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ORIGINAL RESEARCH

A note on the mean square of the greatest divisor of n which is coprime to a fixed integer k

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Abstract Denote by $\delta_k(n)$ the greatest divisor of a positive integer n which is coprime to a given $k \geq 2$. In the case of $k = p$ (a prime) Joshi and Vaidya studied $E_p(x) := \sum_{n \leq x} \delta_p(n) - \frac{p}{2(p+1)}x^2$ (as $x \to \infty$) and obtained $E_p(x) = \Omega_{\pm}(x)$ by an elementary and beautiful approach. Here we study $R_p^{(2)}(x) := \sum_{n \leq x} \delta_p^2(n) - \frac{p^2}{3(p^2+p+1)}x^3 + \frac{p}{6}x$ and show $R_p^{(2)}(x) = \Omega_{\pm}(x^2)$. Moreover, using a method of Adhikari and Balasubramanian we consider a bound of $|R_k^{(2)}(x)|/x^2$ for any square-free integer k.

Keywords The greatest divisor · A mean square of an arithmetical function · Asymptotic behaviour · Estimates of error terms · Omega-results

Mathematics Subject Classification 11A25 · 11A99 · 11N37

1 Introduction

Let $n \geq 1$ and $k \geq 2$ be integers (later k be square-free), and p a prime. In [14], Suryanarayana considered the following arithmetical function:

$$\delta_k(n) := \max\{d \,:\, d|n,\ (d,k) = 1\}, \tag{1}$$

that is, this denotes the greatest divisor of n which is coprime to k. He investigated the behaviour of

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$$\sum_{n \le x} \frac{\delta_k(n)}{n^\sigma}.$$

For example, he obtained the following formula

$$\sum_{n \le x} \delta_k(n) = \frac{k\varphi(k)}{2J_2(k)} x^2 + O\left(2^{v(k)} x \log^2 x\right), \tag{2}$$

uniformly in x and k, where $\varphi(k) = k\prod_{p|k}\left(1 - \frac{1}{p}\right)$ is the Euler function, $J_2(k) = k^2\prod_{p|k}\left(1 - \frac{1}{p^2}\right)$, $v(k) = \sum_{p|k} 1$ is the number of distinct prime divisors of k. In the literature, many researches have considered $\delta_k(n)$, namely, Joshi and Vaidya [10], Ramachandran [12], Herzog and Maxsein [9], Adhikari, Balasubramanian and Sankaranarayanan [3], Adhikari and Balasubramanian [2], Pétermann [11], Adhikari [1], Adhikari and Soundararajan [5], Adhikari and Chakraborty [4], Adhikari and Thangadurai [6], Singh [13], Igawa [7], and Igawa and Sharma [8].

Here, we shall develop some results which are stated in papers [10] and [2] to the square of $\delta_k(n)$ for a fixed square-free integer k. In [10], Joshi and Vaidya considered the error term $E_k(x)$ in the formula (2)

$$E_k(x) := \sum_{n \le x} \delta_k(x) - \frac{k}{2\sigma(k)} x^2,$$

where $\sigma(k) = \sum_{d|k} d$ (note that $\varphi(k)/J_2(k) = 1/\sigma(k)$, if k is square-free), and showed in the case of $k = p$ (any prime)

$$\limsup_{x\to\infty} \frac{E_p(x)}{x} = \frac{p}{p+1}, \quad \liminf_{x\to\infty} \frac{E_p(x)}{x} = -\frac{p}{p+1}.$$

In [9], Herzog and Maxsein, and in [3], Adhikari, Balasubramanian, and Sankaranarayanan showed that for any square-free integer $k \ge 2$,

$$\limsup_{x\to\infty} \frac{E_k(x)}{x} \ge \frac{k}{\sigma(k)}, \quad \liminf_{x\to\infty} \frac{E_k(x)}{x} \le -\frac{k}{\sigma(k)}.$$

For the other direction, Adhikari and Balasubramanian [2] showed

$$\limsup_{x\to\infty} \frac{|E_k(x)|}{x} \le \frac{p}{p+1}\frac{d(k)}{2},$$

where $d(k) = \sum_{d|k} 1$ is the divisor function.

Recently, in [7] Igawa and also in [8] Igawa and Sharma studied behaviour of

$$\sum_{n \le x} \frac{\delta_k^r(n)}{n^\sigma}$$

for any $r \ge 1$ and $\sigma \in \mathbb{R}$. By their statements, for a fixed square-free integer $k \ge 2$ we have

$$F_k^{(2)}(x) := \sum_{n \le x} \delta_k^2(n) = \frac{k^2}{3\sigma(k^2)} x^3 + O_k\left(x^2\right), \tag{3}$$

where the O-constant depends on k (for the detail of the O-term which is uniformly in x and k, see [7, 8]). Using the method of Joshi and Vaidya [10, p. 792, Theorem 1.1] we can obtain (3). We will prove a generalization of (3) in the final section of this paper. So we shall define an error term on the formula (3) by

$$E_k^{(2)}(x) = \sum_{n \le x} \delta_k^2(n) - \frac{k^2}{3\sigma(k^2)} x^3. \tag{4}$$

Moreover, we shall define $R_k^{(2)}(x)$ by

$$R_k^{(2)}(x) = E_k^{(2)}(x) + \frac{kx}{6}. \tag{5}$$

On this $R_k^{(2)}(x)$ we shall prove the following two theorems.

