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ORIGINAL RESEARCH

Numerical implementation of finite-time shadowing of stochastic differential equations

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Abstract This paper focuses on the numerical implementation methods of shadowing theorem of stochastic differential equations. A general shadowing theorem of stochastic differential equations is given, and an explicit bound for shadowing distance is investigated. The main part is numerical implementation methods for shadowing distance in details. Numerical experiments are provided to illustrate the effectiveness of the proposed theorem by the numerical simulations of chaotic orbits of stochastic differential equations.

Keywords Stochastic differential equations · Random shadowing · Numerical implementation · Stochastic Chen system

Mathematics Subject Classification 37H05 · 37C50 · 65C30 · 65P20

1 Introduction

Nowadays, the shadowing property has played an important role in the theory and application of random dynamical systems (RDS), especially in numerical simulations of chaotic systems of stochastic differential equations (SDEs). Numerical computation is in the center in the investigations of chaotic SDEs which describe many natural phenomena, for example, in meteorology and biology [1, 3, 10–12, 17]. However, we mainly concern the reliability and feasibility of numerical computations. These related works are shown in References [5, 8, 13, 19–22].

This work is motivated by two facts. Firstly, the construction of conditions which can assure stochastic shadowing in a class of SDEs has been finished in [19]. Many useful contributions are made to the numerical

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analysis of RDS [2, 14–16]. These results are the foundations of stochastic shadowing. Secondly, the high accurate numerical implementation methods of stochastic shadowing continues to be an interested topic. To the best of our knowledge, no investigations of the numerical implementations of random shadowing of SDEs in a finite time interval exist in the literature.

In this work, we mainly focus on the numerical methods to compute all parameters which are involved in a shadowing theorem of SDEs in a finite time interval. This is the continuation of Paper [19–22] in some sense. The results show that under certain appropriate assumptions the numerical approximate orbits of discrete approximate RDS are more close to the true orbits of the original systems.

The rest of this paper is organized as follows. Section 2 deals with some preliminaries. In Section 3 the theoretical results of the finite time shadowing are summarized. Section 4 presents the details of the numerical implementations. Illustrative numerical experiments for the main theorem are included in Section 5. We demonstrate that the main theorem and numerical implementation methods can be applied to stochastic Chen system (SCS), stochastic Lorenz equations(SLEs) and stochastic Lü equations(SLuEs). Finally, section 6 is addressed to summarize the conclusions of the paper.

2 Preliminaries

Let $W_t, t \in \mathbb{R}^+ := [0, +\infty)$ be a one dimensional Wiener process, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a Wiener space. And we assume that $\Omega := \{\omega \in C(\mathbb{R}^+, \mathbb{R}) : \omega(0) = 0\}$. We consider a class of Stratonovich SDEs as follows,

$$dx_t = f_0(x_t)dt + f_1(x_t) \circ dW_t, \qquad x(0) = x_0(\omega) \in \mathbb{R}^d, \tag{1}$$

where the random variable $x_0(\omega)$ is independent of $\mathcal{F}_0$ and satisfies the inequality $\mathbb{E}|x_0(\omega)|^2 < \infty$. And $f_0 \in \mathcal{C}_b^{1,\delta_1}, f_1 \in \mathcal{C}_b^{2,\delta_1}, \sum_{i=1}^d f_1^i \frac{\partial f_1}{\partial x_i} \in \mathcal{C}_b^{1,\delta_1}$, where the space $\mathcal{C}_b^{k,\delta_1}$ is defined to be the Banach space of function $f : \mathbb{R}^d \to \mathbb{R}^d$, which are in $\mathcal{C}^{k,\delta_1}$ and for which the norm

$$\|f\|_{k,0} := \sup_{x \in \mathbb{R}^d} \frac{\|f(x)\|}{1 + \|x\|} + \sum_{1 \le |\alpha| \le k} \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)|,$$

and

$$\|f\|_{k,\delta_1} := \|f\|_{k,0} + \sum_{|\alpha|=k} \sup_{x \ne y} \frac{\|D^\alpha f(x) - D^\alpha f(y)\|}{\|x - y\|^{\delta_1}}, 0 < \delta_1 \le 1, k = 1, 2,$$

is finite, see References [1, 4].

We define

$$\theta : \mathbb{R}^+ \times \Omega \to \Omega, \theta^t \omega(s) = \omega(t + s) - \omega(t)$$

and $0 \le s \le t$, $s \in \mathbb{R}^+$, $t \in \mathbb{R}^+$. It follows from Theorem 2.3.32 in [1] that SDE (1) generates a unique RDS $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, which is usually written as $\varphi(s, t, \omega)x := \varphi(s, t, \omega, x)$ on the metric dynamical systems $(\Omega, \mathcal{F}, \mathbb{P}, \theta^t)$, and is C^1 with respect to x. The RDS φ is written as follows,

$$\varphi(s, t, \omega)x = x + \int_s^t f_0(\varphi(s, r, \omega)x)dr + \int_s^t f_1(\varphi(s, r, \omega)x) \circ dW_r(\omega).$$

We also make use of the following notations, the norm of a random variable $\|\cdot\|$, the norm of random matrix $\|\cdot\|$ and the norm of a stochastic process $\|x(t, \omega)\|_2$, which are similar to [18, 19].

3 Theoretical foundations of numerical shadowing

3.1 some existed definitions

Definition 1 [19] Let $\mathcal{F}_{t_k}$ be the filtration generated by $W_\tau(\omega), 0 \le \tau \le t_k$, and random variable $x_0(\omega)$ be independent of $\mathcal{F}_{t_0}$. For a given positive number δ and $\mathbb{P}$-almost surely $\omega \in \Omega$, if there is a sequence of time

$\{t_k\}_{k=0}^N, 0 \le t_0 \le t_1 \le, ..., \le t_N$ and a sequence of random variables $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ which means that $u_k(\theta^{t_k}\omega)$ is $\mathcal{F}_{t_k}$-measurable for $k = 0, 1, ..., N$, such that the following inequalities hold

$$\|u_{k+1}(\theta^{t_{k+1}}\omega) - \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)u_k(\theta^{t_k}\omega)\| \le \delta, \quad k = 0, 1, ..., N-1,$$

then the random variables $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ are said to be a (ω, δ)-pseudo orbit of SDE (1) in the mean-square sense, where $\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)u_k(\theta^{t_k}\omega)$ denotes the orbit of RDS φ at the time t_{k+1} which starts from the initial time t_k with the initial value $u_k(\theta^{t_k}\omega)$ and the sample $\theta^{t_k}\omega$, and $u_k(\theta^{t_k}\omega)$ denotes a random variable with the sample $\theta^{t_k}\omega$ which is $\mathcal{F}_{t_k}$-measurable.

