Indian J Pure Appl Math (2021) 52:1004–1020
https://doi.org/10.1007/s13226-021-00042-7

**ORIGINAL RESEARCH**

# Formal Schemes of Rational Degree

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Received: 29 April 2020 / Accepted: 9 September 2020 / Published online: 21 June 2021


**Abstract** Formal schemes in algebraic geometry consist of power series with integer degree, this idea can be naturally carried over to power series with rational degree. In this paper formal schemes with rational degree are constructed. These schemes are non noetherian and thus require slight modification of the standard approach. The first part of the paper constructs non noetherian continuous valuation rings from discrete valuation rings, and these rings are reffered to as 'eka' (one in Hindi) rings. These new rings are designed to carry the properties of discrete valuation to continuous valuation faithfully. The second part of the paper constructs non noetherian formal schemes with rational degree and shows their admissibility. The corresponding flatness and coherence is proved. Finally line bundles of rational degree are constructed and their Čech cohomology computed.

## 1 Introduction

The algebraic geometry over the polynomial ring $R[X]$ is well understood. This idea can be naturally carried over to rational degree polynomials for example $X^{1/2}+1$, and the ring of such polynomials is denoted by $R[X]_{\mathbb{Q}}$. The algebraic geometry of such rings and their line bundles is studied in [1].

In this manuscript we want to study the rational degree analogue of power series ring $R[[X]]$. First, note that it is possible to construct rational degree power series as a group, denoted by $R[[X]]_{\mathbb{Q}}$, with addition as the group operation and with elements of the form $1+X+X^{1/2}+X^{1/3}+\ldots+X^{1/i}+\ldots$. The terms in $f\in R[[X]]_{\mathbb{Q}}$ can be ordered by degree of constituent monomials, using the natural ordering of rational numbers (rational numbers are countable). But, $R[[X]]_{\mathbb{Q}}$ is *not* a ring, see the counter example 4.2.1. In order to construct rings of rational degree power series, ideas of formal schemes and $\pi$ adic convergence are required and these rings will be denoted by $R\langle X\rangle_{\Delta}$.

The first part of the paper constructs continuous valuation rings from discrete valuation rings by attaching roots of a single uniformizer $\pi$. These rings will be called 'eka' rings. The word 'eka' means one in Hindi and refers to a single uniformizer that underlies the continuous valuation ring and forms the neighborhood of zero. The linear topology allows us to transfer this neighborhood to all other elements. The eka rings are continuous valuation avatar of discrete valuation rings but carry over most of the properties faithfully. The equivalence between the discrete valuation rings and their continuous avatar is captured in the following result.

Communicated by Indranil Biswas.

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**Proposition 2.5** *Let* $\mathtt{A}$ *be catgory of discrete valuation rings and morphisms as local homomorphisms and* $\mathcal{E}$ *a functor which attaches dth roots of the uniformizer (a direct limit as constructed in 2.1). Let the resulting category be denoted as* $\mathtt{A}_d$ *with morphisms as local homomorphisms. Then the following results hold*

1. $\mathcal{E}$ *is an equivalence of categories.*
2. $\mathcal{E}$ *preserves unramified maps.*
3. $\mathcal{E}$ *preserves flat maps.*
4. $\mathcal{E}$ *preserves etale maps.*
5. $\mathcal{E}$ *preserves separatedness.*
6. $\mathcal{E}$ *preserves integral closure.*
7. $\mathcal{E}$ *preserves Krull dimension.*
8. $\mathcal{E}$ *preserves units.*

The eka construction is exactly the same as attaching rational degrees. The power series ring $R\langle X\rangle_{\Delta}$ is Non-Noetherian, furthermore if the base ring $R$ eka it is Non-Noetherian too. Therefore, this manuscript gives a natural theory of Non-Noetherian formal schemes.

The restricted power series with rational degree are constructed in 4.1, and the admissibility of these rings is shown in the following lemma.

**Lemma 4.6** *Let* $R$ *be an admissible ring with ideal of definition* $\mathfrak{a}$, *then the ring* $A = R\langle X_1, \ldots, X_r\rangle_{\Delta}$ *where* $\Delta = \mathbb{Q}$ *or* $\mathbb{Z}[1/d]$ *admissible.*

The admissibility above allows us to apply the theory of formal schemes and the admissible formal blow up can be defined naturally see section 5.1. The Non-Noetherian flatness is described in the following lemma.

**Lemma 4.10** *Let* $A$ *be an* $\mathrm{eka}^{d}$ *or* $\mathrm{eka}^{\mathbb{Q}}$ *with uniformizer* $\pi$ *and* $R = A[X_0, \ldots, X_n]$ *or* $A[X_0, \ldots, X_n]_{\Delta}$ *where* $\Delta = \mathbb{Q}$ *or* $\mathbb{Z}[1/p]$. *Let* $J$ *be an ideal of* $R$ *and* $J^{\wedge}$ *and* $R^{\wedge}$ *denote the completion with respect to* $\pi$, *that is*

$$J^{\wedge} = \varprojlim_{i} J/\pi^{i}J = \varprojlim_{i} (J \otimes_R R/\pi^{i}R), \tag{1.1}$$

*then the following hold.*

1. $J \otimes_R R^{\wedge} = J^{\wedge}$.
2. *the ring map* $R \to R^{\wedge}$ *is flat.*

The coherence of rational degree is shown in the following theorem below.

**Theorem 6.1** *Let* $R$ *be an* eka *ring and* $K = R[1/\pi]$ *where* $\pi$ *is the uniformizer in* $R$. *Then the ring is* $R[X_0, \ldots, X_n]_{\Delta}$ *coherent.*

The cohomology of restricted power series with rational degree is computed in the following theorems.

**Theorem 7.5** *Let* $S = R\langle X_0, \ldots, X_n\rangle_{\Delta}$ *and* $X = \operatorname{Proj} S$, *then* $H^i(X, \mathcal{O}_X(m)) = 0$ *for* $0 < i < n$.

**Theorem 7.7** *Let* $S = R\langle X_0, \ldots, X_n\rangle_{\Delta}$ *and* $X = \operatorname{Proj} S$, *then for any* $n \in \Delta$

1. *There is an isomorphism* $S \simeq \oplus_{n\in\Delta} H^0(X, \mathcal{O}_X(n))$.
2. $H^n(X, \mathcal{O}_X(-n-1))$ *is a free module of infinite rank.*

## 2 Discrete to Continuous Valuation

The following constructions are the heart of this manuscript. The word ‘eka’ means one in hindi and refers to a single uniformizer $\pi$ which is modified to get continuous valuations.

Construction 2.1 Let $R$ be a discrete valuation ring with uniformizer $\pi$.

1. Attach all the $d$th roots of $\pi$ as below

$$
\begin{aligned}
&R \hookrightarrow R[\pi^{1/d}] \hookrightarrow R[\pi^{1/d^2}] \hookrightarrow \ldots \hookrightarrow \varinjlim_i R[\pi^{1/d^i}] = \bigcup_i R[\pi^{1/d^i}], \\
&\operatorname{eka}^d R := \varinjlim_i R[\pi^{1/d^i}] = \bigcup_i R[\pi^{1/d^i}].
\end{aligned} \tag{2.1}
$$

The above process can be described algebraically as

$$
\operatorname{eka}^d R := \frac{R[X, X_1, X_2, \ldots, X_i, \ldots]}{\langle X - \pi, X_1^d - X, X_2^d - X_1, \ldots, X_i^d - X_{i-1}, \ldots \rangle}.
$$

Another notation for $\operatorname{eka}^d R$ is $R[\pi^{1/d^\infty}]$ or $R[\pi^{d^{-\infty}}]$.

2. Attach all the rational roots of $\pi$

$$
\begin{aligned}
&R \hookrightarrow R[\pi^{1/2}] \hookrightarrow R[\pi^{1/2}, \pi^{1/3}] \hookrightarrow \ldots \hookrightarrow \varinjlim_i R[\pi^{1/2}, \pi^{1/3}, \ldots, \pi^{1/i}] \\
&= \bigcup_i R[\pi^{1/2}, \pi^{1/3}, \ldots, \pi^{1/i}], \\
&\operatorname{eka}^{\mathbb{Q}} R := \varinjlim_i R[\pi^{1/2}, \pi^{1/3}, \ldots, \pi^{1/i}] = \bigcup_i R[\pi^{1/2}, \pi^{1/3}, \ldots, \pi^{1/i}].
\end{aligned} \tag{2.2}
$$

The above process can be described algebraically as

$$
\operatorname{eka}^{\mathbb{Q}} R := \frac{R[X_1, X_2, \ldots, X_i, \ldots]}{\langle X_1 - \pi, X_2^2 - X_1, \ldots, X_i^i - X_1, \ldots \rangle}.
$$

***Example 2.2***

(i) Let $R = \mathbb{Z}_p$ the ring of $p$ adic integers and $d = p$, then $\operatorname{eka}^p R = \mathbb{Z}_p[p^{1/p^\infty}]$, the $p$ adic completion of this ring yields $\mathbb{Z}_p[p^{1/p^\infty}]^\wedge$, which is the ring of integers of the perfectoid field $\mathbb{Q}_p(p^{1/p^\infty})^\wedge$. Note that $\mathbb{Z}_p[p^{1/p^\infty}] \neq \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$.

