\ \ 10\ \ 7\ \ 6\ \ 20)(2\ \ 18\ \ 4\ \ 14\ \ 11)(3\ \ 8\ \ 16\ \ 5\ \ 21)(9\ \ 15\ \ 12\ \ 19\ \ 13) \in G$ and define $\mathcal{D}_{PSL(3,4)} := Cos(G, H, g)$. By Magma [2], $< H, g > = G$ and $\mathcal{D}_{PSL(3,4)}$ is a pentavalent arc transitive digraph of order $2^6 \cdot 3^2 \cdot 7$ with Aut$(\mathcal{D}_{PSL(3,4)}) = PSL(3, 4)$.

Example 2.15 : Let $G = PGL(3, 4)$. Then $H = < (1\ \ 8\ \ 12\ \ 9\ \ 3)(2\ \ 10\ \ 11\ \ 18\ \ 20)(4\ \ 14\ \ 13\ \ 5\ \ 19)(6\ \ 7\ \ 21\ \ 17\ \ 16) >$ is a subgroup of G. Take $g = (1\ \ 9\ \ 21\ \ 13\ \ 18\ \ 7\ \ 8\ \ 6\ \ 3\ \ 19\ \ 15\ \ 11\ \ 14\ \ 2\ \ 17\ \ 16\ \ 4\ \ 10\ \ 12\ \ 5\ \ 20) \in G$ and define $\mathcal{D}_{PGL(3,4)} := Cos(G, H, g)$. By Magma [2], $< H, g > = G$ and $\mathcal{D}_{PGL(3,4)}$ is a pentavalent arc transitive digraph of order $2^6 \cdot 3^3 \cdot 7$ with Aut$(\mathcal{D}_{PGL(3,4)}) = PGL(3, 4)$.

Example 2.16 : Let $G = M_{12}$. Then $H = < (1\ \ 12\ \ 7\ \ 6\ \ 10)(2\ \ 8\ \ 9\ \ 3\ \ 4) >$ is a subgroup of G. Take $g = (1\ \ 10\ \ 7\ \ 11\ \ 3\ \ 5)(2\ \ 8\ \ 4)(9\ \ 12) \in G$ and define $\mathcal{D}_{M_{12}} := Cos(G, H, g)$. By Magma [2], $< H, g > = G$ and $\mathcal{D}_{M_{12}}$ is a pentavalent arc transitive digraph of order $2^6 \cdot 3^3 \cdot 11$ with Aut$(\mathcal{D}_{M_{12}}) = M_{12}$.

Example 2.17 : Let $G = PSL(2, 3^4)$. Then

$H = < (1\ \ 31\ \ 6\ \ 15\ \ 75)(2\ \ 77\ \ 53\ \ 23\ \ 25)(3\ \ 80\ \ 48\ \ 37\ \ 74)(4\ \ 50\ \ 33\ \ 64\ \ 14)$

$(5\ \ 18\ \ 28\ \ 27\ \ 69)(7\ \ 78\ \ 46\ \ 41\ \ 52)(8\ \ 59\ \ 82\ \ 70\ \ 35)(9\ \ 56\ \ 44\ \ 67\ \ 36)$

$(10\ \ 24\ \ 20\ \ 34\ \ 63)(11\ \ 76\ \ 40\ \ 30\ \ 62)(12\ \ 79\ \ 58\ \ 61\ \ 66)(13\ \ 43\ \ 51\ \ 29\ \ 38)$

$(16\ \ 26\ \ 39\ \ 57\ \ 17)(19\ \ 21\ \ 73\ \ 49\ \ 42)(22\ \ 71\ \ 72\ \ 54\ \ 55)(32\ \ 60\ \ 65\ \ 68\ \ 47) >$

is a subgroup of G. Take

$$g = (1\ 43\ 20\ 45\ 25\ 21\ 17\ 79\ 22\ 81\ 41\ 9\ 31\ 69\ 35\ 2\ 10\ 82\ 74\ 61\ 5\ 78\ 60\ 34\ 27\ 62\ 15\ 8\ 64\ 46\ 37\ 63\ 50\ 42\ 32\ 40\ 7\ 55\ 11\ 33)(3\ 19\ 26\ 71\ 57\ 72\ 29\ 24\ 56\ 4\ 80\ 44\ 38\ 68\ 18\ 13\ 52\ 67\ 53\ 16\ 23\ 39\ 58\ 47\ 49\ 54\ 6\ 76\ 51\ 65\ 12\ 30\ 59\ 73\ 48\ 36\ 70\ 75\ 77\ 66) \in G.$$