Definition 2 [19] For a given positive number ε, $\mathbb{P}$-almost surely $\omega \in \Omega$, and a (ω, δ)-pseudo orbit $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ of SDE (1) with associated times $\{t_k\}_{k=0}^N$, if there is a sequence of time $\{h_k\}_{k=0}^N, 0 \le h_0 = t_0 \le h_1 \le, ..., \le h_N$, which is random, such that the following inequalities hold

$$\|u_k(\theta^{t_k}\omega) - x_k(\theta^{h_k}\omega)\| \le \varepsilon, \quad 0 \le t_k - h_k \le \varepsilon, \qquad k = 0, 1, ..., N, \tag{2}$$

where the random variables $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^N$ are on the true orbit of SDE (1), that is

$$x_{k+1}(\theta^{h_{k+1}}\omega) = \varphi(h_k, h_{k+1}, \theta^{h_k}\omega)x_k(\theta^{h_k}\omega), \quad k = 0, 1, ..., N-1,$$

then the (ω, δ)-pseudo orbit $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ is said to be (ω, ε)-shadowed by a true orbit of SDE (1) in the mean-square sense, where the true orbit of RDS φ is a stochastic process.

3.2 Theoretical results of finite time shadowing

Here we repeat some results in Reference [20]. For a (ω, δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$, we want to prove that it is (ω, ε)-shadowed by a true orbit containing points $\{(\hat{x}_k(\theta^{\hat{h_k}}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ for $\mathbb{P}$-almost surely $\omega \in \Omega$, where $\{\hat{h_k}\}_{k=0}^N = \{t_k\}_{k=0}^N$ in sequels. For simplicity, we define that $t_k := k\Delta t$, Δt is the given time step, $\hat{x}_k(\theta^{\hat{h_k}}\omega)$ is the point on the true orbit, $\mathcal{F}_{t_k}$ is the filtration generated by $W_\tau(\omega), 0 \le \tau \le t_k$, and $N := \frac{t}{\Delta t}$.

We first introduce the function space $\mathcal{X}$ and $\mathcal{Y}$, which are the same as Reference [20]. Specially,

$$y(\omega) =: (y_0(\theta^{t_0}\omega), y_1(\theta^{t_1}\omega), ..., y_N(\theta^{t_N}\omega)) \in \mathbb{R}^{(N+1)d},$$

is a special element of the space $\mathcal{Y}$. And

$$x(\omega) = (x_0(\theta^{h_0}\omega), x_1(\theta^{h_1}\omega), ..., x_N(\theta^{h_N}\omega)) \in \mathbb{R}^{(N+1)d}$$

is a general element of the space $\mathcal{X}$.

Secondly, we define the mapping $G : \mathcal{X} \longrightarrow \mathcal{Y}$, and the kth component of $G(y(\omega))$ is defined for any $y(\omega) \in \mathcal{X}$ to be

$$[G(y(\omega))]_k = y_k(\theta^{t_k}\omega) - \varphi(t_{k-1}, t_k, \theta^{t_{k-1}}\omega)y_{k-1}(\theta^{t_{k-1}}\omega), k = 1, ..., N. \tag{3}$$

Here we consider the first variation $DG(y(\omega))$ of $G(y(\omega))$. By the assumptions that RDS φ generated by SDE (1) is C^1 with respect to x, then the Jacobian matrix $D\varphi(t_{k-1}, t_k, \theta^{t_{k-1}}\omega)y_{k-1}$ with respect to y_{k-1} exists. It follows from $G(y(\omega))$ is C^1 with respect to $y(\omega)$ that the existence of $DG(y(\omega))$ is obtained. Then $DG(y(\omega))$ is defined by the following equality

$$DG(y(\omega)) := \begin{pmatrix} -D\varphi(t_0, t_1, \theta^{t_0}\omega)y_0 & \mathbb{I} & 0 & ..., & 0 & 0 \\ 0 & -D\varphi(t_1, t_2, \theta^{t_1}\omega)y_1 & \mathbb{I} & ..., & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ..., & \mathbb{I} & 0 \\ 0 & 0 & 0 & ..., & -D\varphi(t_{N-1}, t_N, \theta^{t_{N-1}}\omega)y_{N-1} & \mathbb{I} \end{pmatrix}, \tag{4}$$

where

$$D\varphi(t_{k-1},t_k,\theta^{t_{k-1}}\omega)y_{k-1}(\theta^{t_{k-1}}\omega)=\frac{\partial}{\partial y}(\varphi(t_{k-1},t_k,\theta^{t_{k-1}}\omega)y_{k-1}(\theta^{t_{k-1}}\omega)),\ k=1,2,...,N,$$

that is, $DG(y(\omega))$ is a $Nd\times(N+1)d$ random matrix, and $\mathbb{I}$ is the $d\times d$ identity matrix.

Lastly, we wish to find a solution

$$\hat{x}(\omega):=(\hat{x}_0(\theta^{t_0}\omega),\hat{x}_1(\theta^{t_1}\omega),...,\hat{x}_N(\theta^{t_N}\omega))\in\mathcal{X},0\leq t_k-\hat{h_k}\leq\varepsilon$$

of the equation $G(\hat{x}(\omega))=0$ and to prove that it lies in a closed ε-neighborhood of $y(\omega)$.

Here we consider the first variation $DG(y(\omega))$ of $G(y(\omega))$. We define that

$$D\varphi(t_{k-1},t_k,\theta^{t_{k-1}}\omega)y_{k-1}(\theta^{t_{k-1}}\omega)=\frac{\partial}{\partial y}(\varphi(t_{k-1},t_k,\theta^{t_{k-1}}\omega)y_{k-1}(\theta^{t_{k-1}}\omega)),\ k=1,2,...,N.$$

Then we have

$$[DG(y(\omega))\Delta y(\omega)]_k=\Delta y_k(\theta^{t_k}\omega)-D\varphi(t_{k-1},t_k,\theta^{t_{k-1}}\omega)y_{k-1}(\theta^{t_{k-1}}\omega)\cdot\Delta y_{k-1}(\theta^{t_{k-1}}\omega),\ k=1,2,...,N,$$

where

$$\Delta y(\omega)=(\Delta y_0(\theta^{t_0}\omega),\Delta y_1(\theta^{t_1}\omega),...,\Delta y_N(\theta^{t_N}\omega))^T,$$

and

$$\Delta y_k(\theta^{t_k}\omega)=x_k(\theta^{t_k}\omega)-y_k(\theta^{t_k}\omega),\ k=0,1,2,...,N.$$

We assume that $B_k(\theta^{t_k}\omega)$ is a $d\times d$ random matrix obtained by numerical integration and satisfies

$$\|B_k(\theta^{t_k}\omega)-D\varphi(t_{k-1},t_k,\theta^{t_{k-1}}\omega)y_{k-1}(\theta^{t_{k-1}}\omega)\|\leq\delta. \tag{5}$$

