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AN UPPER BOUND ON THE DIAMETER OF A 3-EDGE-CONNECTED C_4-FREE GRAPH

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We give an upper bound on the diameter of a 3-edge-connected C_4-free graph in terms of order. In particular we show that if G is a 3-edge-connected C_4-free graph of order n, and diameter d, then the inequality

$$d \leq \frac{3n}{7} - \frac{3}{7}$$

holds. Moreover, graphs are constructed to show that the bound is almost asymptotically sharp.

Key words : Diameter, edge-connectivity.

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1. INTRODUCTION

Let $G = (V, E)$ be a finite, connected, undirected graph with vertex set V and edge set E. The *distance* $d_G(u, v)$ between two vertices u, v of G is the length of a shortest u-v path in G. The *eccentricity* $\mathrm{ec}(v)$ of a vertex $v \in V$ is the maximum distance between v and any other vertex in G. The value of the maximum eccentricity of the vertices of G is called the diameter of G denoted by $\mathrm{diam}(G)$. The degree $\deg(v)$ of a vertex v of G is the number of edges incident with v. The minimum degree $\delta(G) = \delta$ of G is the minimum of the degrees of vertices in G. We denote by $E(V_1, V_2)$, the set $\{e = xy | x \in V_1, y \in V_2\}$ of edges with one end in V_1 and the other end in V_2. The

[1]The results in this paper are part of the first author's MPhilSc thesis.

edge-connectivity $\lambda(G) = \lambda$ of G is the minimum number of edges whose deletion from G results in a disconnected or trivial graph. A graph is triangle-free if it does not contain C_3 as a subgraph and C_4-free if it does not contain C_4 as a subgraph. For notions not defined here we refer the reader to [1].

In this paper we are concerned with upper bounds on diameter in terms of order and edge-connectivity for C_4-free graphs. Several upper bounds on diameter in terms of other graph parameters are known and below we list a few relevant results. For example an obvious bound is $1 \leq \mathrm{diam}(G) \leq n-1$. Erdős *et al.* [8] answered a question that had seemed, for the better part of three decades, to elude fellow mathematicians. They found out that if G is a connected graph of order n and minimum degree $\delta \geq 2$, then

$$\mathrm{diam}(G) \leq \frac{3n}{\delta+1} - 1$$

and also constructed graphs that, apart from the additive constant, attain the bound. They went further in the very same paper and investigated triangle-free and C_4-free graphs proving that if G is a connected triangle-free graph of order n and minimum degree $\delta \geq 2$, then

$$\mathrm{diam}(G) \leq \frac{n-\delta-1}{2\delta} + 12$$

and that if G is a connected C_4-free graph of order n and minimum degree $\delta \geq 2$, then

$$\mathrm{diam}(G) \leq \frac{5n}{\delta^2 - 2[\delta/2] + 1}.$$

Mukwembi, in [18], showed that if G is a λ-edge-connected graph then

$$\mathrm{diam}(G) \leq \frac{3n}{\lambda+1} - 1$$

and constructed graphs that apart from the additive constant attain the bound. The main theorem of this paper provides an asymptotically sharp upper bound on the diameter of a 3-edge-connected C_4-free graph of prescribed order.

2. RESULTS

Let G be a 3-edge-connected C_4-free graph of order n and diameter $d > 1$. Let $x, y \in V(G)$ be fixed vertices in G such that $d(x,y) = \mathrm{diam}(G) = d$ and let $N_i = \{v \in V(G) | d_G(x,v) = i\}$ for any $0 \leq i \leq d$ and $k_i = |N_i|$. The following observation by Mukwembi [18] is essential to this paper.