Theorem 1.1 (*cf. Joshi and Vaidya* [10, *p.* 804, (4.7), (4.8)]) In the case of $k = p$ (any prime), we have

$$\limsup_{x\to\infty} \frac{R_p^{(2)}(x)}{x^2} = \frac{p^2}{p^2+p+1}, \tag{6}$$

$$\liminf_{x\to\infty} \frac{R_p^{(2)}(x)}{x^2} = -\frac{p^2}{p^2+p+1}, \tag{7}$$

that is,

$$R_p^{(2)}(x) = \Omega_{\pm}\left(x^2\right).$$

Theorem 1.2 (*cf. Adhikari and Balasubramanian* [2, *p.* 38, *Theorem* 1]) *Let $k \ge 2$ be any square-free integer, and p be the smallest prime divisor of k. Then we have*

$$\limsup_{x\to\infty} \frac{|R_k^{(2)}(x)|}{x^2} \le \frac{p^2}{p^2+p+1}\frac{d(k)}{2}, \tag{8}$$

where $d(k) = \sum_{d|k} 1$ is the divisor function.

Remark 1 Note that the integer k of $\delta_k(n)$ is not fixed in Suryanarayana [14], Ramachandran [12], Igawa [7], and Igawa and Sharma [8], but in this present paper, integers k and p are fixed. The situation of Theorems 1.1 and 1.2 are quite different from [12, 14, 7], and [8].

2 Some preliminary lemmas

In order to prove Theorem 1.1, we recall some properties of $\delta_k(n)$ and we prepare some lemmas on $R_p^{(2)}(n)$ for positive integers n. We follow Joshi and Vaidya's beautiful method in [10].

Lemma 2.1 (*Joshi and Vaidya* [10, *p.* 792]) *Let k and n be positive integers and p a prime. Then we have*

(i) $\delta_k(kn) = \delta_k(n)$,
(ii) $\delta_k(pn) = p\delta_k(n)$, if $(p,k) = 1$,
(iii) $\delta_p(n) = n$, if $p \nmid n$,
(iv) $\delta_{kp}(n) = \delta_k(n)$, if $(p,n) = 1$.

We remark that the properties in Lemma 2.1 are very important and are used many times in the following arguments.

Lemma 2.2 (*cf.* [10, *p.* 796, *Theorems* 3.1, 3.2, 3.3]) *Let p be any fixed prime. For any natural numbers N and α, we have*

(i) $R_p^{(2)}(1) = \dfrac{2p^2+3p+3}{3(p^2+p+1)} + \dfrac{p}{6}$,
(ii) $R_p^{(2)}(pN) = R_p^{(2)}(N)$,
(iii) $R_p^{(2)}(p^\alpha - 1) = \dfrac{p^{\alpha+2}(p^\alpha-1)}{p^2+p+1}$.

Proof The first assertion (i) is obtained by the definitions (1), (4), (5) of $\delta_p(n)$, $E_p^{(2)}(n)$, and $R_p^{(2)}(n)$, in fact, $R_p^{(2)}(1) = \delta_p^{(2)}(1) - \frac{p^2}{p^2+p+1} + \frac{p}{6}$. To prove the second assertion (ii), using Lemma 2.1 (i), (iii) we observe that

$$
\begin{aligned}
R_p^{(2)}(pN) &= \sum_{n \le pN} \delta_p^2(n) - \frac{p^2}{3(p^2+p+1)}(pN)^3 + \frac{p(pN)}{6} \\
&= \sum_{\substack{n \le pN \\ p \mid n}} \delta_p^2(n) + \sum_{\substack{n \le pN \\ p \nmid n}} \delta_p^2(n) - \frac{p^5N^3}{3(p^2+p+1)} + \frac{p(pN)}{6} \\
&= \sum_{m \le N} \delta_p^2(m) + \sum_{\substack{n \le pN \\ p \nmid n}} n^2 - \frac{p^5N^3}{3(p^2+p+1)} + \frac{p(pN)}{6} \\
&= -\frac{p^2(p-1)N^3}{3} + E_p^{(2)}(N) + \frac{(pN)(pN+1)(2pN+1)}{6} \\
&\quad - \frac{p^2N(N+1)(2N+1)}{6} + \frac{p(pN)}{6} \\
&= R_p^{(2)}(N).
\end{aligned}
$$

To prove the third assertion (iii), by the definitions (4), (5) of $E_p^{(2)}(n)$, $R_p^{(2)}(n)$, and Lemma 2.2 (i), (ii) we have

$$
\begin{aligned}
R_p^{(2)}(p^\alpha - 1) &= \sum_{n \le p^\alpha} \delta_p^2(n) - 1 - \frac{p^2(p^{3\alpha} - 3p^{2\alpha} + 3p^\alpha - 1)}{3(p^2+p+1)} + \frac{p(p^\alpha - 1)}{6} \\
&= R_p^{(2)}(p^\alpha) - 1 + \frac{p^{\alpha+2}(p^\alpha - 1)}{p^2+p+1} + \frac{p^2}{3(p^2+p+1)} - \frac{p}{6} \\
&= \frac{p^{\alpha+2}(p^\alpha - 1)}{p^2+p+1}.
\end{aligned}
$$

□

By Lemma 2.2 (iii) we get the following result immediately.

Lemma 2.3 *Let α and n be positive integers. For any prime p, we have*

$$\lim_{\alpha \to \infty} \frac{R_p^{(2)}(p^\alpha - 1)}{(p^\alpha - 1)^2} = \frac{p^2}{p^2+p+1},$$

which implies

$$\limsup_{n \to \infty} \frac{R_p^{(2)}(n)}{n^2} \ge \frac{p^2}{p^2+p+1}. \tag{9}$$

Lemma 2.4 (*cf.* [10, *p.* 798, *Theorem* 3.5]) *For any positive integer n we have $R_p^{(2)}(n) \ge R_p^{(2)}(1)$. Moreover, we get*

$$\liminf_{n \to \infty} R_p^{(2)}(n) = R_p^{(2)}(1) = \frac{2p^2+3p+3}{3(p^2+p+1)} + \frac{p}{6}.$$

Proof First, we shall prove $R_p^{(2)}(n) \ge R_p^{(2)}(1)$ for n satisfying $p^\alpha \le n < p^{\alpha+1}$ ($\alpha = 0, 1, \ldots$) by the induction on α.