The detail of the numerical implementation will be shown in Section 4.1. Then we can obtain the numerical approximation $L(y(\omega))$ of $DG(y(\omega))$ and

$$[L(y(\omega))\Delta y(\omega)]_k=\Delta y_k(\theta^{t_k}\omega)-B_k(\theta^{t_k}\omega)\Delta y_{k-1}(\theta^{t_{k-1}}\omega),k=1,2,...,N. \tag{6}$$

Theorem 1 *Let $G:\mathcal{X}\longrightarrow\mathcal{Y}$ be a C^1 map as (3) and $L(y(\omega))$ be the numerical approximation of $DG(y(\omega))$, respectively. And let $y(\omega)\in\mathcal{X}$ be a point such that the pseudo inverses $[DG(y(\omega))]^{-1}$ and $[L(y(\omega))]^{-1}$ exist, and the following inequalities hold*

$$\begin{aligned}\|G(y(\omega))\|&\leq\delta,\\ \|[L(y(\omega))]^{-1}\|&\leq c,\end{aligned} \tag{7}$$

and

$$\|[L(y(\omega))]^{-1}-[DG(y(\omega))]^{-1}\|\leq\alpha \tag{8}$$

for some positive constants δ,c and α.

Furthermore, we let $\varepsilon:=2(\alpha+c)\delta$. If the inequality

$$\|DG(\hat{x}(\omega))-L(y(\omega))\|=\beta\leq\frac{1}{2(\alpha+c)}, \tag{9}$$

holds for $\|\hat{x}(\omega)-y(\omega)\|\leq\varepsilon$, then the equation $G(\hat{x}(\omega))=0$ has one solution $\hat{x}(\omega)$ which satisfies $\|\hat{x}(\omega)-y(\omega)\|\leq\varepsilon$.

Proof The proof is similar to Reference [20], so we omit the detail here. □

4 Numerical implementation methods

This section is the main part of this article. We approximate the local error δ using the local error control mechanism of the numerical scheme. Then we only pay attention to the magnification of the local error, $(\alpha+c)$, that gives the shadowing distance. This is part quote from Paper [20].

4.1 Basic method

Step 1. Appling one-step numerical schemes (EM scheme or Milstein scheme [9]) to simultaneously solve the following equations from t_k to t_{k+1} with the initial values $x(0) = y_k(\theta^{t_k}\omega)$ and $v(0) = \mathbb{I}$,

$$\begin{cases} dx_t = f_0(x_t)dt + f_1(x_t) \circ dW_t(\omega) \\ dv_t = Df_0(x_t)v_t dt + Df_1(x_t)v_t \circ dW_t(\omega), \end{cases}$$

we get the approximations of $\hat{x}_{k+1}(\theta^{t_{k+1}}\omega)$ and $D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)$ respectively,

$$\begin{aligned} &\hat{x}_{k+1}(\theta^{t_{k+1}}\omega) \approx \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega), \\ &D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) \approx v_{k+1}(\theta^{t_{k+1}}\omega) = B_{k+1}(\theta^{t_{k+1}}\omega). \end{aligned}$$

That is, $B_{k+1}(\theta^{t_{k+1}}\omega) := D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)$ is chosen such that the inequality (5) hold by numerical schemes.

Step 2. It follows from Theorem 3.3 that $[L(y(\omega))]^{-1}$ and $[DG(y(\omega))]^{-1}$ both exist. By the oscillation theorem of linear equations, we can find α such that

$$\|[L(y(\omega))]^{-1} - [DG(y(\omega))]^{-1}\| \leq \alpha.$$

The detail of this estimate will be shown in Section 4.2. And the estimate of the parameter c will be shown in Section 4.2, too.

Step 3. By the norm properties, we can find β such that

$$\beta = \|DG(\hat{x}(\omega)) - L(y(\omega))\|.$$

If the inequality (9) is not satisfied, then we will denote this by putting $'-'$ in the table of numerical experiment results. And the detail of the method will be shown in Section 4.3.

Step 4. If the values of c, α, δ and β satisfy all conditions of Theorem 3.1, then the shadowing distance is $\varepsilon = 2(c + \alpha)\delta$.

4.2 Methods to estimate the parameters α and c

Let $A := L(y(\omega))[L(y(\omega))]^T$ and $A + \Delta A := DG(y(\omega))[DG(y(\omega))]^T$. Since $[L(y(\omega))]^{-1} = [L(y(\omega))]^T A^{-1}$, then $[L(y(\omega))]^{-1}$ and $[DG(y(\omega))]^{-1}$ can be formed by the solutions of the following equations, respectively,

$$Ax = b^{(i)}, \quad (A + \Delta A)y = b^{(i)} + \Delta b^{(i)},$$

where $\Delta A = DG(y(\omega))[DG(y(\omega))]^T - L(y(\omega))[L(y(\omega))]^T$, and $b^{(i)} = [L(y(\omega))]_i$ is the i-th column vector of $L(y(\omega))$, $i = 1, 2, ..., (N+1)d$. Furthermore, $\Delta b^{(i)} := \Delta[L(y(\omega))]_i = [L(y(\omega))]_{i+1} - [L(y(\omega))]_i$.

The following lemma is a direct consequence in [6].

Lemma 1 *Suppose* $Ax = b^{(i)}, A \in \mathbb{R}^{Nd\times Nd}, 0 \neq b^{(i)} \in \mathbb{R}^{Nd\times 1}$, *and* $(A + \Delta A)y = b^{(i)} + \Delta b^{(i)}, \Delta A \in \mathbb{R}^{Nd\times Nd}, \Delta b^{(i)} \in \mathbb{R}^{Nd\times 1}$, *with* $\|\Delta A\|_1 \leq \zeta\|A\|_1, \|\Delta b^{(i)}\|_1 \leq \zeta\|b^{(i)}\|_1$. *If* $\zeta\kappa(A) < 1$, *then* $A + \Delta A$ *is nonsingular and*

$$\frac{\|y - x\|_1}{\|x\|_1} \leq \frac{2\zeta}{1 - \zeta\kappa(A)}\kappa(A),$$

where $\kappa(A)$ *is the condition number of the matrix* A, $\zeta > 0$, $\|\cdot\|_1$ *is the* 1*-norm.*

Proposition 1 *Suppose* $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ *is a* (ω, δ)*-pseudo orbit of SDE (1). If* ζ *and* $\kappa(A)$ *satisfy the conditions of Lemma 4.1, then the following conclusions hold*

$$\|[L(y(\omega))]^{-1}\| \leq c \quad \textit{and} \quad \alpha = \frac{2\zeta}{1 - \zeta\kappa(A)}\kappa(A)c,$$

where

$$\zeta = \max\{\zeta_1, \zeta_2\}$$

and

$$\zeta_1 = \sup_i \left\{ \frac{\delta}{\|b^{(i)}\|_1} \right\}, \zeta_2 = \sup_k \left\{ \frac{\delta(\|B_k(\theta^{t_k}\omega)\|_1 + \|B_k(\theta^{t_k}\omega)\|_\infty + \delta + 2)}{\|A\|_1} \right\}. \tag{10}$$

Proof Firstly, it follows from Section 3.2 that $DG(y(\omega))$ exists. Furthermore, Theorem 3.3 shows the existence of $[L(y(\omega))]^{-1}$. Therefore, by the definition of the norm the constant c does exist. And the estimate of $\|[L(y(\omega))]^{-1}\|$ is similar to Reference [20].