Observation : Let G be a λ-edge-connected graph, $V_1, V_2 \subset V(G)$ with $V_1 \cap V_2 = \emptyset$. Clearly $|E(V_1,V_2)| \leq |V_1||V_2|$. If $E(V_1,V_2)$ is a disconnecting set of G, then $|E(V_1,V_2)| \geq \lambda(G)$ so that $|V_1||V_2| \geq \lambda(G)$. Let $v \in V(G)$. Note also that for $i = 0, 1, ..., ec_G(v) - 1$, it is clear that $E(N_i, N_{i+1})$ is a disconnecting set of G.

The following fact follows from the above observation:

Fact 1 : $k_i k_{i+1} \geq \lambda$ for all $i = 1, 2, ..., ec_G(v)$.

For the sake of clarity, we remind the reader that for the purposes of this paper $\lambda = \lambda(G) = 3$. The following useful fact follows from the well known inequality $ab \leq (\frac{a+b}{2})^2$. That is to say, the geometric mean of two (positive) real numbers never exceeds their arithmetic mean.

Fact 2 : For positive integers a and b, if $ab \geq 3$, then $a + b \geq 4$.

Lemma 1 — For each $i \in \{0, 1, ..., d-1\}$, $k_i + k_{i+1} \geq 4$.

PROOF : Note that since G is 3-edge-connected, we have $k_i k_{i+1} \geq 3$ from Fact 1. From this and Fact 2 we have $k_i + k_{i+1} \geq 4$ settling our lemma. □

Lemma 2 — For each $i \in \{1, 2, ..., d-1\}$, if $k_i = 1$, then $k_{i-1} + k_i + k_{i+1} \geq 7$.

PROOF : Note that $k_{i-1} + k_i \geq 4$ and $k_i + k_{i+1} \geq 4$ as a consequence of Lemma 1. Since $k_i = 1$, we obtain $k_{i-1} \geq 4 - k_i = 4 - 1 = 3$ and $k_{i+1} \geq 4 - k_i = 4 - 1 = 3$ which yields $k_{i-1} + k_i + k_{i+1} \geq 3 + 1 + 3 = 7$ as desired. □

Lemma 3 — For each $i \in \{1, 2, ..., d-1\}$, if $k_i = 2$, then $k_{i-1} + k_i + k_{i+1} \geq 7$.

PROOF : Note that $k_{i-1} + k_i \geq 4$ and $k_i + k_{i+1} \geq 4$ as a consequence of Lemma 1. Since $k_i = 2$, we obtain $k_{i-1} \geq 4 - k_i = 4 - 2 = 2$ and $k_{i+1} \geq 4 - k_i = 4 - 2 = 2$ which yields $k_{i-1} + k_i + k_{i+1} \geq 2 + 2 + 2 = 6$. Now, to prove the current lemma, it remains to show that with the stated conditions, $k_{i-1} + k_i + k_{i+1} \neq 6$. Suppose to the contrary that $k_{i-1} + k_i + k_{i+1} = 6$. We have already shown that $k_{i-1} \geq 2$ and $k_{i+1} \geq 2$ hence our supposition only holds true when $k_{i-1}, k_{i+1} = 2$. Observe that when a graph is forbidden from having a C_4 cycle as its subgraph (this is the case in the current paper), it immediately becomes illegal for two vertices to share more than one neighbour. Now, since G is 3-edge connected, $|E(N_{i-1}, N_i)| \geq 3$ and $|E(N_i, N_{i+1})| \geq 3$. From this and since there are exactly two vertices in each of N_{i-1}, N_i and N_{i+1}, we observe that the two vertices in N_i necessarily share at least a neighbour in N_{i-1} and similarly, the same two vertices in N_i unavoidably share a neighbour in N_{i+1}. This is, as stated before, illegal and immediately proves that under the conditions of the lemma $k_{i-1} + k_i + k_{i+1} \neq 6$ thereby settling our lemma. □

Lemma 4 — For $i \in \{0, 1, ..., d-2\}$, if $k_i = 1$ then $k_{i+2} \geq k_{i+1}$.