(ii) Let $R = \mathbb{Z}_p$ the ring of $p$ adic integers, then the value group of $\operatorname{eka}^{\mathbb{Q}} R$ is $\mathbb{Q}$ which is same as the value group of $\mathbb{C}_p$.

The valuation rings and their associated fields are very closely related as shown below.

**Proposition 2.3** *Let $R$ be a valuation ring of arbitary height, and $\pi \in \mathfrak{m}_R \backslash \{0\}$, then the following are equivalent*

a) *$R$ is $\pi$ adically separated, that is $\cap_{i \in \mathbb{Z}_{>0}} \pi^i = 0$.*

b) *$R\left[\dfrac{1}{\pi}\right]$ is a field (thus, fraction field of $R$).*

*Proof* See [5, page 135, proposition 6.7.2]. □

The above proposition is used critically for the following result.

**Proposition 2.4** *Let $\operatorname{eka}^d R$ and $\operatorname{eka}^{\mathbb{Q}} R$ be as in construction 2.1, then the following hold*

(i) *The rings $\operatorname{eka}^d R$, $\operatorname{eka}^{\mathbb{Q}} R$ are non notherian.*
(ii) *The maximal ideals of $\operatorname{eka}^d R$, $\operatorname{eka}^{\mathbb{Q}} R$ are given as $\bigcup_i [\pi^{1/d^i}], \bigcup_i [\pi^{1/2}, \pi^{1/3}, \ldots, \pi^{1/i}]$ respectively.*
(iii) *The rings $\operatorname{eka}^d R$, $\operatorname{eka}^{\mathbb{Q}} R$ have Krull dimension one.*
(iv) *The rings $\operatorname{eka}^d R$, $\operatorname{eka}^{\mathbb{Q}} R$ are not regular.*
(v) *The rings $\operatorname{eka}^d R$, $\operatorname{eka}^{\mathbb{Q}} R$ are separated.*
(vi) *The* $\operatorname{Frac}(\operatorname{eka}^d R)$ and $\operatorname{Frac}(\operatorname{eka}^{\mathbb{Q}} R)$ *are* $\operatorname{eka}^d R[1/\pi]$ *and* $\operatorname{eka}^{\mathbb{Q}} R[1/\pi]$ *respectively.*
(vii) *The* $\operatorname{Frac}(\operatorname{eka}^d R)$ and $\operatorname{Frac}(\operatorname{eka}^{\mathbb{Q}} R)$ *have continuous valuation $\mathbb{Z}[1/d]$ and $\mathbb{Q}$ respectively.*

*Proof* Let $R$ be a discrete valuation ring with uniformizer $\pi$ and residue field $\kappa = R/\langle \pi \rangle$.

(i) This follows from the non terminating chain of ideals

$$\langle\pi\rangle \subset \langle\pi^{1/d}\rangle \subset \langle\pi^{1/d^2}\rangle \subset \cdots$$

(ii) The only elements added to $R$ are the rational roots of $\pi$, thus modding out what is added gives the same field.

$$\kappa = \frac{R}{\langle\pi\rangle} = \frac{\mathrm{eka}^{\,d}R}{\bigcup_i [\pi^{1/d^i}]} = \frac{\mathrm{eka}^{\,\mathbb{Q}}R}{\bigcup_i [\pi^{1/2}, \pi^{1/3}, \ldots, \pi^{1/i}]}.$$

Since, $\kappa$ is a field the result follows. Furthermore note that

$$\kappa = \frac{R}{\langle\pi\rangle} = \frac{R[\pi^{1/d}]}{\langle\pi^{1/d}\rangle} = \ldots = \frac{R[\pi^{1/d^i}]}{\langle\pi^{1/d^i}\rangle} = \ldots$$

(iii) The Krull dimension of a discrete valuation ring $R$ is one and is given as $0 \subset \langle\pi\rangle \subset A$, this transforms to $0 \subset \cup_i\langle\pi^{1/d^i}\rangle \subset \mathrm{eka}^{\,d}R$, thus preserving Krull dimension. Note that $\langle\pi\rangle$ is an ideal in $\mathrm{eka}^{\,d}R$ but is no longer prime since $\pi^{1/d}\cdot\pi^{1-1/d} \in \langle\pi\rangle$, but neither $\pi^{1/d}$ nor $\pi^{1-1/d}$ are in $\langle\pi\rangle$. Notice that $\mathrm{rad}\,\langle\pi\rangle = \cup_i\langle\pi^{1/d^i}\rangle$.

(iv) Let $\mathfrak{m}$ be $\bigcup_i[\pi^{1/d^i}]$ or $\bigcup_i[\pi^{1/2}, \pi^{1/3}, \ldots, \pi^{1/i}]$, then $\mathfrak{m} = \mathfrak{m}^2$, giving $\mathfrak{m}/\mathfrak{m}^2 = 0$, but Krull dimension of $\mathrm{eka}^{\,d}R$ and $\mathrm{eka}^{\,\mathbb{Q}}R$ is one, implying the result.

(v) Separatedness in $R$ means $\cap_{i\in\mathbb{Z}_{>0}}\pi^i = 0$. It can be expressed as a descending sequence

$$\pi \supset \pi^2 \supset \pi^3 \supset \cdots \supset \bigcap_{i\in\mathbb{Z}_{>0}} \pi^i = 0. \tag{2.3}$$

The corresponding object is $\mathrm{eka}^{\,d}R$ will have the descending sequence

$$\cdots \supset \pi^{1/d^2} \supset \pi^{1/d} \supset \pi \supset \pi^2 \supset \pi^3 \supset \cdots \supset \bigcap_i \pi^i = 0. \tag{2.4}$$

with addition of elements at top of the chain, thus the intersection is still zero. Similar, argument follows for $\mathrm{eka}^{\,\mathbb{Q}}R$ from the descending sequence

$$\cdots \supset \langle\pi^{1/2}, \pi^{1/3}\rangle \supset \langle\pi^{1/2}\rangle \supset \pi \supset \pi^2 \supset \pi^3 \supset \cdots \supset \bigcap_i \pi^i = 0. \tag{2.5}$$

(vi) Follows from above and proposition 2.3.

(vii) Let $u, \pi^{1/d^i} \in \mathrm{eka}^{\,d}R$ consider

$$\frac{u}{\pi^{1/d^i}} = \frac{u}{\pi^{1/d^i}}\cdot\frac{\pi^{1-1/d^i}}{\pi^{1-1/d^i}} = \frac{u\cdot\pi^{1-1/d^i}}{\pi} \in \mathrm{Frac}\,(\mathrm{eka}^{\,d}R) = \mathrm{eka}^{\,d}R[1/\pi] \tag{2.6}$$

If $u$ is a unit $v(u/\pi^{1/d^i}) = -1/d^i$ and $v$ is valuation. Hence, proceeding exactly as in discrete valuation rings gives a surjective function $v : \mathrm{Frac}\,(\mathrm{eka}^{\,d}R)^\times \to \mathbb{Z}[1/d]$ and $v : \mathrm{Frac}\,(\mathrm{eka}^{\,\mathbb{Q}}R)^\times \to \mathbb{Q}$

□

It is also possible to compare morphisms between discrete valuation rings and their continous avatar. This is explored in the next result.