Secondly, we estimate the parameter α [6]. On the one hand, note that the quantities $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ and $L(y(\omega))$ can be calculated by one-step numerical scheme as shown in Section 4.1. Then the condition number $\kappa(A)$ can be found by the Matlab code during our computations. We need to estimate the supermum norm of $L(y(\omega))$. It follows from (6) and the definition of the 1-norm that $\|\Delta b^{(i)}\|_1 := \|B_{i+1}(\theta^{t_{i+1}}\omega) - B_i(\theta^{t_i}\omega)\|_1 = \delta$. Note that we can calculate every row of $b^{(i)}$ and obtain $\zeta_1 = \max\left\{\frac{\delta}{\|b^{(i)}\|_1}\right\}$ such that $\|\Delta b^{(i)}\|_1 \leq \zeta_1 \|b^{(i)}\|_1$, where

$$\|b^{(i)}\|_1 = \sum_{j=1}^{Nd} \|(L(y(\omega)))_{ij}\|.$$

On the other hand, we have

$$\|D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k\|_1 \leq \|B_k(\theta^{t_k}\omega)\|_1 + \delta.$$

Therefore, it follows from the property of norm that we get

$$\begin{aligned}
&\|D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k \cdot [D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k]^T - B_k(\theta^{t_k}\omega) \cdot [B_k(\theta^{t_k}\omega)]^T\|_1 \\
&\quad = \|D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k \cdot [D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k]^T \\
&\qquad - [B_k(\theta^{t_k}\omega) - D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k + D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k] \cdot [B_k(\theta^{t_k}\omega)]^T\|_1 \\
&\quad \leq \|D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k \cdot [[D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k]^T - [B_k(\theta^{t_k}\omega)]^T]\|_1 \\
&\qquad + \|[D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k - B_k(\theta^{t_k}\omega)] \cdot [B_k(\theta^{t_k}\omega)]^T\|_1 \\
&\quad \leq \delta(\|B_k(\theta^{t_k}\omega)\|_1 + \|B_k(\theta^{t_k}\omega)\|_\infty + \delta) := C_1^{(k)},
\end{aligned}$$

where $\|[B_k(\theta^{t_k}\omega)]^T\|_1 \leq \|B_k(\theta^{t_k}\omega)\|_\infty$, and $\|\cdot\|_\infty$ is the infinite norm.

Note that A is block tridiagonal, it follows from the random matrix computation that we obtain

$$\|\Delta A\|_1 = \sup_k\{2\delta + C_1^{(k)}\}$$

and the second part of (10) such that $\|\Delta A\|_1 \leq \zeta_2\|A\|_1$, where

$$\|B_k(\theta^{t_k}\omega)\|_\infty = \max\left\{ \sum_{j=1}^d \|(B_k(\theta^{t_k}\omega))_{1j}\|, \sum_{j=1}^d \|(B_k(\theta^{t_k}\omega))_{2j}\|, \ldots, \sum_{j=1}^d \|(B_k(\theta^{t_k}\omega))_{dj}\| \right\}.$$

We choose $\zeta = \max\{\zeta_1, \zeta_2\}$ such that $\|\Delta b^{(i)}\|_1 \leq \zeta\|b^{(i)}\|_1$ and $\|\Delta A\|_1 \leq \zeta\|A\|_1$. It follows from Lemma 4.1 that

$$\alpha = \frac{2\zeta}{1 - \zeta\kappa(A)}\kappa(A)c.$$

This proof is finished. □

4.3 Method to estimate the parameter β

This part will prove that the parameter β can be explicitly estimated.

Proposition 2 *Assume that the functions f_0 and f_1 satisfies the conditions in Section 2. In the proper ε_0-neighbourhood of the (ω,δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega),\mathcal{F}_{t_k})\}_{k=0}^{N}$, let the local Lipschitz constants of f_0, f_1 and f_2 be K_1, and the local Lipschitz constants of Df_0 and Df_1 be K_2, where the parameters K_1 and K_2 are positive and independent of δ and t_k, and $f_2 = f_0 + \frac{1}{2}f_1'f_1$.*

Then the following equality holds

$$\beta = K_3 + \delta,$$

where

$$K_3 = \sup_k K_2\sqrt{2\Delta t_{k+1}(\Delta t_{k+1}+1)K_4}, \tag{11}$$

$$K_4 = \sup_k \left[3\varepsilon_0^2 \exp[3\Delta t_{k+1}(\Delta t_{k+1}+1)K_1^2]\right] \quad and \quad \Delta t_{k+1} = t_{k+1} - t_k. \tag{12}$$

Proof It follows from the definition of DG that

$$\|DG(\hat{x}(\omega)) - L(y(\omega))\| \le K_5 + \delta,$$

where

$$K_5 = \sup_k \|D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega) - D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)\|.$$

First of all, we estimate the error caused by different initial value. The constant $\varepsilon_0 \in (0,\varepsilon]$ is given such that the inequality $\|y_k(\theta^{t_k}\omega) - \hat{x}_k(\theta^{t_k}\omega)\| \le \varepsilon_0$ hold, then we have

$$\mathbb{E}\Big[\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 \le I_1 + I_2 + I_3,$$

where

$$I_1 = 3\mathbb{E}\Big[y_k(\theta^{t_k}\omega) - \hat{x}_k(\theta^{t_k}\omega)\Big]^2,$$
$$I_2 = 3\mathbb{E}\Big[\int_{t_k}^{t_{k+1}} [f_2(\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)) - f_2(\varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega))]ds\Big]^2$$

and

$$I_3 = 3\mathbb{E}\Big[\int_{t_k}^{t_{k+1}} [f_1(\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)) - f_1(\varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega))]dW_s(\omega)\Big]^2.$$

Utilizing Cauchy-Schwarz inequality and the local Lipschitz condition of f_2, we have

$$I_2 \le 3\Delta t_{k+1}K_1^2 \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - \varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 ds.$$