In the case of $\alpha = 0$, that is, $1 \le n < p$, since the positive integers $m \le n$ are not divided by p, we see that $\delta_p^2(m) = m^2$ by Lemma 2.1 (iii). Hence

$$
\begin{aligned}
R_p^{(2)}(n) &= \sum_{m \le n} \delta_p^2(m) - \frac{p^2 n^3}{3(p^2+p+1)} + \frac{pn}{6} \\
&= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - \frac{p^2 n^3}{3(p^2+p+1)} + \frac{pn}{6} \\
&= \frac{(p+1)n^3}{3(p^2+p+1)} + \frac{(3p^2+3p+3)n^2}{6(p^2+p+1)} + \frac{(p^2+p+1)n}{6(p^2+p+1)} + \frac{pn}{6}.
\end{aligned} \tag{10}
$$

In the right hand side of (10), we set $n = 1$, then

$$
\begin{aligned}
R_p^{(2)}(n) &\ge \frac{p+1}{3(p^2+p+1)} + \frac{3p^2+3p+3}{6(p^2+p+1)} + \frac{p^2+p+1}{6(p^2+p+1)} + \frac{p}{6} \\
&= \frac{4p^2+6p+6}{6(p^2+p+1)} + \frac{p}{6} \\
&= R_p^{(2)}(1),
\end{aligned}
$$

here, in the last equality we use Lemma 2.2 (i).

Now assume that the assertion is true for $p^\alpha \le n < p^{\alpha+1}$ $(\alpha \ge 0)$. Let m be any integer such that $p^{\alpha+1} \le m < p^{\alpha+2}$. Note that we can write $m = np + r$ $(p^\alpha \le n < p^{\alpha+1}, 0 \le r \le p-1)$. By Lemma 2.1 (iii) we observe that

$$
\begin{aligned}
R_p^{(2)}(m) &= \sum_{j \le m} \delta_p^2(j) - \frac{p^2 m^3}{3(p^2+p+1)} + \frac{pm}{6} \\
&= \sum_{j \le np} \delta_p^2(j) + \sum_{np < j \le np+r} \delta_p^2(j) - \frac{p^2(np+r)^3}{3(p^2+p+1)} + \frac{p(np+r)}{6} \\
&= \frac{p^2}{3\sigma(p^2)}(np)^3 + E_p^{(2)}(np) + \sum_{np < j \le np+r} \delta_p^2(j) - \frac{p^2(np+r)^3}{3(p^2+p+1)} \\
&\quad + \frac{p(np)}{6} + \frac{pr}{6} \\
&= R_p^{(2)}(np) + \sum_{np < j \le np+r} j^2 - \frac{3(np)^2 p^2 r + 3(np)p^2 r^2 + p^2 r^3}{3(p^2+p+1)} + \frac{pr}{6}.
\end{aligned} \tag{11}
$$

Here, we see that

$$
\begin{aligned}
\sum_{np < j \le np+r} j^2 &= \frac{6r(np)^2 + (6r^2+6r)(np) + 2r^3 + 3r^2 + r}{6} \times \frac{p^2+p+1}{p^2+p+1} \\
&= \frac{rp^2(np)^2}{p^2+p+1} + \frac{r^2p^2(np)}{p^2+p+1} + \frac{r^3 p}{3(p^2+p+1)} \\
&\quad + \frac{r(p+1)(np)^2}{p^2+p+1} + \frac{r^2(p+1)(np)}{p^2+p+1} + r(np) + \frac{r^3(p+1)}{3(p^2+p+1)} + \frac{3r^2+r}{6}.
\end{aligned} \tag{12}
$$

Hence by (11), (12), Lemma 2.2 (ii), and the assertion of induction we have

$$
\begin{aligned}
R_p^{(2)}(m) &= R_p^{(2)}(n) + \frac{r(p+1)(np)^2}{p^2+p+1} + \frac{r^2(p+1)(np)}{p^2+p+1} + r(np) \\
&\quad + \frac{r^3(p+1)}{3(p^2+p+1)} + \frac{3r^2+r}{6} + \frac{pr}{6} \\
&\ge R_p^{(2)}(n) \\
&\ge R_p^{(2)}(1).
\end{aligned} \tag{13}
$$

Therefore, we get the first assertion. The second assertion is trivial. Actually, by Lemma 2.2 (ii) we see that $R_p^{(2)}(1) = R_p^{(2)}(p) = R_p^{(2)}(p^2) = \cdots$, and hence by the first assertion we obtain the second assertion. □

Lemma 2.5 (*cf.* [10, *p.* 796, *Theorem* 3.4]) *For any integer n satisfying* $p^\alpha \le n < p^{\alpha+1}$ $(\alpha = 0, 1, 2, \ldots)$ *we have*

$$R_p^{(2)}(n) \le R_p^{(2)}\left(p^{\alpha+1} - 1\right). \tag{14}$$

Proof We shall prove the assertion (14) by the induction on $\alpha \ge 0$.

Let $\alpha = 0$ (i.e., $1 \le n < p$). On the right hand side of (10) we put $n = p - 1$ and use Lemma 2.2 (iii) then

$$\begin{aligned} R_p^{(2)}(n) &\le \frac{(p+1)(p-1)^3}{3(p^2+p+1)} + \frac{(3p^2+3p+3)(p-1)^2}{6(p^2+p+1)} + \frac{p^3-1}{6(p^2+p+1)} + \frac{p(p-1)}{6} \\ &= \frac{p^3(p-1)}{p^2+p+1} \\ &\le R_p^{(2)}(p-1) \end{aligned}$$

this shows (14) is true for $\alpha = 0$.