**Proposition 2.5** *Let* $\mathtt{A}$ *be catgory of discrete valuation rings and morphisms as local homomorphisms and* $\mathcal{E}$ *a functor which attaches dth roots of the uniformizer ( a direct limit as constructed in 2.1). Let the resulting category be denoted as* $\mathtt{A}_d$ *with morphisms as local homomorphisms. Then the following results hold*

1. $\mathcal{E}$ *is an equivalence of categories.*
2. $\mathcal{E}$ *preserves unramified maps.*
3. $\mathcal{E}$ *preserves flat maps.*
4. $\mathcal{E}$ *preserves etale maps.*
5. $\mathcal{E}$ *preserves separatedness.*
6. $\mathcal{E}$ *preserves integral closure.*
7. $\mathcal{E}$ *preserves Krull dimension.*
8. $\mathcal{E}$ *preserves units.*

*Proof*

1. In order to show equivalence of categories one needs to show essential surjectivity, fullness and faithfulness.

   **essential surjectivity** This follows from construction, the objects of $\mathtt{A}_d$ are obtained from $\mathtt{A}$ by applying $\mathcal{E}$.
   **full and faithful** Let $R \to R'$ be a map in category $\mathtt{A}$, then the local homomorphism condition forces $\langle\pi\rangle \mapsto \langle\pi'\rangle$ where $\pi, \pi'$ are uniformizers of the rings $R, R'$. This precisely corresponds to the map $\mathrm{eka}^d R \to \mathrm{eka}^d R'$ in category $\mathtt{A}_d$ with local homomorphism

$$\langle \pi, \pi^{1/d}, \pi^{1/d^2}, \ldots \rangle \to \langle \pi', \pi'^{1/d}, \pi'^{1/d^2}, \ldots \rangle. \tag{2.7}$$

Let $a \in R \backslash \langle\pi\rangle, a' \in R' \backslash \langle\pi'\rangle$ and $a \mapsto a' \in \mathrm{Mor}(R, R')$, then $a \mapsto a' \in \mathrm{Mor}(\mathrm{eka}^d R, \mathrm{eka}^d R')$. Since, the rings $R, R'$, $\mathrm{eka}^d R$, $\mathrm{eka}^d R'$ are local rings, the elements either lie in the maximal ideal or do not (hence are units). We have covered both the cases for morphisms and shown there is one to one correspondence.

2. Let $\phi : R \to R'$ be a morphism in $\mathtt{A}$ and $\pi, \pi'$ be uniformizers of $R$ and $R'$ respectively. Conisder the residue fields $\kappa = R/\langle\pi\rangle$ and $\kappa' = R'/\langle\pi'\rangle$. The map $\phi$ is unramified if $\kappa'$ is a finite separable extension of $\kappa$. Applying the functor $\mathcal{E}$ gives the morphism $\mathrm{eka}^d R \to \mathrm{eka}^d R'$ but the residue fields remain the same since

$$\kappa = \frac{R}{\pi} = \frac{\bigcup_i R[\pi^{1/d^i}]}{\bigcup_i \pi^{1/d^i}}, \tag{2.8}$$

and thus the separable extension remains as such.

3. Let $\phi : R \to R'$ be flat in $\mathtt{A}$. Since, $\mathcal{E}$ is a direct limit functor it preserves flatness. Another direct proof can be given by observing that flatness implies injectivity $\langle\pi\rangle \otimes_R R' \hookrightarrow R'$, applying the direct limit gives the injective map

$$\overset{lim}{\underset{i}{\to}} \langle \pi^{1/d^i} \rangle \otimes_{\overset{lim}{\underset{i}{\to}} R[\pi^{1/d^i}]} \overset{lim}{\underset{i}{\to}} R'[\pi'^{1/d^i}] \hookrightarrow \overset{lim}{\underset{i}{\to}} R'[\pi'^{1/d^i}], \tag{2.9}$$

implying flatness of $\mathrm{eka}^d R \to \mathrm{eka}^d R'$ where $\mathrm{eka}^d R' = \overset{lim}{\underset{i}{\to}} R'[\pi'^{1/d^i}]$.

4. Let $\phi : R \to R'$ be etale in $\mathtt{A}$, that is $\phi$ is flat and unramified. Since, unramified and flat maps are preseved by $\mathcal{E}$, the result follows.
5. Follows from proposition 2.4 , (v).
6. Let $R$ be a discrete valuation ring, it is well known that it is integrally closed. The $\mathrm{eka}^d R$ avatar of $R$ is obtained via a series of algebraic extensions

$$A[\pi^{1/d}] = \frac{A[X]}{X^d - \pi}, A[\pi^{1/d^2}] = \frac{A[X]}{X^{d^2} - \pi}, \ldots, A[\pi^{1/d^i}] = \frac{A[X]}{X^{d^i} - \pi}, \ldots. \tag{2.10}$$

Since, every added element (that is $\pi^{1/d^i}$) is integral, the ring $\mathrm{eka}^d R$ is integral.

7. Follows from proposition 2.4 , (iii).
8. Let $u \in R$ be a unit, then $u \in R \backslash \langle\pi\rangle$. The functor $\mathcal{E}$ carries $u \in R$ to $u \in \mathrm{eka}^d R$ such that $u \in \mathrm{eka}^d R \backslash \cup_i \pi^{1/d^i}$ (the functor preserves everything except the maximal ideal). This implies that $u$ is a unit, since non units form the maximal ideal in the local ring.

□

***Remark 2.6*** The above proof carries over word for word for the rings $\mathrm{eka}^{\mathbb{Q}} R$.

## 3 Complete Valuation Rings

**Definition 3.1** The rings $\mathrm{eka}^d R^\wedge$ and $\mathrm{eka}^{\mathbb{Q}} R^\wedge$ will denote the $\pi$ adic separated completion of continuous valuation rings $\mathrm{eka}^d R$ and $\mathrm{eka}^{\mathbb{Q}} R$.

Affirmation 3.2

1. The rings $\mathrm{eka}^{d}R^{\wedge}$ and $\mathrm{eka}^{\mathbb{Q}}R^{\wedge}$ are linearly topologized with ideal of definition $\langle\pi\rangle$, and thus $\{\pi^n\}_{n\geq 0}$ forms a fundamental system of neighbourhoods of 0.
2. The rings $\mathrm{eka}^{d}R^{\wedge}$ and $\mathrm{eka}^{\mathbb{Q}}R^{\wedge}$ are adic and admissible.

### 3.1 Continuous Maps

The $\mathrm{eka}^{d}R^{\wedge}$ give the simplest examples of continuous homomorphisms between linearly topologized rings.

***Example 3.3***

1. $\mathrm{eka}^{3}R^{\wedge} \to \mathrm{eka}^{6}R^{\wedge}$, the ring $\mathrm{eka}^{6}R^{\wedge}$ dominates $\mathrm{eka}^{3}R^{\wedge}$ and the fundamental neighbourhoods of zero are generated by $\langle\pi\rangle$.
2. $\mathrm{eka}^{2}R^{\wedge} \to \mathrm{eka}^{6}R^{\wedge}$, the ring $\mathrm{eka}^{6}R^{\wedge}$ dominates $\mathrm{eka}^{3}R^{\wedge}$ and the fundamental neighbourhoods of zero are generated by $\langle\pi\rangle$.
3. If $n_1|n_2$, then we have a continuous homomorphism $\mathrm{eka}^{n_1}R^{\wedge} \to \mathrm{eka}^{n_2}R^{\wedge}$, the ring $\mathrm{eka}^{n_2}R^{\wedge}$ dominates $\mathrm{eka}^{n_1}R^{\wedge}$ and the fundamental neighbourhoods of zero are generated by $\langle\pi\rangle$.

Counterexample 3.4 We do not have a homomorphism from $\mathrm{eka}^{2}R^{\wedge} \to \mathrm{eka}^{3}R^{\wedge}$ since, any such non trivial homomorphisms would force $\pi^{1/2}\mapsto\pi^{1/3}$ or $\pi\mapsto\pi^{2/3}$ a contradiction. Notice that the fundamental neighbourhoods of zero are still generated by $\langle\pi\rangle$.

***Remark 3.5***

(i) Let $R^{\wedge}$ be a complete discrete valuation ring, then it is possible to form the co-cartesian square as follows.

$$\begin{tikzcd} R^{\wedge} \arrow[r] \arrow[d] & \mathrm{eka}^{d_1} R^{\wedge} \arrow[d] \\ \mathrm{eka}^{d_2} R^{\wedge} \arrow[r] & \mathrm{eka}^{d_1} R^{\wedge} \otimes_{R^{\wedge}} \mathrm{eka}^{d_2} R^{\wedge} \end{tikzcd} \tag{3.1}$$

The topology is $\pi$ adic for all the rings.

(ii) It is also possible to directly form the co-cartesian square, where $\ell = \mathrm{lcm}(d_1, d_2)$,

$$\begin{tikzcd} R^{\wedge} \arrow[r] \arrow[d] & \mathrm{eka}^{d_1} R^{\wedge} \arrow[d] \\ \mathrm{eka}^{d_2} R^{\wedge} \arrow[r] & \mathrm{eka}^{\ell} R^{\wedge} \end{tikzcd} \tag{3.2}$$

for example,

$$\begin{tikzcd} R^{\wedge} \arrow[r] \arrow[d] & \mathrm{eka}^{2} R^{\wedge} \arrow[d] \\ \mathrm{eka}^{3} R^{\wedge} \arrow[r] & \mathrm{eka}^{6} R^{\wedge} \end{tikzcd} \tag{3.3}$$

## 4 Formal Schemes

In this section let $A = \mathrm{eka}^{d}R^{\wedge}$ or $A = \mathrm{eka}^{\mathbb{Q}}R^{\wedge}$, then we can consider the following constructions associated with the polynomial algebra $A[X_0, \ldots, X_n]$.