It follows from the local Lipschitz condition of f_1 and the Itô isometry that we have

$$I_3 \le 3K_1^2 \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - \varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 ds.$$

Therefore, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 \\
&\le 3\mathbb{E}\Big[y_k(\theta^{t_k}\omega) - \hat{x}_k(\theta^{t_k}\omega)\Big]^2 \\
&\quad + 3(\Delta t_{k+1}+1)K_1^2 \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - \varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 ds.
\end{aligned}$$

It follows from the Gronwall inequality and (12) that we obtain

$$\mathbb{E}\Big[\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 \le K_4. \tag{13}$$

Secondly, we estimate the value of K_5. The expression K_5 implies

$$\mathbb{E}\Big[D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 \le I_4 + I_5,$$

where

$$I_4 = 2\mathbb{E}\Big[\int_{t_k}^{t_{k+1}} [Df_0(\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)) \cdot D\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - Df_0(\varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)) \cdot D\varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)]ds\Big]^2$$

and

$$I_5 = 2\mathbb{E}\Big[\int_{t_k}^{t_{k+1}} [Df_1(\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)) \cdot D\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - Df_1(\varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)) \cdot D\varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)] \circ dW_s(\omega)\Big]^2.$$

Utilizing the Cauchy-Schwarz inequality and the local Lipschitz condition of Df_0, we have

$$I_4 \le 2\Delta t_{k+1}K_2^2 \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - \varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 ds.$$

It follows from the Itô isometry and the local Lipschitz condition of Df_1 that we have

$$I_5 \le 2K_2^2 \int_{t_k}^{t_{k+1}} \mathbb{E}\Big[\varphi(t_k, s, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)) - \varphi(t_k, s, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 ds.$$

Using these conclusions and (13), we obtain

$$\mathbb{E}\Big[D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\Big]^2 \le 2\Delta t_{k+1}(\Delta t_{k+1} + 1)K_2^2 K_4. \tag{14}$$

Therefore, by (11) and (14) we arrive at the inequality

$$\|D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)\hat{x}_k(\theta^{t_k}\omega)\| \le K_3. \tag{15}$$

By the conclusions of (11) and (15), we obtain $K_5 \le K_3$. Therefore, we only need to choose

$$\beta = K_3 + \delta.$$

This completes the proof. □

5 Numerical experiments

5.1 Experimental preparation

We consider the Stratonovich SCS

$$dX_t = f_0(X_t)dt + f_1(X_t) \circ dW_t(\omega), \quad X(0) = x_0 \in \mathbb{R}^3 \tag{16}$$

where $X_t = (x(t), y(t), z(t))^T \in \mathbb{R}^3$, $x(t)$, $y(t)$ and $z(t)$ make up the system state, σ, ρ and β are the system parameters, and

$$f_0(X_t) = \begin{pmatrix} -\sigma x(t) + \sigma y(t) \\ (\rho - \sigma)x(t) + \rho y(t) - x(t)z(t) \\ -\beta z(t) + x(t)y(t) \end{pmatrix}, f_1(X_t) = \begin{pmatrix} \mu x(t) \\ \mu y(t) \\ \mu z(t) \end{pmatrix}.$$

In this experiment, we choose the classic parameters $\sigma = 35, \rho = 28, \beta = 3$. A RDS (φ, θ) can be generated by the solution operator of SDE (16).

The preset initial conditions are as follows: the initial value is $x_0 = -10.0, y_0 = 1.0, z_0 = 37.0$. SDE (16) is discreted by Euler-Maruyama(EM) scheme, we obtain that

$$\begin{cases} x_{k+1} = x_k + (\sigma(-x_k + y_k) + 0.5\mu^2 x_k)\Delta t_k + \mu x_k \Delta W_k, \\ y_{k+1} = y_k + (-x_k z_k - (\rho - \sigma)x_k + \rho y_k + 0.5\mu^2 y_k)\Delta t_k + \mu y_k \Delta W_k, \\ z_{k+1} = z_k + (x_k y_k - \beta z_k + 0.5\mu^2 z_k)\Delta t_k + \mu z_k \Delta W_k. \end{cases} \tag{17}$$

5.2 Numerical results

This section will provide numerical experiments to compute the shadowing distance of SDE (16). The classical parameters are chosen and the initial value is $x_0 = -10.0, y_0 = 1.0, z_0 = 37.0$.

Firstly, in order to show the influence of noise on the pseudo orbits, we choose various size of noise, such as $\mu = 0$(the deterministic case), $\mu = 1.0$, $\mu = 2.0$ and $\mu = 4.0$. And we take the temporal step-size $\Delta t = 1.05e - 3$ and $N = 1.0e + 4$. Taking the numerical solution for an example, Fig. 1 shows the perturbation of the pseudo orbits of SDE (16) by EM scheme corresponding to different scales of noise. As we can see from Fig. 1, due to the increase of the noise scale, that is, the value of μ is increased from 0 to 4.0, the perturbation of the pseudo orbits becomes much more seriously in x and y directions, respectively. That is to say, the pseudo orbits in x direction of Fig. 1 become more and more dense, which illustrates that the perturbation of the pseudo orbits in x direction become more and more larger. It is similar to the case in y direction.

Secondly, we focus on the numerically performing the shadowing distance, $\varepsilon = 2(\alpha + c)\delta$. The amplification factors α and c are computed quite accurately, and the local error tolerance δ is approximately determined by the numerical scheme. Applying EM scheme and Milstein scheme to SCS, we present the discrete forms respectively.

SDE (16) is discreted by Milstein scheme as follows,

$$\begin{cases} x_{k+1} = x_k + (\sigma(-x_k + y_k) + 0.5\mu^2 x_k)\Delta t_k + \mu x_k \Delta W_k + 0.5\mu^2 x_k((\Delta W_k)^2 - \Delta t_k), \\ y_{k+1} = y_k + (-x_k z_k - (\rho - \sigma)x_k + \rho y_k + 0.5\mu^2 y_k)\Delta t_k + \mu y_k \Delta W_k + 0.5\mu^2 y_k((\Delta W_k)^2 - \Delta t_k), \\ z_{k+1} = z_k + (x_k y_k - \beta z_k + 0.5\mu^2 z_k)\Delta t_k + \mu z_k \Delta W_k + 0.5\mu^2 z_k((\Delta W_k)^2 - \Delta t_k). \end{cases} \tag{18}$$

Table 1 presents the results that δ is the local error and ε is the shadowing distance. We can obtain that the existence of shadowing orbit and the effectiveness of the numerical method. As we know, the shadowing error δ obtained by Milstein scheme is smaller than that by EM scheme. Therefore, shadowing distance ε is significantly smaller in case of Milstein scheme compared to EM scheme. Because we do not consider the reparameterization of time, the value of shadowing distance ε are not small with respect to the local error δ.