Let n be any integer satisfying $p^\alpha \le n < p^{\alpha+1}$ $(\alpha \ge 0)$. Assume that (14) is true. For m such that $p^{\alpha+1} \le m < p^{\alpha+2}$, we shall write $m = np + r$ $(p^\alpha \le n < p^{\alpha+1}, 0 \le r \le p-1)$. In (13) we shall use the assumption, then

$$\begin{aligned} R_p^{(2)}(m) \le R_p^{(2)}\left(p^{\alpha+1} - 1\right) + \frac{r(p+1)(np)^2}{p^2+p+1} + \frac{r^2(p+1)(np)}{p^2+p+1} \\ + r(np) + \frac{r^3(p+1)}{3(p^2+p+1)} + \frac{3r^2+r}{6} + \frac{pr}{6}. \end{aligned} \tag{15}$$

Here, in the right hand side of (15) we put $r = p - 1$ and $n = p^{\alpha+1} - 1$, and use Lemma 2.2 (iii) then

$$\begin{aligned} &\le R_p^{(2)}\left(p^{\alpha+1} - 1\right) + R_p^{(2)}\left(p^{\alpha+2} - 1\right) - R_p^{(2)}\left(p^{\alpha+1} - 1\right) \\ &\quad - \frac{p^{\alpha+5} - p^{\alpha+2}}{p^2+p+1} + \frac{p^3 - p}{p^2+p+1} \\ &\quad + \frac{(p^2-p)(p^{\alpha+1}-1)(p^2+p+1)}{p^2+p+1} \\ &\quad + \frac{(p-1)^3(p+1)}{3(p^2+p+1)} + \frac{3(p-1)^2 + (p-1)}{6} + \frac{p(p-1)}{6} \\ &= R_p^{(2)}\left(p^{\alpha+2} - 1\right) - \frac{p^4 - p^3}{p^2+p+1} + \frac{(p-1)^3(p+1)}{3(p^2+p+1)} \\ &\quad + \frac{3(p-1)^2 + (p-1)}{6} + \frac{p(p-1)}{6} \\ &= R_p^{(2)}\left(p^{\alpha+2} - 1\right). \end{aligned}$$

Therefore, we obtain the assertion (14) for any $\alpha \ge 0$. □

Lemma 2.6 (*cf.* [10, *p.* 800–801]) *For any prime p we put*

$$C_p := \frac{(p-1)^2(p+1)}{p^2+p+1} + (p-1) + \frac{(p-1)^3(p+1)}{3(p^2+p+1)} + \frac{3(p-1)^2 + (p-1)}{6} + \frac{p(p-1)}{6}.$$

Let α *be any non-negative integer. For any positive integer n satisfying* $p^\alpha \le n < p^{\alpha+1}$ *we have the following inequality.*

$$\frac{R_p^{(2)}(n)}{n^2} \leq \frac{p^2}{p^2+p+1} + \frac{1}{14n} + \frac{(\alpha+1)C_p}{p^\alpha}. \tag{16}$$

Proof Let $\alpha = 0$ $(1 \leq n < p)$. In the proof of Lemma 2.3 we divide (10) by n^2 and add $-1/(14n)$, then we have

$$\begin{aligned}\frac{R_p^{(2)}(n)}{n^2} - \frac{1}{14n} &\leq \frac{p^2-1}{3(p^2+p+1)} + \frac{3(p^2+p+1)}{6(p^2+p+1)} - \frac{1}{14n} + \frac{p+1}{6n} \\ &= \frac{p^2}{p^2+p+1} + \frac{-p^2+3p+1}{6(p^2+p+1)} - \frac{1}{14n} + \frac{p+1}{6n}.\end{aligned}$$

Here note that if $p \geq 5$, then $-p^2+3p+1 < 0$, moreover, if $p = 2$ (in this case $n = 1$), we see

$$\frac{-p^2+3p+1}{6(p^2+p+1)} - \frac{1}{14n} = 0,$$

and if $p = 3$ (in this case $n = 1, 2$) we observe that

$$\frac{-p^2+3p+1}{6(p^2+p+1)} - \frac{1}{14n} < 0.$$

Hence, we get

$$\begin{aligned}\frac{R_p^{(2)}(n)}{n^2} - \frac{1}{14n} &\leq \frac{p^2}{p^2+p+1} + \frac{p+1}{6n} \\ &\leq \frac{p^2}{p^2+p+1} + \frac{p+1}{6} + \frac{5p-7}{6} \\ &= \frac{p^2}{p^2+p+1} + p - 1 \\ &\leq \frac{p^2}{p^2+p+1} + C_p.\end{aligned}$$

We see that the assertion (16) is true for $\alpha = 0$.

Assume that (16) is true for $p^\alpha \leq n < p^{\alpha+1}$ $(\alpha \geq 0)$. Let m be any integer such that $p^{\alpha+1} \leq m < p^{\alpha+2}$. As was done in the proofs of previous lemmas, we write $m = np + r$ $(p^\alpha \leq n < p^{\alpha+1}$, $0 \leq r \leq p-1)$. By (13) we get

$$\begin{aligned}\frac{R_p^{(2)}(m)}{m^2} - \frac{1}{14m} = &-\frac{1}{14m} \\ &+ \frac{1}{m^2}\left(R_p^{(2)}(n) + \frac{r(p+1)(np)^2}{p^2+p+1}\right) \\ &+ \frac{1}{m^2}\left(\frac{r^2(p+1)(np)}{p^2+p+1} + r(np)\right) \\ &+ \frac{1}{m^2}\left(\frac{r^3(p+1)}{3(p^2+p+1)} + \frac{3r^2+r}{6} + \frac{pr}{6}\right).\end{aligned}$$