### 4.1 Poynomial Algebras

Construction 4.1

1. Consider the inclusions $L_0 \subset L_1 \subset L_2 \subset \ldots L_i \subset \cdots$ where

$$\begin{aligned} L_0 &= A[X_0, \ldots, X_n] \\ L_1 &= A[X_0^{1/d}, \ldots, X_n^{1/d}] \\ L_2 &= A[X_0^{1/d^2}, \ldots, X_n^{1/d^2}] \\ \vdots &= \qquad \vdots \\ L_i &= A[X_0^{1/d^i}, \ldots, X_n^{1/d^i}] \\ \vdots &= \qquad \vdots \\ \varinjlim_i L_i &= \bigcup_i A[X_0^{1/d^i}, \ldots, X_n^{1/d^i}] \\ A[X_0, \ldots, X_n]_{\mathbb{Z}[1/d]} &:= \bigcup_i A[X_0^{1/d^i}, \ldots, X_n^{1/d^i}] \end{aligned} \tag{4.1}$$

Another notation for $\varinjlim_i L_i$ is $A[X_0^{1/d^\infty}, \ldots, X_n^{1/d^\infty}]$.

2. Consider the following inclusions $Q_1 \subset Q_2 \subset \ldots Q_i \subset \cdots$ where $\ell_i = \mathrm{lcm}(1, 2, \ldots, i)$

$$\begin{aligned} Q_1 &= A[X_0, \ldots, X_n] \\ Q_2 &= A[X_0^{1/2}, \ldots, X_n^{1/2}], \quad \ell_2 = \mathrm{lcm}(1, 2) = 2 \\ Q_3 &= A[X_0^{1/6}, \ldots, X_n^{1/6}], \quad \ell_3 = \mathrm{lcm}(1, 2, 3) = 6 \\ \vdots &= \qquad \vdots \\ Q_i &= A[X_0^{1/\ell^i}, \ldots, X_n^{1/\ell^i}] \\ \vdots &= \qquad \vdots \\ \varinjlim_i Q_i &= \bigcup_i A[X_0^{1/\ell^i}, \ldots, X_n^{1/\ell^i}] \\ A[X_0, \ldots, X_n]_{\mathbb{Q}} &:= \bigcup_i A[X_0^{1/\ell^i}, \ldots, X_n^{1/\ell^i}] \end{aligned} \tag{4.2}$$

The ring $A[X_0, \ldots, X_n]_{\mathbb{Q}}$ contains all rational powers.

**Lemma 4.2** *Let $\Delta = \mathbb{Q}$ or $\mathbb{Z}[1/d]$ and $k$ a field, then the ring $k[X_0, \ldots, X_n]_{\mathbb{Q}}$ is non noetherian.*

*Proof* This follows from the non terminating sequence $\langle X_0 \rangle \subset \langle X_0^{1/d} \rangle \cdots \subset \langle X_0^{1/d^i} \rangle \subset \cdots$. □

### 4.2 Restricted Power series

The construction of Restricted power series is given in [6, § 7.5] or [4, pp 212-213].

Construction 4.3 The polynomial rings constructed in 4.1 can be completed $\pi$ adically to get restricted power series. These series are analogues of Tate algebras with integer degrees as described in [3, pp 13].

1. $A\langle X_0, \ldots, X_n\rangle_{\mathbb{Z}[1/d]} := \overset{\lim}{\longleftarrow} A[X_0, \ldots, X_n]_{\mathbb{Z}[1/d]}/\pi^i$. More concretely, these are power series with elements of the form

$$\sum_{I \in \mathbb{Z}[1/p]^n_{\geq 0}} a_I \xi^I \text{ such that } \forall n \geq 0, \forall^{\text{almost}}, \quad v_\pi(a_I) > n, a_I \in A, \tag{4.3}$$

where $\xi$ represent $\{X_0, \ldots, X_n\}$. In words the above equation means that except for a finitely many $a_I$, all other $a_I$ have high $\pi$ adic valuation (or converge $\pi$ adically). For example,

$$1 + \pi X^{1/d} + \pi^2 X^{1/d^2} + \ldots + \pi^i X^{1/d^i} + \ldots.$$

2. $A\langle X_0, \ldots, X_n\rangle_{\mathbb{Q}} := \overset{\lim}{\longleftarrow} A[X_0, \ldots, X_n]_{\mathbb{Q}}/\pi^i$. More concretely, these are power series with elements of the form

$$\sum_{I \in \mathbb{Q}^n_{\geq 0}} a_I \xi^I \text{ such that } \forall n \geq 0, \forall^{\text{almost}}, \quad v_\pi(a_I) > n, a_I \in A, \tag{4.4}$$

where $\xi$ represent $\{X_0, \ldots, X_n\}$. In words the above equation means that except for a finitely many $a_I$, all other $a_I$ have high $\pi$ adic valuation (or converge $\pi$ adically).For example,

$$1 + \pi X + \pi^2 X^{1/2} + \ldots + \pi^i X^{1/i} + \ldots.$$

***Remark 4.4***

(a) The monomial terms in the series can be ordered either using the lexicographic order or by using the standard ordering of rational numbers appearing as degree.

(b) Let $f \in k\langle X_0, \ldots, X_n\rangle_\Delta$ (where $\Delta = \mathbb{Q}$ or $\mathbb{Z}[1/d]$), then $f \mod \pi \in k[X_0, \ldots, X_n]_\Delta$, since all the terms in the tail have high $\pi$ adic valuation they vanish modulo $\pi$.

#### *4.2.1 Critical Counterexample*

The $\pi$ adic completion is critical to define power series ring. In particular there is no power series ring of the form $A[[X]]_{\mathbb{Q}}$ or $A[[X]]_{\mathbb{Z}[1/d]}$. To see the this consider the elements

$$\begin{aligned} f &= 1 + X^{1/p} + X^{1/p^2} + \ldots + X^{1/p^i} + \ldots \text{ and} \\ g &= 1 + X^{1-1/p} + X^{1-1/p^2} + \ldots + X^{1-1/p^i} + \ldots, \end{aligned} \tag{4.5}$$

then $f \cdot g$ will contain the term

$$\underbrace{(1 + 1 + \ldots + 1 + \ldots)}_{\text{infinitely many}} X \tag{4.6}$$

a term that does not make sense in any ring. The problem arises from the fact that the terms of $f$ and $g$ do not converge to zero $\pi$ adically. The coefficients $a_n$ in our example are all equal to 1, that is $a_n = 1$, thus $v_\pi(a_n) = 0$ or $|a_n|_\pi = |1|_\pi = 1$. Therefore, $\lim_{n\to\infty} |a_n|_\pi = 1 \not\to 0$. On the other hand the geometric series $1 + \pi + \pi^2 + \ldots + \pi^i + \ldots$ converges $\pi$ adically.

Notice that we can still consider $A[[X]]_{\mathbb{Q}}$ and $A[[X]]_{\mathbb{Z}[1/d]}$ as a group with group operation $+$.

### 4.3 Linear Topology

As is customary, the story begins by recalling a few facts from [6, pp. 60 §7](also denoted as EGA0 and EGA1).

**Definition 4.5** In the linearly topologised ring $A$, we say that an ideal $I$ is an ideal of definition if $I$ is open and if, for all neighborhoods $V$ of 0, there is an integer $n > 0$ such that $I^n \subset V$ (by abuse of language, we say that the sequence $(I^n)$ tends to 0). We say that linearly topologised ring $A$ is preadmissible if there exists in $A$ an ideal of definition; we say that A is admissible if it is preadmissible and if it is also separated and complete.

The linear topology allows to transfer neighborhoods of one point to another (linearly), thus only neighborhoods of zero (or some other fixed element) are required to define neighborhood of any other element.