Define $\mathcal{D}_{M_{12}} := Cos(G, H, g)$. By Magma [2], $< H, g > = G$ and $\mathcal{D}_{M_{12}}$ is a pentavalent arc transitive digraph of order $2^6 \cdot 3^3 \cdot 11$ with Aut$(\mathcal{D}_{M_{12}}) = M_{12}$.

Example 2.18 : Let $G = Suz(8)$. Then G contains the following elements:

$$a = \begin{bmatrix} 1 & x^5 & 0 & x^2 \\ x^2 & 0 & x^6 & 0 \\ 0 & x & 0 & 0 \\ x^5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} x & x^3 & x^4 & x^6 \\ 0 & x^3 & x^4 & 0 \\ x^6 & x & x & x^3 \\ 1 & x^4 & x^5 & x \end{bmatrix}.$$

Let $H = \ < a >$. By Magma [2], we have $G = \ < H, g >$ and $\mathcal{D}_{Suz(8)} := Cos(G, H, g)$ is a pentavalent arc transitive digraph of order $2^6 \cdot 7 \cdot 13$ and Aut$(\mathcal{D}_{Suz(8)}) = Suz(8)$.

3. Main Results

In this section, we assume that $\overrightarrow{\Gamma}$ is a strongly connected $(G, 1)$-transitive 5-valent digraph of order $2^a p^b q$, where p and q are distinct odd primes, $a \in \{3, \ldots, 8\}, b \in \{1, \ldots, 4\}$. Thus $\mid G_v \mid = 5$ for every vertex $v \in V$, by [14, Theorem 1.2]. Then $\mid G \mid = 2^a \cdot 5 \cdot p^b \cdot q$. We also assume that G is non-solvable. The following result shows that G has no non-trivial solvable normal minimal subgroups.

Lemma 3.1 — Let $\overrightarrow{\Gamma}$ is a strongly connected G-arc transitive 5-valent digraph and N be a solvable non-trivial normal minimal subgroup of G. Then N is an elementary abelian 5-group and $Aut(\overrightarrow{\Gamma}) = S_5$ wr $\mathbb{Z}_r$ for some $r \geq 3$.

PROOF : Let N be a solvable non-trivial normal minimal subgroup of G. By [5, Corollary 4.3A] N is an elementary abelian p-group for some prime p. Since G is arc transitive, G_v is primitive on $\Gamma^+(v)$, for each vertex $v \in V$. So $\mid N_v \mid = 1$ or $5 \mid \mid N_v \mid$. In the former case, as $\overrightarrow{\Gamma}$ is strongly connected, we get $N = 1$, which contradicts our assumption. In the latter case, N is a elementary abelian 5-group. So by [15, Theorem 2.9], $\overrightarrow{\Gamma} = C_r(5, t)$, where $r = \mid G : G^* \mid \geq 3$, and $1 \leq t \leq r - 1$. Therefore by [15, Theorem 2.8], Aut$(\overrightarrow{\Gamma}) = S_5$ wr $\mathbb{Z}_r$.