As for the right hand side of the above identity, in the second line we put $m = np$ and $r = p-1$, in the third line, replace np by m, and we put $r = p-1$, in the fourth line, we put $r = p-1$, and replace m^2 by m, then

$$\begin{aligned}
&\le -\frac{1}{14m}\\
&\quad+\frac{1}{p^2}\frac{R_p^{(2)}(n)}{n^2}+\frac{p^2-1}{p^2+p+1}\\
&\quad+\frac{1}{m}\left(\frac{(p-1)^2(p+1)}{p^2+p+1}+(p-1)\right)\\
&\quad+\frac{1}{m}\left(\frac{(p-1)^3(p+1)}{3(p^2+p+1)}+\frac{3(p-1)^2+(p-1)}{6}+\frac{p(p-1)}{6}\right).
\end{aligned}$$

Next, in the second line we shall use the assumption, in the third and fourth lines, we put $m=p^{\alpha+1}$. Therefore

$$\begin{aligned}
&\le -\frac{1}{14m}+\frac{p^2}{p^2+p+1}+\frac{1}{14np^2}+\frac{(\alpha+1)C_p}{p^{\alpha+2}}+\frac{C_p}{p^{\alpha+1}}\\
&\le -\frac{1}{14m}+\frac{1}{14np^2}+\frac{p^2}{p^2+p+1}+\frac{(\alpha+2)C_p}{p^{\alpha+1}}.
\end{aligned}$$

Now we shall remark that

$$\begin{aligned}
-\frac{1}{14m}+\frac{1}{14np^2}&=\frac{1}{14}\left(\frac{1}{np^2}-\frac{1}{np+r}\right)\le\frac{1}{14}\left(\frac{1}{np^2}-\frac{1}{np+p-1}\right)\\
&=\frac{1}{14}\frac{(1-p)(np-1)}{(np^2)(np+p-1)}<0.
\end{aligned}$$

So we obtain

$$\frac{R_p^{(2)}(m)}{m^2}\le\frac{p^2}{p^2+p+1}+\frac{1}{14m}+\frac{(\alpha+2)C_p}{p^{\alpha+1}},$$

and finally we get the assertion (16) for any $\alpha\ge 0$. □

Theorem 2.7 (*cf.* [10, *p.* 799, *Theorem* 3.6]) *For any prime p we have*

$$\limsup_{n\to\infty}\frac{R_p^{(2)}(n)}{n^2}=\frac{p^2}{p^2+p+1}. \tag{17}$$

Proof By Lemma 2.6, for any integer $n\ge 1$ satisfying $p^\alpha\le n<p^{\alpha+1}$, since $\alpha\le\frac{\log n}{\log p}$ and $\frac{1}{p^\alpha}<\frac{p}{n}$ we see that

$$\frac{R_p^{(2)}(n)}{n^2}\le\frac{p^2}{p^2+p+1}+\frac{1}{14n}+\frac{p}{n}\left(\frac{\log n}{\log p}+1\right)C_p.$$

Then by the inequality we obtain

$$\limsup_{n\to\infty}\frac{R_p^{(2)}(n)}{n^2}\le\frac{p^2}{p^2+p+1}.$$

From this and (9) we reach the assertion (17). □

3 Proof of Theorem 1.1

In the previous section we obtained some results on $R_p^{(2)}(n)$ for positive integers n. To prove Theorem 1.1 we need one more lemma on $R_p^{(2)}(x)$ for the continuous variable $x>0$ similar to [10, Section 4].

Lemma 3.1 *Let $N\ge 1$ be any integer. For $x\in[N,N+1)$ we have*

(i) $R_p^{(2)}(x)$ *is continuous*,

(ii) $R_p^{(2)}(x) = R_p^{(2)}([x]) + \left(-\frac{p^2[x]^2}{p^2+p+1} + \frac{p}{6}\right)\{x\} - \frac{p^2}{p^2+p+1}\left([x]\{x\}^2 + \frac{\{x\}^3}{3}\right)$ *with* $\{x\} = x - [x]$ (*here* $[x]$ *is the largest integer not exceeding* x).

Proof The first assertion (i) is trivial from the definition (5) of $R_k^{(2)}(x)$. Moreover, by the definition (5) we have the second assertion (ii) easily. □

Proof of Theorem 1.1 Assume $N \geq \sqrt{\frac{p^3+p^2+p}{6p^2}}$ be any integer, by Lemma 3.1 we see that $R_p^{(2)}(x)$ is decreasing on $[N, N+1)$, and

$$\frac{R_p^{(2)}(x)}{x^2} \leq \frac{R_p^{(2)}([x])}{[x]^2}$$

for any $x \in [N, N+1)$.

Then by Lemma 3.1 and Theorem 2.7 we get the assertion (6) as follows.

$$\begin{aligned}\limsup_{x\to\infty}\frac{R_p^{(2)}(x)}{x^2} &= \lim_{x\to\infty}\left\{\sup_{t>x}\left\{\frac{R_p^{(2)}(t)}{t^2}\right\}\right\}\\ &= \lim_{x\to\infty}\left\{\sup_{N>x}\left\{\frac{R_p^{(2)}(N)}{N^2}\right\}\right\}\\ &= \limsup_{N\to\infty}\frac{R_p^{(2)}(N)}{N^2}\\ &= \frac{p^2}{p^2+p+1}.\end{aligned}$$

Next we shall show the assertion (7). By Lemma 3.1 (ii) and the first assertion of Lemma 2.4, we have

$$\begin{aligned}\frac{R_p^{(2)}(x)}{x^2} &\geq \frac{R_p^{(2)}(1)}{x^2} - \frac{p^2}{p^2+p+1}\frac{\left(x^2 - 2x\{x\} + \{x\}^2\right)\{x\}}{x^2} - \frac{p^2}{p^2+p+1}\frac{x+1/3}{x^2}\\ &\geq -\frac{p^2}{p^2+p+1} - \frac{p^2}{p^2+p+1}\left(\frac{1}{x} + \frac{1}{3x^2}\right).\end{aligned}$$