*4.3.1 Examples*

1. $\mathbb{Z}_p$ has infinitely many ideals of definition $p^i\mathbb{Z}_p, i \in \mathbb{Z}_{>0}$ and the maximal among them is $p\mathbb{Z}_p$ (using $p$ adic convergence to zero or algebraically $p$ adic filtration).
2. $\mathbb{Z}_p[p^{1/p^\infty}]$ contains infinitely many ideals of definition $p^i\mathbb{Z}_p, i \in \mathbb{Z}[1/p]_{>0}$. The maximal ideal is $I = \cup_i p^{1/p^i}$ coming from the chain $p \subset p^{1/p} \subset p^{1/p^2} \subset \cdots$. But, the maximal ideal is not an ideal of definition, since $I^n = I$ and hence does not tend to zero.
3. $\mathbb{Z}_p[\zeta^{1/p^\infty}]$ contains infinitely many ideals of definition $p^i\mathbb{Z}_p, i \in \mathbb{Z}_{>0}$ ($p$ adic convergence) forming the neighborhood of zero. Similar to above $I = \cup_i \zeta^{1/p^i}$, where $\zeta$ is the root of unity, is not an ideal of definition (forming the neighborhood of one), since $I^n = I$ and it does not tend to one.

By virtue of corollary below the topologies are independent of the choice of ideal of definition $I$. For semi local rings it is customary to take ideal of definition of a topological ring $A$ as radical of $A$ denoted as $\operatorname{rad}(A)$. In this tract the focus is on linearly topologised rings with finitely generated ideal of definition.

**Corollary** *(Corollaire (7.1.8) [6, pp. 61]). If a preadmissible ring A is such that, for an ideal of definition I, the powers $I^n (n > 0)$ form a fundamental system of neighborhoods of 0, it is the same as the powers $I'^n$ of all ideal of definition $I'$ of A.*

**Lemma 4.6** *Let R be an admissible ring with ideal of definition $\mathfrak{a}$, then the ring $A = R\langle X_1, \ldots, X_r\rangle_\Delta$ where $\Delta = \mathbb{Q}$ or $\mathbb{Z}[1/d]$ admissible.*

*Proof* The first proof follows [6, §7.5]. $A$ is the sub ring of the group of formal series $R[[X_1, \ldots, X_r]]_\Delta$ formed by formal series $\sum_\alpha c_\alpha \xi^\alpha$ (with $\alpha = (\alpha_1, \ldots, \alpha_r) \in (\Delta^r_{\geq 0})$ such that $\lim c_\alpha = 0$; (ordering the monomials by rational degree). The neighborhoods of 0 can be defined explicitly in $A$. For all neighborhoods $V$ of 0 in $R$, let $V'$ be the set of $x = \sum_\alpha c_\alpha \xi^\alpha \in A$ such that $c_\alpha \in V$ for all $\alpha$. The $V'$ form a fundamental system of neighborhoods of 0 defining on $A$ a topology of a separated ring.

Another proof can be given by adapting the proof of Proposition 3 from [4, pp 213] by changing the map $\phi^{\mathfrak{a}}_{n_1,\ldots,n_r}$ with $(n_1, \ldots, n_r) \in \Delta_{\geq 0}$,

$$\phi^{\mathfrak{a}}_{n_1,\ldots,n_r} : (A/\mathfrak{a})[X_1, \ldots, X_r]_\Delta \to A/\mathfrak{a} \tag{4.7}$$

which maps every polynomial to the coefficient of $X_1^{n_1} \cdots X_r^{n_r}$ in this polynomial. The inverse limit is formed by $\mathfrak{a}$ over the neighborhood of zero as in the cited reference. □

**Corollary 4.7** *Let R be an admissible discrete valuation ring, then $\pi$ adically completed $\mathrm{eka}^d R$ and $\mathrm{eka}^{\mathbb{Q}} R$ are admissible.*

*Proof* The lemma 4.6 gives the admissibility of the rings $R\langle X\rangle_\Delta$, setting $X = \pi$ gives the result. □

Now that we have an admissible non noetherian ring, the results of [3, pp 158-160] can be adapted to the case at hand.

Notation 4.8

1. Let $\operatorname{Spf} A$ denote the set of open prime ideals of $A$, that is the prime ideals that contain the uniformizer $\pi$, then $\operatorname{Spf} A$ can be canonically identified with $\operatorname{Spec} A/\pi$.
2. For $f \in A$, there is a sheaf associated with the open subset $D(f)$ given as
$$D(f) \mapsto A\langle f^{-1}\rangle = \varprojlim_n (A/\pi^n[f^{-1}]). \tag{4.8}$$
3. Let $x \in \operatorname{Spf} A$ correspond to an open prime ideal $\mathfrak{p}_x$, then the stalk
$$\mathcal{O}_x = \varinjlim_{x \in D(f)} A\langle f^{-1}\rangle \text{ is a local ring with maximal ideal } \mathfrak{m}_x = \mathfrak{p}_x\mathcal{O}_x \tag{4.9}$$
4. Fibre product to two affine formal schemes $\operatorname{Spf} A$ and $\operatorname{Spf} B$ over $\operatorname{Spf} R$ is given by $\operatorname{Spf}(A\hat{\otimes}_R B)$ where $\hat{\otimes}$ denotes completed tensor product given as

$$A\hat{\otimes}_R B = \overset{\lim}{\underset{n}{\longleftarrow}}(A/\mathfrak{a}^n \otimes_R B/\mathfrak{b}^n), \tag{4.10}$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of definition of the ring $A$ and $B$ respectively. If these ideals are finitely generated then the ideal of definition of $A\hat{\otimes}_R B$ is $\mathfrak{a} \otimes_R B + A \otimes_R \mathfrak{b}$. For the case at hand if $A$ and $B$ are formal schemes obtained by $\pi$ adic completion of rational degree polynomial algebras, the ideal of definition becomes $\pi \otimes_R B + A \otimes_R \pi$.

***Remark 4.9*** Let $K$ be a perfectoid field with ring of integers $\mathfrak{o}_K$, then setting $\Delta = \mathbb{Z}[1/p]$ and $p$ adically completing the ring $\mathfrak{o}_K[X_1, \ldots, X_n]_{\mathbb{Z}[1/p]}$ leads to perfectoid Tate algebras. This theory can be developed without using Huber rings, Witt vectors or analysis and is presented in [2] (compare with [8]). Furthermore, a simple modulo $p$ map gives equivalence between categories of $\operatorname{char} 0$ and $\operatorname{char} p$.

## 4.4 Flatness

In this section we give a direct proof of flatness which does not rely on finiteness or noetherian conditions. The proof below relies on $\pi$ adic completion of eka rings and the fact that there is no nontrivial ideal in $\cap_i \pi^i = 0$.

**Lemma 4.10** *Let $A$ be an* eka$^d$ *or* eka$^{\mathbb{Q}}$ *with uniformizer $\pi$ and $R = A[X_0, \ldots, X_n]$ or $A[X_0, \ldots, X_n]_\Delta$ where $\Delta = \mathbb{Q}$ or $\mathbb{Z}[1/p]$. Let $J$ be an ideal of $R$ and $J^\wedge$ and $R^\wedge$ denote the completion with respect to $\pi$, that is*

$$J^\wedge = \overset{\lim}{\longleftarrow} J/\pi^i J = \overset{\lim}{\longleftarrow}(J \otimes_R R/\pi^i R), \tag{4.11}$$

*then the following hold.*

1. $J \otimes_R R^\wedge = J^\wedge$.
2. *the ring map $R \to R^\wedge$ is flat.*

*Proof*

1. The elements of $R^\wedge$ are of the form $(r_0, r_1, r_2, \ldots)$ such that

$$\begin{aligned} r_0 &\equiv r_1 \bmod \pi, \\ r_1 &\equiv r_2 \bmod \pi^2, \\ &\vdots \\ r_i &\equiv r_{i+1} \bmod \pi^{i+1}, \\ &\vdots \end{aligned} \tag{4.12}$$

The elements of $J \otimes_R R^\wedge$ are of the form $(Jr_0, Jr_1, Jr_2, \ldots)$ such that

$$\begin{aligned} Jr_0 &\equiv Jr_1 \bmod \pi, \\ Jr_1 &\equiv Jr_2 \bmod \pi^2, \\ &\vdots \\ Jr_i &\equiv Jr_{i+1} \bmod \pi^{i+1}, \\ &\vdots \end{aligned} \tag{4.13}$$

which is precisely the description of $J^\wedge$. Note that there is no non trivial ideal lying in $\cap_i \pi^i$ by assumption.
2. Let $J$ be an arbitrary ideal of $R$, consider the map $J \otimes_R R^\wedge \to R \otimes_R R^\wedge = R^\wedge$. According to above this map becomes $J^\wedge \to R^\wedge$ which is injective giving flatness.