PROOF : Let a be the only vertex in N_i. Observe that every vertex in N_{i+1} is necessarily adjacent to a since a is the only vertex in N_i. From Whitney's inequality, we have $\delta(G) \geq \lambda(G) \geq 3$, hence every vertex in N_{i+1} has at least two other neighbours besides a. Further, since G is C_4-free, a vertex in N_{i+1} can only be adjacent to one other vertex in the same set. Hence, all the vertices in N_{i+1} are adjacent to at least one other vertex in N_{i+2}. Note that all the neighbours of vertices in N_{i+1} that are in N_{i+2} are distinct, otherwise a C_4-cycle is formed, which is illegal. Thus N_{i+2} has at least as many vertices as N_{i+1} and $k_{i+2} \geq k_{i+1}$ as desired. □

Lemma 5 — For each $i \in \{1, 2, ..., d-1\}$, if $k_i \geq 3$, then $k_{i-1} + k_i + k_{i+1} \geq 7$.

PROOF : If $k_{i-1} = 1$, then, by the condition of the lemma and by Lemma 4, $k_{i-1} + k_i + k_{i+1} \geq 1 + 3 + 3 = 7$ as desired. So suppose $k_{i-1} \geq 2$. If $k_{i+1} \geq 2$, then, by our supposition, $k_{i-1} + k_i + k_{i+1} \geq 2 + 3 + 2 = 7$ as desired, so suppose now that $k_{i+1} = 1$. If $k_i \geq 4$, then $k_{i-1} + k_i + k_{i+1} \geq 2 + 4 + 1 = 7$ as desired. Hence suppose that $k_i = 3$. Thus it remains to prove that if $k_i = 3$ and $k_{i+1} = 1$, then $k_{i-1} + k_i + k_{i+1} \geq 7$. Note that since G is 3-edge-connected, every vertex in N_i is necessarily adjacent to the single vertex in N_{i+1}, say b. Now, since $\delta(G) \geq 3$, every vertex in N_i has at least two other neighbours besides b. Further, since G is C_4-free, a vertex in N_i can only be adjacent to one other vertex in the same set. Hence, all the vertices in N_i are adjacent to at least one other vertex in N_{i-1}. Note that all the neighbours of vertices in N_i that are in N_{i-1} are distinct, otherwise a C_4-cycle is formed, which is illegal. Thus N_{i-1} has at least as many vertices as N_i and $k_{i-1} \geq k_i = 3$ which yields $k_{i-1} + k_i + k_{i+1} \geq 3 + 3 + 1 = 7$ as desired. □

Lemma 6 — For each $i \in \{1, 2, ..., d-1\}$, $k_{i-1} + k_i + k_{i+1} \geq 7$.

PROOF : There are three cases to consider.

Case 1 : $k_i = 1$.

If this is so, then we are done by Lemma 2.

Case 2 : $k_i = 2$.

If this is so, then we are done by Lemma 3.

Case 3 : $k_i \geq 3$.

If this is so, then we are done by Lemma 5. □

Throughout the rest of this paper we define $c \in \{0, 1, 2\}$ as a number such that $d - c \equiv 2 \mod 3$ where $d = \mathrm{diam}(G)$.

***Theorem* 1** — *Let G be a 3-edge-connected C_4-free graph of order n. Then*

$$\operatorname{diam}(G) \leq \frac{3n}{7} - \frac{3}{7}.$$

This inequality is almost best possible.

PROOF : Note that $n = \sum_{i=0}^{d} k_i = \sum_{i=0}^{\frac{d-c-2}{3}} (k_{3i} + k_{3i+1} + k_{3i+2}) + (k_{d-c+1} + k_{d-c+2})$ where $k_i = 0$ whenever $i > d$. Hence there are three cases to consider:

Case $1 : d \equiv 0 \mod 3$. If this is so then $c = 1$ and we have the following, by Lemma 6 and the fact that $k_d \geq 1$ since N_d is non empty:

$$\begin{aligned}
n &= \sum_{i=0}^{d} k_i \\
&= \sum_{i=0}^{\frac{d-1-2}{3}} (k_{3i} + k_{3i+1} + k_{3i+2}) + k_d \\
&\geq \sum_{i=0}^{\frac{d-3}{3}} (k_{3i} + k_{3i+1} + k_{3i+2}) + k_d \\
&\geq \sum_{i=0}^{\frac{d-3}{3}} (7) + 1 \\
&= (\frac{d-3}{3} + 1)(7) + 1 = \frac{7d}{3} + 1
\end{aligned}$$

and making d subject of the formula, we obtain $d \leq \frac{3n}{7} - \frac{3}{7}$.