Hence we get that

$$\liminf_{x\to\infty}\frac{R_p^{(2)}(x)}{x^2} = \lim_{x\to\infty}\left\{\inf_{t>x}\left\{\frac{R_p^{(2)}(t)}{t^2}\right\}\right\} \geq -\frac{p^2}{p^2+p+1}. \tag{18}$$

On the other hand, since by Lemma 3.1 (ii) we observe that

$$\frac{R_p^{(2)}(x)}{x^2} \leq \frac{R_p^{(2)}([x])}{x^2} - \frac{p^2}{p^2+p+1}\frac{[x]^2\{x\}}{x^2} + \frac{p\{x\}}{6x^2}.$$

Now we take a sequence $x_v = p^v + \theta_v$, where $\theta_v = 1 - 1/(2v)$ $(v = 1, 2, \ldots)$. Since $[x_v] = p^v$, $\{x_v\} = 1 - 1/(2v)$, and Lemma 2.2 (ii) we have

$$\frac{R_p^{(2)}(x_v)}{x_v^2} \leq \frac{R_p^{(2)}(1)}{p^{2v}} - \frac{p^2}{p^2+p+1}\frac{p^{2v}\left(1-\frac{1}{2v}\right)}{(p^v+1)^2} + \frac{p}{6p^{2v}}. \tag{19}$$

Hence we get

$$\liminf_{v\to\infty}\frac{R_p^{(2)}(x_v)}{x_v^2}\leq \liminf_{v\to\infty}\left(\frac{R_p^{(2)}(1)}{p^{2v}}-\frac{p^2}{p^2+p+1}\frac{p^{2v}\left(1-\frac{1}{2v}\right)}{(p^v+1)^2}+\frac{p}{6p^{2v}}\right)$$
$$=\lim_{v\to\infty}\left(\frac{R_p^{(2)}(1)}{p^{2v}}-\frac{p^2}{p^2+p+1}\frac{p^{2v}\left(1-\frac{1}{2v}\right)}{(p^v+1)^2}+\frac{p}{6p^{2v}}\right)$$
$$=-\frac{p^2}{p^2+p+1}.$$

By this estimate we obtain

$$\begin{aligned}\liminf_{x\to\infty}\frac{R_p^{(2)}(x)}{x^2}&=\lim_{x\to\infty}\left\{\inf_{t>x}\left\{\frac{R_p^{(2)}(t)}{t^2}\right\}\right\}\\&\leq\lim_{x\to\infty}\left\{\inf_{v>x}\left\{\frac{R_p^{(2)}(x_v)}{x_v^2}\right\}\right\}\\&=\liminf_{v\to\infty}\frac{R_p^{(2)}(x_v)}{x_v^2}\\&\leq-\frac{p^2}{p^2+p+1}.\end{aligned}\tag{20}$$

By (18) and (20) we reach the assertion (7) of Theorem 1.1. □

4 proof of Theorem 1.2

We shall prove Theorem 1.2 using Adhikari and Balasubramanian's method in [2] and Theorem 1.1. To this end, we prepare the following lemma which is an analogue of (2.1) in [2] (see also [10, p. 793, Theorem 2.1]).

Lemma 4.1 (*Adhikari and Balasubramanian* [2, *p.* 39]) *Let k be any square-free integer ≥ 2. We write $k=p_1p_2\cdots p_r$ (p_i are primes such that $p_1<p_2<\cdots<p_r$), and $m=k/p_r$. As for $F_k^{(2)}(x)$ which is defined in* (3) *we have*

$$\begin{aligned}F_k^{(2)}(x)-F_k^{(2)}\left(\frac{x}{p_r}\right)=\frac{m^2}{3\sigma(m^2)}\left(1-\frac{1}{p_r}\right)x^3+R_m^{(2)}(x)-p_r^2R_m^{(2)}\left(\frac{x}{p_r}\right)\\-\frac{m(1-p_r)x}{6}.\end{aligned}\tag{21}$$

Moreover, let β be the positive integer satisfying $p_r^{\beta+1}\leq x$. We have

$$\begin{aligned}F_k^{(2)}(x)-F_k^{(2)}\left(\frac{x}{p_r^{\beta+1}}\right)&=\frac{k^2}{3\sigma(k^2)}\left(1-\frac{1}{p_r^{3\beta+3}}\right)x^3\\&+R_m^{(2)}(x)+(1-p_r^2)\left(R_m^{(2)}\left(\frac{x}{p_r}\right)+\cdots+R_m^{(2)}\left(\frac{x}{p_r^{\beta}}\right)\right)\\&-p_r^2R_m^{(2)}\left(\frac{x}{p_r^{\beta+1}}\right)+\frac{k}{6}\left(1-\frac{1}{p_r^{\beta+1}}\right)x.\end{aligned}\tag{22}$$

Proof By a similar result of Lemma 2.1 (i) and Lemma 2.1 (ii), (iv) we have

$$F_k^{(2)}(x) = \sum_{\substack{n \le x \\ p_r \mid n}} \delta_k^2(n) + \sum_{\substack{n \le x \\ p_r \nmid n}} \delta_k^2(n) = \sum_{l \le x/p_r} \delta_k^2(p_r l) + \sum_{\substack{n \le x \\ p_r \nmid n}} \delta_m^2(n)$$
$$= F_k^{(2)}\left(\frac{x}{p_r}\right) + F_m^{(2)}(x) - p_r^2 \sum_{l \le x/p_r} \delta_m^2(l),$$

and we get

$$F_k^{(2)}(x) - F_k^{(2)}\left(\frac{x}{p_r}\right) = F_m^{(2)}(x) - p_r^2 F_m^{(2)}\left(\frac{x}{p_r}\right).$$