□

**Proposition 4.11** *Let $R$ be* $\mathrm{eka}^d$ *or* $\mathrm{eka}^{\mathbb{Q}}$ *with uniformizer $\pi$ and the completions are carried out with respect to $\pi$*

1. *The map $R \to R[X_0,\ldots,X_n]_\Delta$ is flat.*
2. *The map $R[X_0,\ldots,X_n]_\Delta \to R\langle X_0,\ldots,X_n\rangle_\Delta$ is flat.*
3. *The map $R \to R\langle X_0,\ldots,X_n\rangle_\Delta$ is flat.*

*Proof*

1. This follows from the fact that free modules are flat.
2. This is lemma 4.10.
3. Follows from above two.

□

## 5 Eka Topologically finite

**Definition 5.1** Let $R$ be an $\mathrm{eka}^d$ or $\mathrm{eka}^{\mathbb{Q}}$ with uniformizer $\pi$ and $A = R\langle X_0,\ldots,X_n\rangle_\Delta$. The $R$ algebra of the form $A/\mathfrak{a}$ where $\mathfrak{a}$ is a finitely generated ideal of $A$ will be called topologically eka of finite presentation.

### 5.1 Formal Blow Up

Recall that scheme theoretic blow up for an ideal $\mathfrak{I}$ is given as $\mathrm{Proj}\ \oplus_{d=0}^{\infty} \mathfrak{I}^d$, this needs to be translated to the case at hand.

***Remark 5.2***

1. The ring $R\langle X_0,\ldots,X_n\rangle_\Delta$ consists of elements which are formal series with almost all terms having high $\pi$ adic valuation. Thus $f \bmod \pi^i, i \in \mathbb{Z}_{>0}$ is always a polynomial. Thus, there are polynomial rings of the form
$$\frac{R}{\pi^n} \otimes_R R\langle X_0,\ldots,X_n\rangle_\Delta = (R/\pi^n)[X_0,\ldots,X_n]_\Delta. \tag{5.1}$$
The scheme theoretic blow up can be defined for each of these rings, and a direct limit gives the blow up for formal scheme.
2. Let $A$ be a topologically eka of finite presentation and $\mathfrak{a}$ a coherent ideal open ideal of $A$.

$$X_{\mathfrak{a},n} := \mathrm{Proj}\left(\bigoplus_{d=0}^{\infty} \mathfrak{a}^d \otimes_R \frac{R}{\pi^n}\right)$$
$$X_{\mathfrak{a},1} \subseteq X_{\mathfrak{a},2} \subseteq \cdots \subseteq X_{\mathfrak{a},n} \subseteq \cdots \subseteq \underset{(}{\overset{\lim}{\to}} n) X_{\mathfrak{a},n} \tag{5.2}$$
$$X_{\mathfrak{a}} := \underset{(}{\overset{\lim}{\to}} n) X_{\mathfrak{a},n} = \underset{(}{\overset{\lim}{\to}} n)\, \mathrm{Proj}\left(\bigoplus_{d=0}^{\infty} \mathfrak{a}^d \otimes_R \frac{R}{\pi^n}\right)$$

**Definition 5.3** Let $X = \mathrm{Spf}\, A$ where $A$ is topologically eka of finite presentation and $\mathfrak{a}$ a coherent open ideal of $A$. The formal $R$ schemes $X_{\mathfrak{a}}$ along with canonical projection $X_{\mathfrak{a}} \to X$ is called the formal blowing up of $\mathfrak{a}$ on $X$. Any blow up of this form is called admissible formal blowing up of $X$.

$$X_{\mathfrak{a}} = \underset{(}{\overset{\lim}{\to}} n)\, \mathrm{Proj}\left(\bigoplus_{d=0}^{\infty} \mathfrak{a}^d \otimes_R (R/\pi^n)\right) \tag{5.3}$$

The correct set up formal admissible blow up allows to transfer some results for Noetherian formal schemes to the Non Noetherian set up in our case. For example, the following proposition carried over verbatim.

**Proposition 5.4** *Admissible formal blowing up commutes with flat base change.*

*Proof* [3, pp 185] □

### 5.2 Problems of Finiteness

Since the underlying rings $R[X_0,\ldots,X_n]_\Delta$ are Non Noetherian there are problems with finiteness. Denote $A' := R[X_0,\ldots,X_n]_\Delta$ and $\mathfrak{a}'$ an ideal of $A'$.

1. The $R$ algebra $A'/\mathfrak{a}'$ is not necessarily finite. In fact, in most of the cases it will have infinite many copies of $R$. Thus, a product topology of the form $\prod_i R_i$ where each $R_i = R$ is required.
2. In the case of topologically finite algebras the flattening techniques of Raynaud Gruson are used for proving the separatedness and completion of topological algebras. These techniques are not available for the case at hand.
3. The Artin-Rees lemma works only for Noetherian rings, thus there is a requirement for a simpler analogue that will work for Non-Noetherian rings.

***Example 5.5*** This example illustrates the difficulty associated with finite generation and finite presentation.

Let $k$ be a field, consider the following short exact sequence

$$0 \to \langle X, X^{1/2}, X^{1/3}, \ldots, X^{1/i}\ldots\rangle \to k[X]_{\mathbb{Q}} \to \frac{k[X]_{\mathbb{Q}}}{\langle X, X^{1/2}, X^{1/3}, \ldots, X^{1/i}\ldots\rangle} \to 0. \tag{5.4}$$

$k[X]_{\mathbb{Q}}/\langle X, X^{1/2}, X^{1/3}, \ldots\rangle$ is a finite $k[X]_{\mathbb{Q}}$ module, but is not finitely presented since the kernel$\langle X, X^{1/2}, X^{1/3}, \ldots, X^{1/i}\ldots\rangle$ is not finitely generated.

#### *5.2.1 Artin Rees Lemma*

The Artin Rees Lemma holds only for Noetherian Rings, here we give an analogue for non Noetherian rings.

**Proposition 5.6** *Let $A$ be a complete ring equipped with $\pi$ adic topology and $M$ be a $A$ module also equipped with $\pi$ adic topolgy. Let $N \subset M$ be an open submodule, then the $\pi$ adic topology on $N$ coincides with the topology induced on $N$ via inclusion in $M$.*

*Proof* Since $N \subset M$ this gives

$$\pi^n N \subset \pi^n M \cap N. \tag{5.5}$$

Since $N$ is a open submodule of $M$, there exists $n \in \mathbb{N}$ such that $\pi^n M \subset N$ implying

$$\pi^{n+k} M \subset \pi^k N \subset \pi^k M \subset N \text{ for all } k \geq n, \tag{5.6}$$

implying the assertion. □

**Definition 5.7** Let $A := R\langle X_0,\ldots,X_n\rangle_\Delta$ (with $R$ eka complete) and $\mathfrak{a}$ an ideal of $A$ such that all the ideals of $A/\mathfrak{a}$ are open will be called topologically eka algebras. Note that the ideals of $A/\mathfrak{a}$ are equipped with $\pi$ adic topology.

**Lemma 5.8** *The topologically eka algebras are complete and separated.*

*Proof* The definition of topologically eka algebras is designed so that the proposition 5.6 applies and the lemma holds. □

## 6 Coherence

### 6.1 Mittag Leffler System

The inverse limit functor $\overset{\lim}{\longleftarrow}$ is not an exact functor, but it becomes an exact functor if it satisfies Mittag Leffler condition. For the case at hand the inverse limit is simply a $\pi$ adic completion of the chosen eka ring $R$. The following cases satisfy Mittag Leffler and will be used later.

1. Let $(f_1,\ldots,f_j)$ be a finitely generated ideal in the ring $R\langle X_0,\ldots,X_n\rangle_\Delta$, then this ideal can be approximated as a Mittag Leffler system

$$\cdots \to \frac{(f_1,\ldots,f_j)}{\pi^i} \to \cdots \to \frac{(f_1,\ldots,f_j)}{\pi^2} \xrightarrow{\text{mod } \pi} (f_1,\ldots,f_j)\pi. \tag{6.1}$$

2. The ring $R\langle X_0,\ldots,X_n\rangle_\Delta$ is obtained from $\pi$ adic completion of $R[X_0,\ldots,X_n]_\Delta$ given by the Mittag Leffler system

$$\cdots \to \frac{R[X_0,\ldots,X_n]_\Delta}{\pi^i} \to \cdots \to \frac{R[X_0,\ldots,X_n]_\Delta}{\pi^2} \xrightarrow{\text{mod } \pi} R[X_0,\ldots,X_n]_\Delta\pi. \tag{6.2}$$

**Theorem 6.1** *Let $R$ be an* eka *ring and $K = R[1/\pi]$ where $\pi$ is the uniformizer in $R$. Then the ring $R[X_0,\ldots,X_n]_\Delta$ is coherent.*