Finally, the approximation results of shadowing obtained from the shadowing method is shown in Fig. 2. The blue cycle indicate the shadowing orbits and the black stars represent the (ω, δ)-pseudo orbits. We choose 50 points in the computational result. It presents that there are the shadowing orbits in the proper neighbourhood of the (ω, δ)-pseudo orbits. Therefore, the theoretical results can give information on chaotic orbits by the existence of shadowing orbits and the computation of shadowing distances.

5.3 Compare with results of other SDEs

5.3.1 Results on stochastic Lorenz equations

We consider the Stratonovich SLEs

$$dX_t = f_0(X_t)dt + f_1(X_t) \circ dW_t(\omega), \quad X(0) = x_0 \in \mathbb{R}^3 \tag{19}$$

where the parameters are similar with SCS(16) and

$$f_0(X_t) = \begin{pmatrix} -\sigma x(t) + \sigma y(t) \\ \rho x(t) - y(t) - x(t)z(t) \\ -\beta z(t) + x(t)y(t) \end{pmatrix}, f_1(X_t) = \begin{pmatrix} \mu x(t) \\ \mu y(t) \\ \mu z(t) \end{pmatrix}.$$

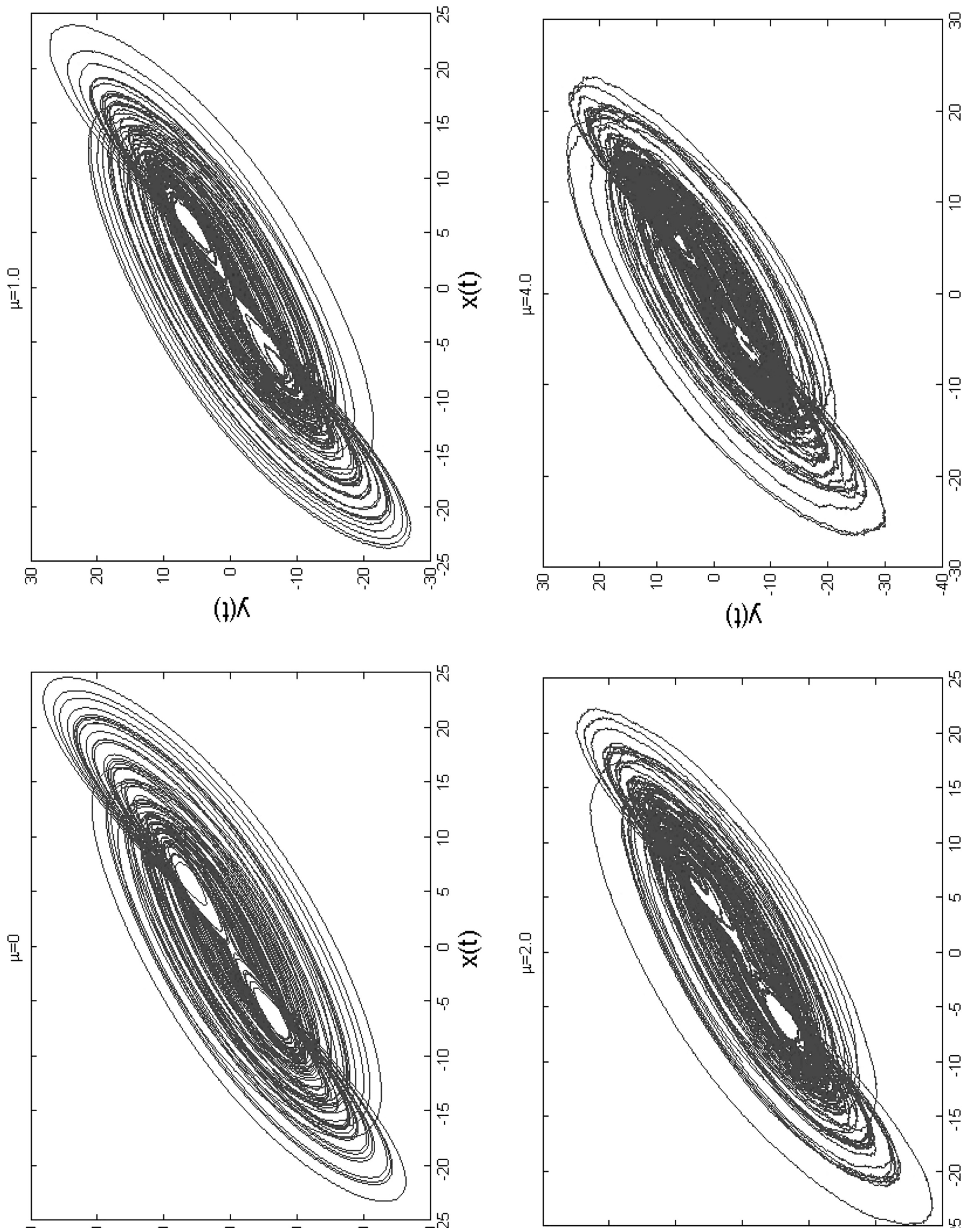

Fig. 1 Pseudo orbits for different sizes of noise $\mu = 0, \mu = 1.0, \mu = 2.0$ *and* $\mu = 4.0$

Table 1 Summaries of the parameters for SCS

μ	EM	T = 10.5			
	c	α	δ	β	ε
0	325.1272	1.4603	1.0500e-3	8.7832e-4	0.6858
1.0	327.3332	13.2132	0.0324	0.0008	22.0674
2.0	358.8756	91.1213	0.0324	0.0009	29.1598
4.0	377.5459	178.7624	0.0324	0.0008	36.0488

μ	Milstein	T = 10.5			
	c	α	δ	β	ε
0	325.1272	1.4603	1.0500e-3	8.7832e-4	0.6858
1.0	332.3267	10.1223	1.0500e-3	0.0009	0.7191
2.0	368.8352	89.5351	1.0500e-3	0.0010	0.9626
4.0	398.7845	139.0756	1.0500e-3	0.0009	1.1295

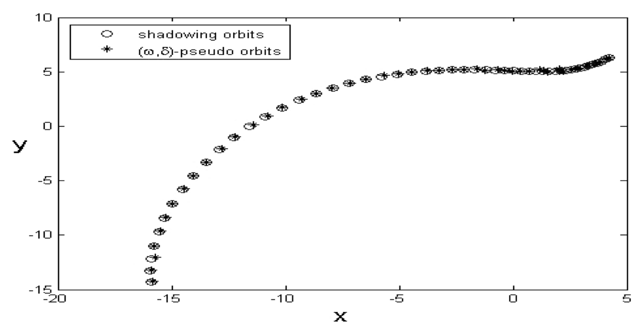

Fig. 2 Zoom in parts of the approximation shadowing trajectory on SCS

Table 2 Summaries of the parameters for SLEs

μ	EM	T = 21.0			
	c	α	δ	β	ε
0	468.9008	1.7147	1.0500e-3	0.0009	0.9883
0.5	451.2325	13.6754	0.0324	0.0010	30.1261
1.0	383.2516	84.3276	0.0324	0.0009	30.2991
1.3	356.3169	123.5653	0.0324	0.0008	31.0964