Case $2 : d \equiv 1 \mod 3$. If this is so then $c = 2$ and we have the following, by Lemma 6 and Lemma 1:

$$\begin{aligned}
n &= \sum_{i=0}^{d} k_i \\
&= \sum_{i=0}^{\frac{d-2-2}{3}} (k_{3i} + k_{3i+1} + k_{3i+2}) + (k_{d-1} + k_d) \\
&\geq \sum_{i=0}^{\frac{d-4}{3}} (k_{3i} + k_{3i+1} + k_{3i+2}) + (k_{d-1} + k_d)
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=0}^{\frac{d-4}{3}}(7)+4\\
&=(\frac{d-4}{3}+1)(7)+4\\
&=\frac{7d-7}{3}+4
\end{aligned}$$

and making d subject of the formula, we obtain $d \leq \frac{3n}{7}-\frac{5}{7}$.

Case 3 : $d \equiv 2 \mod 3$. If this is so then $c=0$ and we have the following, by Lemma 6:

$$\begin{aligned}
n &= \sum_{i=0}^{d} k_i\\
&= \sum_{i=0}^{\frac{d-0-2}{3}} (k_{3i}+k_{3i+1}+k_{3i+2})\\
&\geq \sum_{i=0}^{\frac{d-2}{3}}(k_{3i}+k_{3i+1}+k_{3i+2})\\
&\geq \sum_{i=0}^{\frac{d-2}{3}}(7)\\
&=(\frac{d-2}{3}+1)(7)\\
&=\frac{7d+7}{3}
\end{aligned}$$

and making d subject of the formula, we obtain $d \leq \frac{3n}{7}-1$.
Hence, considering all three cases, we obtain $d \leq \frac{3n}{7}-\frac{3}{7}$ as desired. To see that this bound is almost tight, consider the following graph.

Let A be the graph in Figure 1 and B be the graph in Figure 2. To form $A \oplus B$ we join A and B by introducing a new vertex r and adding the edges uw, vx, ur, vr, xr, and wr. To form $B \oplus A$ we join A and B by introducing a new vertex r and adding the edges uy, vz, ur, vr, zr, and yr. To form $B \oplus B$ we join B and B by introducing a new vertex r and adding the edges yw, zx, yr, zr, xr, and wr. As an example consider $B \oplus B$ in Figure 3. Let $H := A \oplus B \oplus B \oplus ... \oplus B \oplus A$. Then H is a 3-edge-connected C_4-free graph and

$$\mathrm{diam}(H)=\frac{3n}{8}-\frac{5}{8}$$

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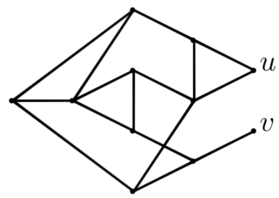

Figure 1: The graph A

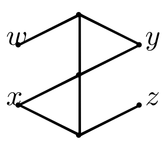

Figure 2: The graph B

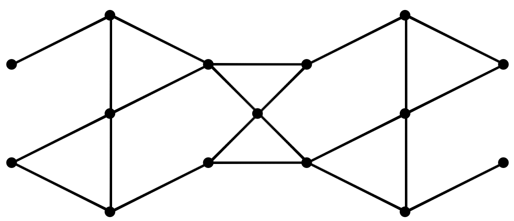

Figure 3: $B \oplus B$

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