*Proof* Since $R$ is an eka ring it is of the form $R := \bigcup_{i\geq 0} R_i$ with $R_0$ a discrete valuation ring with uniformizer $\pi$ and each $R_i$ is Noetherian. Let $(f_1,\ldots,f_j)$ be a finitely generated ideal in the ring $R[X_0,\ldots,X_n]_\Delta$, thus it will lie in some ring

$$B_i = \begin{cases} R_i[X_0^{1/d^i},\ldots,X_n^{1/d^i}] \text{ if } \Delta = \mathbb{Z}[1/d] \\ R_i[X_0^{1/\ell^i},\ldots,X_n^{1/\ell^i}] \text{ if } \Delta = \mathbb{Q} \end{cases} \tag{6.3}$$

Thus, there is a finite presentation of the form

$$B_i^m \to B_i^n \to (f_1,\ldots,f_j) \to 0. \tag{6.4}$$

Applying the direct limit to (6.4) and noticing that direct limit commutes with direct sum gives the following

$$\begin{array}{c} (\varinjlim_i B_i)^m \longrightarrow (\varinjlim_i B_i)^n \longrightarrow \varinjlim_i (f_1,\ldots,f_j) \longrightarrow 0 \\ (R[X_0,\ldots,X_n]_\Delta)^m \longrightarrow (R[X_0,\ldots,X_n]_\Delta)^n \longrightarrow (f_1,\ldots,f_j) \longrightarrow 0 \end{array} \tag{6.5}$$

□

## 7 Čech Cohomology

***Remark 7.1*** This section uses results from [1] for $\Delta = \mathbb{Q}$ and [2] for $\Delta = \mathbb{Z}[1/p]$ which can be transferred verbatim to $\Delta = \mathbb{Z}[1/d]$. The Čech cohomology associated with the polynomial rings $R[X_0,\ldots,X_n]_\mathbb{Q}$ and $R[X_0,\ldots,X_n]_{\mathbb{Z}[1/p]}$ is constructed in the above mentioned papers along with explanatory examples. These results will now be used in this section.

### 7.1 Homogeneous elements of degree $d$ and Proj

Let $(R[X_0,\ldots,X_n]_\Delta)_d$ be the space of homogeneous polynomials of degree $d$ in $n+1$ variables. This space is infinite dimensional and generated by $X_0^{a_0}X_1^{a_1}\cdots X_n^{a_n}$ with $\sum_i a_i = d$ and each $a_i \in \Delta_{\geq 0}$ (where $\Delta = \mathbb{Z}[1/d]$ or $\mathbb{Q}$). Similarly, let $(R\langle X_0,\ldots,X_n\rangle_\Delta)_d$ be the space of homogeneous restricted series of degree $d$ in $n+1$ variables. This space is also infinitely generated with the same basis as $(R[X_0,\ldots,X_n]_\Delta)_d$. Furthermore, the rings $(K[X_0,\ldots,X_n]_\Delta)_d$ and $(K\langle X_0,\ldots,X_n\rangle_\Delta)_d$ also have the basis generated by $X_0^{a_0}X_1^{a_1}\cdots X_n^{a_n}$ with $\sum_i a_i = d$ and each $a_i \in \Delta_{\geq 0}$, only the coefficients change from $R$ to $K$, where $K = R[1/\pi]$.

***Example 7.2*** $(R[X,Y]_\Delta)_1$ is infinitely generated by $X, Y, X^a, Y^b$ such that $a+b=1$. The same elements also generate $(R\langle X,Y\rangle_\Delta)_1$. Let $\pi$ be a unifomizer, the following are examples of degree one elements.

$$\begin{aligned} &X+Y+\pi^2X^{1/p^2}Y^{1-1/p^2} \in (R[X,Y]_\Delta)_1 \subset (K[X,Y]_\Delta)_1, \\ &X+Y+\pi X^{1/p}Y^{1-1/p}+\pi^2X^{1/p^2}Y^{1-1/p^2}+\ldots+\pi^iX^{1/p^i}Y^{1-1/p^i} \\ &\quad +\ldots \in (R\langle X,Y\rangle_\Delta)_1 \subset (K\langle X,Y\rangle_\Delta)_1. \end{aligned} \tag{7.1}$$

Let $A_d = (K\langle X_0,\ldots,X_n\rangle_\Delta)_d$ and $A=\prod_{d\geq 0}A_d, A_+=\prod_{d>0}A_d$.

$$\operatorname{Proj} A = \{\text{Set of homogeneous prime ideals of } A \text{ that do not contain } A_+\} \tag{7.2}$$

7.2 Zero Cohomology

**Proposition 7.3** *Let $R$ be an admissible eka ring, $A=R[X_0,\ldots,X_n]_\Delta, B=R\langle X_0,\ldots,X_n\rangle_\Delta$ and $D(\cdot)$ denote open sets of $B$.*

1. *Then the localizations associated with Čech complex of $B$ are obtained as the following $\pi$ adic completions*

$$\begin{aligned} D(X_i) \to B\langle 1/X_i\rangle &= \varprojlim_s (B/\pi^s[1/X_i]) = \varprojlim_s (A/\pi^s[1/X_i]) \\ D(X_iX_j) \to B\langle 1/(X_iX_j)\rangle &= \varprojlim_s (B/\pi^s[1/(X_iX_j)]) = \varprojlim_s (A/\pi^s[1/(X_iX_j)]) \\ \vdots \quad &= \quad \vdots \\ D(X_0\cdots X_n) \to B\langle 1/(X_0\cdots X_n)\rangle &= \varprojlim_s (B/\pi^s[1/(X_0\cdots X_n)]) = \varprojlim_s (A/\pi^s[1/(X_0\cdots X_n)]). \end{aligned} \tag{7.3}$$

2. *Each of the rings $B\langle 1/X_i\rangle, B\langle 1/(X_iX_j)\rangle,\ldots,B\langle 1/(X_0\cdots X_n)\rangle$ is obtained via a Mittag-Leffler system.*
3. *The Čech complex of $B$ can be obtained from $\pi$ adic completion of Čech complex of $A$.*

*Proof*

1. The first equality follows from the fact that $B$ is admissible as shown in lemma 4.6. The second equality follows from the fact that $B$ is $\pi$ adic completion of $A$.
2. The Mittag Leffler system is obtained via the surjective $\bmod \pi^i$ maps

$$\cdots \to \frac{A}{\pi^{i+1}}\left[\frac{1}{f}\right] \underset{\cdots}{\xrightarrow{\bmod \pi^i}} \to \frac{A}{\pi^2}\left[\frac{1}{f}\right] \xrightarrow{\bmod \pi} A\pi\left[\frac{1}{f}\right], \tag{7.4}$$

where $f$ could be any element from the set $\{X_i, X_iX_j,\ldots,X_0\cdots X_n\}$.
3. As shown above each of localizations of $B$ in the Čech complex can be obtained from $\pi$ adic completion of $A$. Note that each of product in Čech complex is finite. Therefore, $\pi$ adic completion of Čech complex of $A$ gives Čech complex of $B$.

□

***Remark 7.4*** Let us describe the localized rings a bit more concretly. If $B=R\langle X_0,\ldots,X_n\rangle_\Delta$, then

$$\begin{aligned} B\langle 1/X_i\rangle &= \varprojlim_s R/\pi^s[(X_0/X_1),\ldots,(X_n/X_i)]_\Delta \\ &= R\langle (X_0/X_i),\ldots,(X_n/X_i)\rangle_\Delta. \end{aligned} \tag{7.5}$$

**Theorem 7.5** *Let $S=R\langle X_0,\ldots,X_n\rangle_\Delta$ and $X=\operatorname{Proj} S$, then $H^i(X,\mathscr{O}_X(m))=0$ for $0<i<n$.*

*Proof* The Čech complex of $R[X_0,\ldots,X_n]_\Delta$ gives zero cohomology groups $(0<i<n)$, see papers mentioned in remark 7.1. But $\pi$ adic completion is an exact functor since it satsifies the Mittag-Leffler condition (as given in proposition 7.3 ). Thus, the Čech complex of $R\langle X_0,\ldots,X_n\rangle_\Delta$ again gives zero cohomology for $0<i<n$. □

***Remark 7.6*** Note that inverse limit is left exact and that is enough for the proof of theorem above. After all the end terms are not being considered (only $0<i<n$). Since, $K\otimes -$ is right exact, the theorem also carries over to the ring $K\langle X_0,\ldots,X_n\rangle_\Delta$.

### 7.3 Non Zero Cohomology

This proof is adapted from similar results obtained as in remark 7.1.