μ	Milstein	T = 21.0			
	c	α	δ	β	ε
0	468.9008	1.7147	1.0500e-3	0.0009	0.9883
0.5	458.3582	12.4722	1.0500e-3	0.0010	0.9887
1.0	417.7546	67.9523	1.0500e-3	0.0010	1.0199
1.3	378.6774	129.7812	1.0500e-3	0.0009	1.0678

Here we choose the parameters $\sigma = 10, \rho = 28, \beta = 8/3$ and the initial value $x_0 = 0, y_0 = 1, z_0 = 0$. Table 2 contains some numerical results we obtained for SLEs.

Figures 3 and 4 show the pseudo orbits and the shadowing orbits for SLEs. As we can see, there are similar results as the results of SCS.

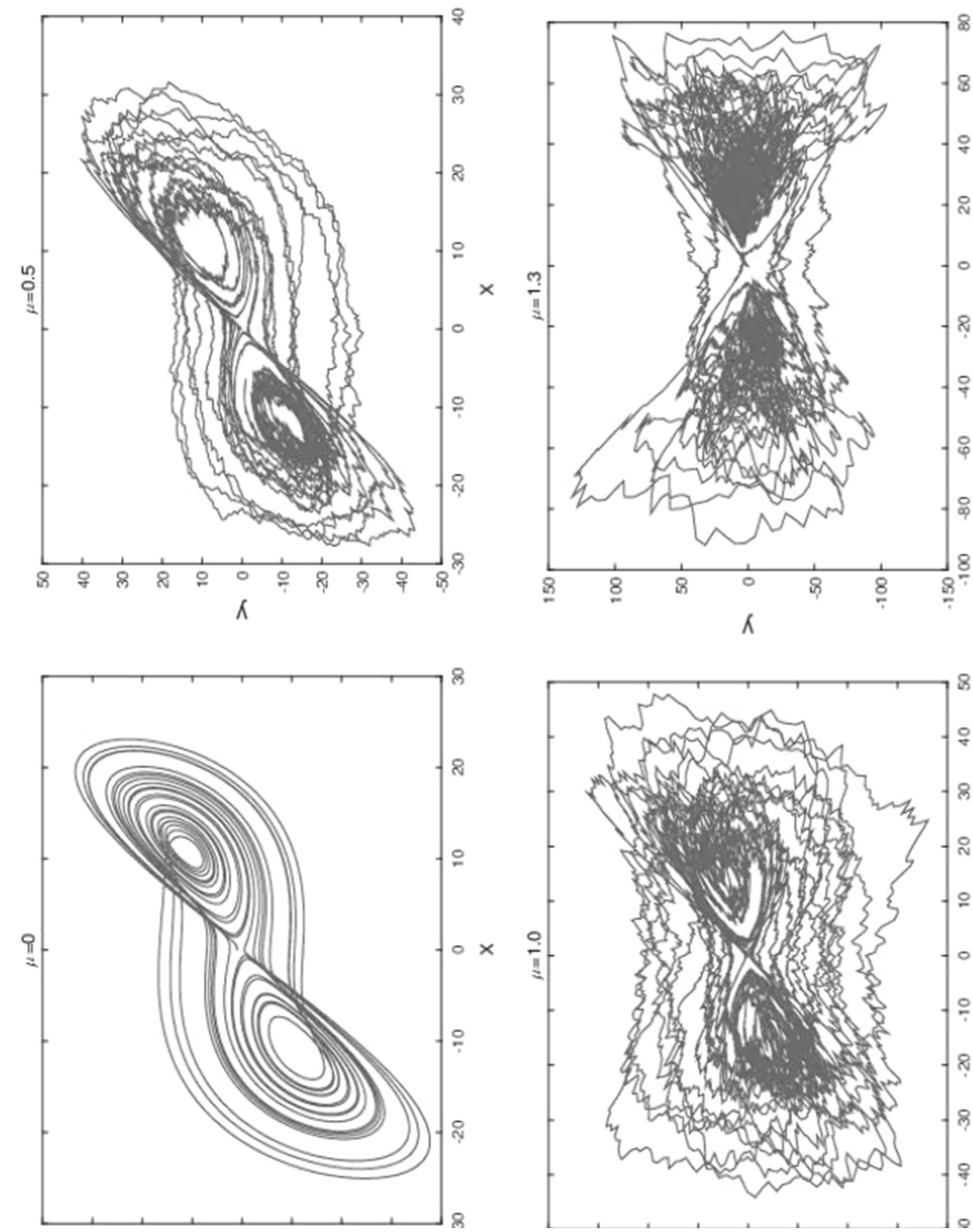

Fig. 3 Pseudo orbits for different sizes of noise for $\mu = 0$, $\mu = 0.5$, $\mu = 1.0$, and $\mu = 1.3$

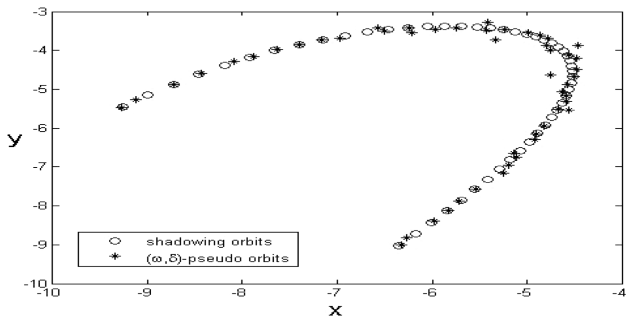

Fig. 4 Zoom in parts of the approximation shadowing trajectory on the SLEs

Table 3 Summaries of the parameters for SLuEs

μ	EM	T = 10.5			
	c	α	δ	β	ε
0	456.1289	1.4728	1.0500e-3	0.0008	0.9609
0.5	449.8121	10.6576	0.0324	0.0009	29.8384
1.0	399.5433	75.8756	0.0324	0.0010	30.8071
1.3	366.7623	110.7654	0.0324	0.0010	30.9438

μ	Milstein	T = 10.5			
	c	α	δ	β	ε
0	456.1289	1.4728	1.0500e-3	0.0008	0.9609
0.5	448.4532	10.3534	1.0500e-3	0.0010	0.9635
1.0	388.3222	74.7751	1.0500e-3	0.0009	0.9725
1.3	365.8712	109.9121	1.0500e-3	0.0010	0.9991

5.3.2 Results on stochastic lü equations

We consider the Stratonovich SLuEs [7]

$$dX_t = f_0(X_t)dt + f_1(X_t) \circ dW_t(\omega), \quad X(0) = x_0 \in \mathbb{R}^3 \tag{20}$$

where the parameters are similar with SCS(16) and

$$f_0(X_t) = \begin{pmatrix} -\sigma x(t) + \sigma y(t) \\ \rho y(t) - x(t)z(t) \\ -\beta z(t) + x(t)y(t) \end{pmatrix}, f_1(X_t) = \begin{pmatrix} \mu x(t) \\ \mu y(t) \\ \mu z(t) \end{pmatrix}.$$

Here we choose the parameters $\sigma = 36, \rho = 15, \beta = 3$ and the initial value $x_0 = 0, y_0 = 1, z_0 = 0$. Table 3 contains the numerical results we obtained for SLuEs.