On the right hand side of this identity, we shall use the definition (5) of $R_k^{(2)}(x)$, then we have the first assertion (21). Now we take a positive integer β such that $p_r^{\beta+1} \le x$. For the identity (21) we shall replace x by x/p_r successively β-times, then

$$F_k^{(2)}\left(\frac{x}{p_r^l}\right) - F_k^{(2)}\left(\frac{x}{p_r^{l+1}}\right) = \frac{m^2}{3\sigma(m^2)}\left(1 - \frac{1}{p_r}\right)\left(\frac{x}{p_r^l}\right)^3$$
$$+ R_m^{(2)}\left(\frac{x}{p_r^l}\right) - p_r^2 R_m^{(2)}\left(\frac{x}{p_r^{l+1}}\right) - \frac{m(1-p_r)}{6}\frac{x}{p_r^l},$$

where $l = 0, 1, \ldots, \beta$. Summing up all these $(\beta+1)$-identities we get the second assertion (22). □

Proof of Theorem 1.2 We shall prove by induction on the number of distinct prime factors of square-free integer k. In the case of $\nu(k) = 1$, the assertion (8) is true by Theorem 1.1. So we shall assume that the assertion (8) is true for $\nu(k) = r - 1$:

$$\limsup_{x\to\infty} \frac{\left|R_k^{(2)}(x)\right|}{x^2} \le \frac{p^2}{p^2+p+1}\frac{d(k)}{2}, \tag{23}$$

where p is the smallest prime divisor of k.

Now we shall take a square-free integer k such that $\nu(k) = r$. We write $k = p_1 p_2 \cdots p_r = m p_r$ $(p_1 < p_2 < \cdots < p_r)$. In this setting, we use the definition (5) of $R_k^{(2)}(x)$ in the left hand side of (22), then

$$\frac{R_k^{(2)}(x)}{x^2} = \frac{k}{3x}\left(1 - \frac{1}{p_r^{\beta+1}}\right) + \frac{1}{x^2}R_k^{(2)}\left(\frac{x}{p_r^{\beta+1}}\right) - \frac{p_r^2}{x^2}R_m\left(\frac{x}{p_r^{\beta+1}}\right)$$
$$+ \frac{R_m^{(2)}(x)}{x^2} + \frac{(1-p_r^2)}{x^2}\left(R_m^{(2)}\left(\frac{x}{p_r}\right) + R_m^{(2)}\left(\frac{x}{p_r^2}\right) + \cdots + R_m^{(2)}\left(\frac{x}{p_r^\beta}\right)\right).$$

From this we have

$$\frac{|R_k^{(2)}(x)|}{x^2} \le \frac{k}{3x} + \frac{1}{x^2}\left|R_k^{(2)}\left(\frac{x}{p_r^{\beta+1}}\right)\right| + \frac{p_r^2}{x^2}\left|R_m\left(\frac{x}{p_r^{\beta+1}}\right)\right|$$
$$+ \frac{|R_m^{(2)}(x)|}{x^2} + \left(p_r^2 - 1\right)\left(\frac{1}{p_r^2}\frac{\left|R_m^{(2)}\left(\frac{x}{p_r}\right)\right|}{\left(\frac{x}{p_r}\right)^2} + \frac{1}{p_r^4}\frac{\left|R_m^{(2)}\left(\frac{x}{p_r^2}\right)\right|}{\left(\frac{x}{p_r^2}\right)^2} + \cdots + \frac{1}{p_r^{2\beta}}\frac{\left|R_m^{(2)}\left(\frac{x}{p_r^\beta}\right)\right|}{\left(\frac{x}{p_r^\beta}\right)^2}\right)$$

We shall take any $\varepsilon > 0$. Since $R_k^{(2)}(x) = O_k(x^2)$, for sufficiently large integer β (keeping $p_r^{\beta+1} \le x$, so x is also sufficiently large) we have

$$\frac{k}{3x} + \frac{1}{x^2}\left|R_k\left(\frac{x}{p_r^{\beta+1}}\right)\right| + \frac{p_r^2}{x^2}\left|R_m^{(2)}\left(\frac{x}{p^{\beta+1}}\right)\right| \le \varepsilon.$$

By the assumption of induction, for the same $\varepsilon > 0$ we have

$$\frac{|R_m^2(x)|}{x^2} \le \frac{p^2}{p^2+p+1}\frac{d(m)}{2} + \varepsilon \quad \textit{for any } x \ge x_0.$$

So, for sufficiently large x satisfying $x/p_r^{\beta} \ge x_0$ we observe that

$$\begin{aligned}
&\frac{|R_m^{(2)}(x)|}{x^2} + (p_r^2-1)\left(\frac{1}{p_r^2}\frac{\left|R_m^{(2)}\left(\frac{x}{p_r}\right)\right|}{\left(\frac{x}{p_r}\right)^2} + \frac{1}{p_r^4}\frac{\left|R_m^{(2)}\left(\frac{x}{p_r^2}\right)\right|}{\left(\frac{x}{p_r^2}\right)^2} + \cdots + \frac{1}{p_r^{2\beta}}\frac{\left|R_m^{(2)}\left(\frac{x}{p_r^\beta}\right)\right|}{\left(\frac{x}{p_r^\beta}\right)^2}\right)\\
&\le \frac{p^2}{p^2+p+1}\frac{d(m)}{2} + \varepsilon\\
&\quad + (p_r^2-1)\left(\left(\frac{1}{p_r^2} + \frac{1}{p_r^4} + \cdots + \frac{1}{p_r^{2\beta}}\right)\frac{p^2}{p^2+p+1}\frac{d(m)}{2} + \left(\frac{1}{p_r^2} + \frac{1}{p_r^4} + \cdots + \frac{1}{p_r^{2\beta}}\right)\varepsilon\right)\\
&\le \frac{p^2}{p^2+p+1}(1+1)\frac{d(m)}{2} + 2\varepsilon\\
&= \frac{p^2}{p^2+p+1}\frac{d(k)}{2} + 2\varepsilon.
\end{aligned}$$