**Theorem 7.7** *Let* $S=R\langle X_0,\ldots,X_n\rangle_\Delta$ *and* $X=\operatorname{Proj} S$, *then for any* $n\in\Delta$

1. *There is an isomorphism* $S\simeq\oplus_{n\in\Delta}H^0(X,\mathcal{O}_X(n))$.
2. $H^n(X,\mathcal{O}_X(-n-1))$ *is a free module of infinite rank.*

*Proof*

1. Take the standard cover by affine sets $\mathfrak{U}=\{U_i\}_i$ where each $U_i=D(X_i), i=0,\ldots,n$. The global sections are given as the kernel of the following map

$$\prod S_{X_{i_0}}\longrightarrow\prod S_{X_{i_0}X_{i_1}}\tag{7.6}$$

The element mapping to the Kernel has to lie in all the intersections $S=\cap_i S_{X_i}$, as given on [7, pp 118] and is thus the ring $S$ itself. Note that

$$S_{X_i}=R\langle(X_0/X_i),\ldots,(X_n/X_i)\rangle_\Delta\tag{7.7}$$

as shown in remark 7.4.

2. $H^n(X,\mathcal{O}_X(-m))$ is the cokernel of the map

$$d^{n-1}:\prod_k S_{X_0\cdots\hat{X}_k\cdots X_n}\longrightarrow S_{X_0\cdots X_n}\tag{7.8}$$

$S_{X_0\cdots X_n}$ is a free $R$ module with basis $X_0^{l_0}\cdots X_n^{l_n}$ with each $l_i\in\Delta$. The image of $d^{n-1}$ is the free submodule generated by those basis elements with atleast one $l_i\geq 0$. Thus $H^n$ is the free module with basis as negative monomials

$$\{X_0^{l_0}\cdots X_n^{l_n}\}\text{ such that }l_i<0\tag{7.9}$$

The grading is given by $\sum l_i$ and there are infinitely many monomials with degree $-n-\epsilon$ where $\epsilon$ is something very small and $\epsilon\in\Delta$. Recall, that in the standard coherent cohomology there is only one such monomial $X_0^{-1}\cdots X_n^{-1}$ .

□

***Remark 7.8*** The proof of above theorem carries word for word to the ring $K\langle X_0,\ldots,X_n\rangle_\Delta$.

## 8 Field Extensions

The idea of eka rings helps build correspondence of field extensions.

Let $R$ be a discrete valuation ring with uniformizer $\pi$ and residue field $\kappa=R/\pi$. The local homomorphism of rings describes the homomorphism of discrete valuation rings.

**Lemma 8.1** *Let* $R$, $S$ *be local rings and* $\phi:R\to S$ *a local ring map, then the following are equivalent.*

1. $\phi(\mathfrak{m}_R)\subset\mathfrak{m}_S$.
2. $\phi^{-1}(\mathfrak{m}_S)=\mathfrak{m}_R$.
3. *For any* $x\in R$, *if* $\phi(x)$ *is invertible in* $S$, *then* $x$ *is invertible in* $R$.

*Proof* [9, tag/07BH] □

**Corollary 8.2** *Let* $R$ *be a discrete valuation ring with uniformizer* $\pi$ *and a local homomorphism* $\varphi: R[\pi^{1/d_1}]\to R[\pi^{1/d_2}]$ *where* $d_1|d_2$. *Then*

1. *$\varphi$ is completely determined by $u \mapsto u'$, where $u, u'$ are units in $R$.*
2. *The map $\varphi$ induces $\phi : \kappa[[t^{1/d_1}]]^\wedge \to \kappa[[t^{1/d_2}]]^\wedge$, where $^\wedge$ denotes $t$ adic completion and $\kappa = R/\pi$.*
3. *There is an isomorphism $\mathrm{Mor}\,(R[\pi^{1/d_1}], R[\pi^{1/d_2}]) \simeq \mathrm{Mor}\,(\kappa[[t^{1/d_1}]]^\wedge, \kappa[[t^{1/d_2}]]^\wedge)$.*
4. *Let $K = \mathrm{Frac}\,(R[\pi^{1/d_1}]), K' = \mathrm{Frac}\,(R[\pi^{1/d_2}])$ and $k = \mathrm{Frac}\,(\kappa[[t^{1/d_1}]]^\wedge), k' = \mathrm{Frac}\,(\kappa[[t^{1/d_2}]]^\wedge)$, then there is an isomorphism between field extensions $K'/K$ and $k'/k$.*
5. *Let $\bar{K} = \mathrm{Frac}\,(\,\mathrm{eka}^{\mathbb{Q}}R)$ and $\bar{k} = \mathrm{Frac}\,(\kappa[[t]]^\wedge_{\mathbb{Q}})$ then there is an isomorphism between field extensions $\bar{K}/K$ and $\bar{k}/k$.*

*Proof* First notice that $R[\pi^{1/d_i}]$ is a discrete valuation ring with uniformizer $\pi^{1/d_i}$ and same residue field $\kappa = R/\pi = R[\pi^{1/d_i}]/\pi^{1/d_i}$.

1. The lemma 8.1 implies that every local homomorphism will have $\pi^{1/d_1} \hookrightarrow \pi^{1/d_2}$ and we already know that $R^\times = R \backslash \langle \pi \rangle = R[\pi^{1/d}] \backslash \langle \pi^{1/d} \rangle$. Thus, the local homomorphism will be determined by where the units map to.
2. Every local homomorphism $\phi$ will have $t^{1/d_1} \hookrightarrow t^{1/d_2}$, and thus is completely determined by the mapping of units $u \mapsto u'$ where $u, u' \in \kappa \backslash \{0\}$. But the units are $R^\times = R \backslash \langle \pi \rangle = R[\pi^{1/d}] \backslash \langle \pi^{1/d} \rangle = R/\pi = \kappa$.
3. The above two imply that $\varphi \simeq \phi$ since both are determined by the mapping $u \mapsto u'$.

$$
\begin{array}{cccc}
u & R[\pi^{1/d_1}] & \kappa[[t^{1/d_1}]]^\wedge & u \\
\downarrow & \downarrow & \downarrow & \downarrow \\
u' & R[\pi^{1/d_2}] & \kappa[[t^{1/d_2}]]^\wedge & u'
\end{array}
\tag{8.1}
$$

4. Recall that $\mathrm{Frac}\,(R[\pi^{1/d_i}]) = R[\pi^{1/d^i}]\left[\dfrac{1}{\pi}\right]$ and $\mathrm{Frac}\,(\kappa[[t^{1/d_i}]]^\wedge) = \kappa[[t^{1/d_i}]]^\wedge\left[\dfrac{1}{t}\right]$, thus the correspondence below implies the result.

$$
\begin{array}{cccc}
u\pi^a & R[\pi^{1/d_1}]\left[\dfrac{1}{\pi}\right] & \kappa[[t^{1/d_1}]]^\wedge\left[\dfrac{1}{t}\right] & ut^a \\
\downarrow & \downarrow & \downarrow & \downarrow \\
u'\pi^a & R[\pi^{1/d_2}]\left[\dfrac{1}{\pi}\right] & \kappa[[t^{1/d_2}]]^\wedge\left[\dfrac{1}{t}\right] & u't^a
\end{array}
\tag{8.2}
$$

Notice that the uniformizer embeds in the bigger field as it is and the field extension comes from mapping the units of $R$.

5. Follows exactly as above.

□

***Remark 8.3*** Let $\ell_i = \mathrm{lcm}(1, 2, \ldots, i)$, there is a sequence of embeddings

$$\begin{tikzcd}
R[\pi] \arrow[d, hook] \arrow[dd, bend right=60] \arrow[dddd, bend right=70] \arrow[dddddd, bend right=80] & \kappa[[t]] \arrow[d, hook] \arrow[dd, bend left=60] \arrow[dddd, bend left=70] \arrow[dddddd, bend left=80] \\
R[\pi^{1/2}] \arrow[d, hook] & \kappa[[t^{1/2}]]^{\wedge} \arrow[d, hook] \\
R[\pi^{1/6}] \arrow[d, hook] & \kappa[[t^{1/6}]]^{\wedge} \arrow[d, hook] \\
\vdots \arrow[d, hook] & \vdots \arrow[d, hook] \\
R[\pi^{1/\ell_i}] \arrow[d, hook] & \kappa[[t^{1/\ell_i}]]^{\wedge} \arrow[d, hook] \\
\vdots \arrow[d, hook] & \vdots \arrow[d, hook] \\
\mathrm{eka}^{\mathbb{Q}} R & \kappa[[t]]^{\wedge}_{\mathbb{Q}}
\end{tikzcd} \tag{8.3}$$

**Acknowledgements** I am grateful to Timothy Ferdinands for the counter example 4.2.1, and Matthew Morrow for comments.

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