Figures 5 and 6 show the pseudo orbits and the shadowing orbits for SLuEs. As we can see, there are similar results as the result of SCS.

As can be seen from these numerical results, there is an explicit dependent relationship between the shadowing distance and the local error, and there exists a true orbit in the appropriate neighbourhood of a (ω, δ)-pseudo orbit of SDE (16), (19) and (20), respectively. The numerically detected behavior of the system indeed reflects its real dynamical behavior which can be described more accurately.

5.4 The features of my device

And the feathers of my device are as follows, in which the computations are performed (Table 4).

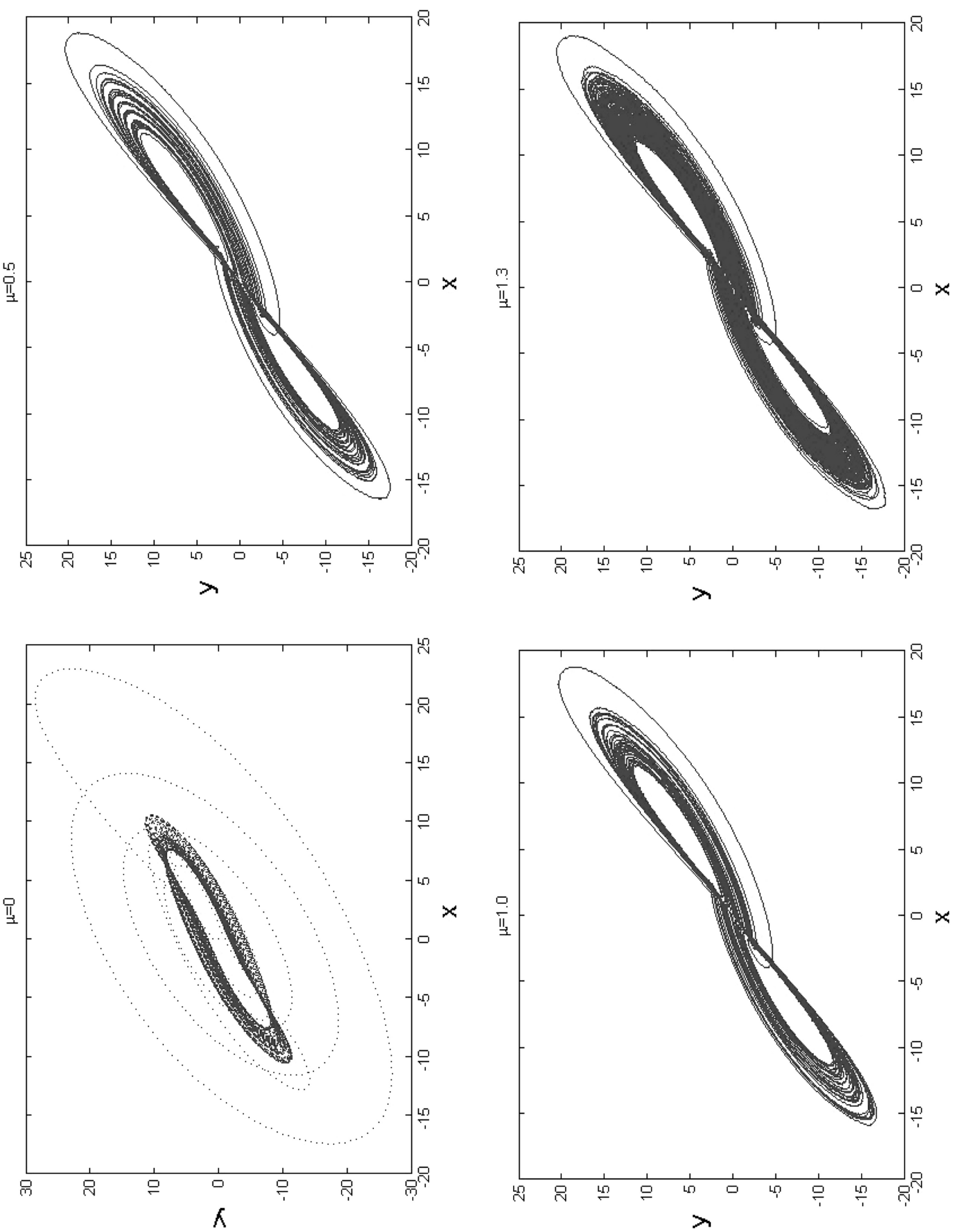

Fig. 5 Pseudo orbits for different sizes of noise for $\mu = 0$, $\mu = 0.5$, $\mu = 1.0$, and $\mu = 1.3$

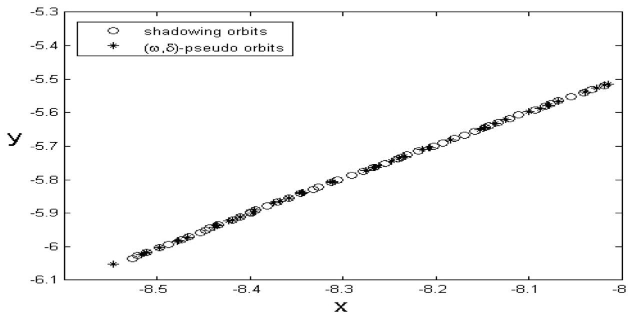

Fig. 6 Zoom in parts of the approximation shadowing trajectory on the SLuEs

Table 4 The features of my device

My device	Precision	RAM	Processor speed
Lenovo-PC	64-bit operating systems	6.00GB	2.20GHz

6 Conclusion

The main result of this paper is the numerical implementation of shadowing theorem for finite time of SDEs. It focuses on the mathematical approaches to estimate the shadowing distance in the neighborhood of a (ω, δ)-pseudo orbit. The results show that the methods are effective and the numerical experiments are performed and match the results of theoretical analysis. More simple and high-performance methods will be shown in our further work.

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