Therefore, we see that for any $\varepsilon > 0$ there exists a sufficiently large $x_1 > 0$ satisfying

$$\frac{\left|R_k^{(2)}(x)\right|}{x^2} \le \frac{p^2}{p^2+p+1}\frac{d(k)}{2} + \varepsilon \quad \textit{for any } x \ge x_1,$$

which shows the assertion (8) is true for $v(k) = r$. □

5 A generalization of Joshi and Vaidya's theorem

Finally, as we mentioned in the introduction, using the method of [10] we shall show a generalization of (3) (when $k \ge 2$ is a fixed square-free integer). In the following theorem, if we set $r = 1$ we obtain Theorem 1.1 of [10], and if we set $r = 2$ we get (3).

Theorem 5.1 (*cf. Joshi and Vaidya* [10, *Theorem* 1.1]) *Let $r \ge 1$ be a fixed integer and $k \ge 2$ be a fixed square-free integer. Moreover, we write $F_k^{(r)}(x) = \sum_{n \le x} \delta_k^r(n)$. Then, we have*

$$F_k^{(r)}(x) = \frac{k^r}{(r+1)\sigma(k^r)}x^{r+1} + O_k(x^r) \quad (x \to \infty), \tag{24}$$

where $\sigma(m) = \sum_{d|m} d$.

Proof We shall prove the formula (24) by the induction with respect to the number of distinct prime factors of k. First, in the case of $v(k) = 1$ we shall deduce

$$F_p^{(r)}(x) = \frac{p^r}{(r+1)\sigma(p^r)}x^{r+1} + O(x^r). \tag{25}$$

By Lemma 2.1 (i), (iii) we observe that

$$
\begin{aligned}
F_p^{(r)}(x) &= \sum_{\substack{n\le x\\ p\mid n}} \delta_p^r(n) + \sum_{\substack{n\le x\\ p\nmid n}} \delta_p^r(n) \\
&= \sum_{m\le x/p} \delta_p^r(m) + \sum_{n\le x} n^r - \sum_{\substack{n\le x\\ p\mid n}} n^r.
\end{aligned} \tag{26}
$$

Since, by partial summation it is easily seen that

$$
\sum_{n\le x} n^r = \frac{1}{r+1} x^{r+1} + O(x^r), \tag{27}
$$

we obtain

$$
\sum_{\substack{n\le x\\ p\mid n}} n^r = p^r \sum_{m\le x/p} m^r = \frac{1}{(r+1)p} x^{r+1} + O(x^r). \tag{28}
$$

Hence, by (26), (27), and (28) we have

$$
F_p^{(r)}(x) - F_p^{(r)}\left(\frac{x}{p}\right) = \frac{1}{r+1}\left(1-\frac{1}{p}\right)x^{r+1} + O(x^r) \tag{29}
$$

which is a generalization of (2.1) in [10, p. 793]. As for (29), by a similar argument to obtain (22) we get

$$
\begin{aligned}
F_p^{(r)}(x) &= \frac{1}{r+1}\left(1-\frac{1}{p}\right)\left(1+\frac{1}{p^{r+1}}+\frac{1}{(p^{r+1})^2}+\cdots+\frac{1}{(p^{r+1})^\alpha}\right)x^{r+1} \\
&\quad + O\left(x^r\left(1+\frac{1}{p^r}+\cdots+\frac{1}{(p^r)^\alpha}\right)\right) \\
&= \frac{1}{(r+1)}\frac{p^r}{p^r+p^{r-1}+\cdots+1}x^{r+1} + O(x^r)
\end{aligned}
$$

which implies (25), here α is the positive integer satisfying $p^\alpha \le x$ and $p^{\alpha+1} > x$.

Since the formula (25) has been shown, next we assume that the assertion (24) is true for $k = p_1 p_2 \cdots p_l$ (here p_i $(i = 1, \ldots, l)$ are distinct primes). We shall consider the case $v(k) = l+1$, $k = p_1 \cdots p_{l+1} \cdot p_{l+1} = Kp_{l+1}$. By an argument similar to the case $k = p$ we obtain

$$
F_{Kp_{l+1}}^{(r)}(x) = F_{Kp_{l+1}}^{(r)}\left(\frac{x}{p_{l+1}}\right) + F_K^{(r)}(x) - p_{l+1}^r \sum_{m\le x/p_{l+1}} \delta_K^r(m)
$$

and by the assumption of the induction we get

$$
F_{Kp_{l+1}}^{(r)} - F_{Kp_{l+1}}^{(r)}\left(\frac{x}{p_{l+1}}\right) = \frac{K^r}{(r+1)\sigma(K^r)}\left(1-\frac{1}{p_{l+1}}\right)x^r + O(x^r). \tag{30}
$$

By a similar method in the proof of Theorem 4.1 (22) to the above (30), we have

$$
\begin{aligned}
F_{Kp_{l+1}}^{(r)}(x) &= \frac{K^r}{(r+1)\sigma(K^r)}\left(1-\frac{1}{p_{l+1}}\right)x^{r+1}\left(\frac{p_{l+1}^{r+1}}{p_{l+1}^{r+1}-1}+O\left(\frac{1}{x^{r+1}}\right)\right) + O(x^r) \\
&= \frac{(Kp_{l+1})^r}{(r+1)\sigma((Kp_{l+1})^r)}x^{r+1} + O(x^r).
\end{aligned}
$$

Therefore, we reach the assertion (24). □

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