By [15, Theorem 2.8], the digraph $C_r(5, t-1)$ is pentavalent and G-arc transitive but not $(G, 2)$-arc transitive, where $G = S_5$ wr $\mathbb{Z}_r$. So $\mid G \mid = 2^{3r} \cdot 3^r \cdot 5^r \cdot r$. This conclude that G contains no

Table 1: Non-abelian simple $\{2,p,q,5\}$-groups

T	Order	G
PSL$(2,2^4)$	$2^4\cdot 3\cdot 5\cdot 17$	PSL$(2,2^4)$ - PSL$(2,2^4)\rtimes\mathbb{Z}_2$ - PSL$(2,2^4)\rtimes\mathbb{Z}_4$
PSL$(2,3^4)$	$2^4\cdot 3^4\cdot 5\cdot 41$	PSL$(2,3^4)$ - PSL$(2,3^4)\rtimes K$ $\ (K\leqslant\mathbb{Z}_4\times\mathbb{Z}_2)$
PSL$(2,11)$	$2^2\cdot 3\cdot 5\cdot 11$	PSL$(2,11)$- PGL$(2,11)$
PSL$(2,19)$	$2^2\cdot 3^2\cdot 5\cdot 19$	PSL$(2,19)$ - PGL$(2,19)$
PSL$(2,31)$	$2^5\cdot 3\cdot 5\cdot 31$	PSL$(2,31)$ - PGL$(2,31)$
A_7	$2^3\cdot 3^2\cdot 5\cdot 7$	A_7- S_7
A_8	$2^6\cdot 3^2\cdot 5\cdot 7$	A_8 - S_8
A_9	$2^6\cdot 3^4\cdot 5\cdot 7$	A_9 - S_9
PSL$(3,4)$	$2^6\cdot 3^2\cdot 5\cdot 7$	PSL$(3,4)$ - PSL$(3,4)\rtimes K$ $\ (K\leqslant D_{12})$
M_{11}	$2^4\cdot 3^2\cdot 5\cdot 11$	M_{11}
M_{12}	$2^6\cdot 3^3\cdot 5\cdot 11$	M_{12} - $\mathrm{M}_{12}\rtimes\mathbb{Z}_2$
Suz(8)	$2^6\cdot 5\cdot 7\cdot 13$	Suz(8)

solvable non-trivial normal minimal subgroup. We are now going to prove the main result.

***Theorem* 3.2** — *Let $\overrightarrow{\Gamma}$ be a strongly connected $(G,1)$-transitive 5-valent digraph of order 2^ap^bq, where p and q are distinct odd primes, $a\in\{3,\dots,8\}$, $b\in\{1,\dots,4\}$, and G is non-solvable. Then G is described in Table 1, and $q=3$ or 13 and $p\in\{7,11,17,19,31,41\}$.*

PROOF : Let T be the group generated by all minimal normal subgroups of G. So $T=Soc(G)$. Since G contains no non-trivial abelian normal subgroups, by [5, Theorem 4.3A], T is a direct product of non-abelian simple groups. Since $\mid G\mid = 2^a\cdot 5\cdot p^b\cdot q$, and since the order of every non-abelian simple group is divisible by at least three distinct primes, T is a non-abelian simple normal subgroup of G. Set $C:=C_G(T)$. So we have $\dfrac{N_G(T)}{C_G(T)}=\dfrac{G}{C}\leqslant Aut(T)$. Since $C=1$, by [5, Corollary 4. 3A], we have $T\leqslant G\leqslant Aut(T)$. Therefore by [4, 8], we can list all groups T and G in Table 1.

Since $\mid G_v\mid=5$ for each $v\in V$, we have $H:=G_v\simeq\mathbb{Z}_5$. By Magma [2], each G is determined in Table 1, has one conjugacy class of subgroups isomorphic to $\mathbb{Z}_5$ and H is Sylow 5-subgroup of G. Therefore $\overrightarrow{\Gamma}$ can be represented as a coset digraph $Cos(G,H,g)$, for some $g\in G\setminus H$ such that $g^{-1}\notin HgH$ and $G=<H,g>$.

Assume $G\cong PSL(2,11), PGL(2,11), PSL(2,19), PGL(2,19), PSL(2,2^4), A_7,$
$S_7, PSL(2,31), M_{11}$. So by Magma [2] (see the Magma code in Appendix), $\overrightarrow{\Gamma}$ is isomorphic to one of the digraphs in Examples 2.1-2.18.

The coset digraphs corresponding to the rest of groups G in Table 1 can be obtained by using the Magma code in Appendix. (The available version of Magma does not allow us to compute all the digraphs corresponding to the rest of groups G in Table 1). This concludes the proof of Theorem 3.2. □

APPENDIX

MAGMA CODE

```
/*
Input : a finite group G and a positive integer n (G is one of the groups in Table 1, and n = 5)
Output : all pentavalent arc transitive coset digraphs with automorphism group G
*/
Graph:= function(G, n);
graph := [ ];
i := 0;
P := SylowSubgroup(G, n);
l := {a : a in G | sub< G|a, P > eq G};
L := {a : ainl |  a^-1notin {h * a * k : h, k in P}};
T := Transversal(G, P);
TH := CosetTable(G, P);
V := {t : t in T};
for x in L do
    D := {h * x * k : h, k in P};
    E := {[TH(1, b), TH(1, d * b)] : b in T, d in D};
    X := Digraph< V | E >;
    A:=AutomorphismGroup(X);
    if (not exists{r : r in graph | IsIsomorphic(X, r) eq true}) then
    Append ( ~graph, X);
  i := i + 1;
 end if;
end for;
return i, graph;
end function;
```

REFERENCES

1. M. Alaeiyan and M. Akbarizadeh. Classification of the pentavalent symmetric graphs of order 18p, *Indian Journal of Pure and Applied Mathematics*, **50**(2) (2019), 485-497.
2. W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system I: The user language, *Journal of Symbolic Computation*, **24**(3-4) (1997), 235-265.
3. Y. Q. Chen, On hamiltonicity of vertex-transitive graphs and digraphs of order p_4, *Journal of Combinatorial Theory Series B*, **72**(1) (1998), 110-121.
4. J. H. Conway, R. T. Curtis, and S. P. Norton, *Atlas of finite groups: Maximal subgroups and ordinary characters for simple groups*.
5. J. D. Dixon and B. Mortimer, *Permutation groups*, vol. **163**, Springer Science & Business Media, 1996.
6. S.-T. Guo, J.-X. Zhou, and Y.-Q. Feng, Pentavalent symmetric graphs of order 12p, *The Electronic Journal of Combinatorics* **18**(1) (2011), P. 233.
7. X.-H. Hua, Y.-Q. Feng, and J. Lee, Pentavalent symmetric graphs of order 2pq, *Discrete Mathematics*, **311**(20) (2011), 2259-2267.
8. B. Huppert and W. Lempken, *Simple groups of order divisible by at most four primes*, IEM, 2000.
9. B. Ling, Classifying pentavalent symmetric graphs of order 24p, *Bulletin of the Iranian Mathematical Society*, **43**(6) (2017), 1855-1866.
10. B. Ling, C. X., Wu, and B. G. Lou, Pentavalent symmetric graphs of order 30p, *Bulletin of the Australian Mathematical Society*, **90**(3) (2014), 353-362.
11. J. Pan, B. Lou, and C. Liu, Arc-transitive pentavalent graphs of order 4pq, *The Electronic Journal of Combinatorics*, **20**(1) (2013), P. 36.
12. P. Potocnik, and K. Bercic, Two-arc-transitive two-valent digraphs of certain orders, *Ars Mathematica Contemporanea*, **11**(1) (2015), 127-146.
13. P. Potocnik, P. Spiga, and G. Verret, *A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two*, arXiv preprint arXiv:1310.6543 (2013).
14. P. Potocnik and G. Verret, On the vertex-stabiliser in arc-transitive digraphs, *Journal of Combinatorial Theory*, Series B **100**(6) (2010), 497-509.
15. C. E. Praeger, Highly arc transitive digraphs, *European Journal of Combinatorics*, **10**(3) (1989), 281-292.
16. D.-W. Yang, Y.-Q. Feng, and J.-L. Du, Pentavalent symmetric graphs of order 2pqr, *Discrete Mathematics*, **339**(2) (2016), 522-532.