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ORIGINAL RESEARCH

On some infinite families of congruences for [j, k]-partitions into even parts distinct

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Abstract In this paper, we define the partition function $ped_{j,k}(n)$, the number of [j, k]-partitions of n into even parts distinct, where none of the parts are congruent to j (mod k) (where $k > j \geq 1$). We obtain many infinite families of congruences modulo powers of 2 for $ped_{3,6}(n)$ and congruences modulo powers of 2 and 3 for $ped_{9,18}(n)$. For example, for all $n \geq 0$ and $\alpha, \beta \geq 0$,

$$ped_{9,18}\left(2 \cdot 3^{4\alpha+4} \cdot 7^{2\beta+1}(7n+s) + \frac{11 \cdot 3^{4\alpha+3} \cdot 7^{2\beta+1} + 1}{4}\right) \equiv 0 \pmod{16},$$

where $s = 0, 2, 3, 4, 5, 6$.

Keywords Congruences · Partitions with even parts distinct · [j k]-partitions

Mathematics Subject Classification 05A17 · 11P83

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. An ℓ-regular partition is a partition in which none of the parts are divisible by ℓ. Arithmetic properties of ℓ-regular partition functions have been studied by a number of mathematicians. We can see [4, 8, 12, 16].

For positive integers j and k such that $k > j \geq 1$, a partition of n is a [j, k]-partitions if none of the parts are congruent to j (mod k). Let $p_{j,k}(n)$ denote the number of such partitions of n with $p_{j,k}(0) = 1$. The generating function for $p_{j,k}(n)$ is given by

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$$\sum_{n=0}^{\infty} p_{j,k}(n)q^n = \frac{(q^j;q^k)_\infty}{(q;q)_\infty}, \tag{1.1}$$

where $(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots$.

In 2010, Andrews, Hirschhorn and Sellers [1] introduced the partition function $ped(n)$, the number of partitions of a positive integer n with even parts distinct (the odd parts are unrestricted). The generating function for $ped(n)$ is given by

$$\sum_{n=0}^{\infty} ped(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty}. \tag{1.2}$$

They obtained many Ramanujan-type congruences for $ped(n)$. For example, for all $n \geq 0$,

$$ped(9n+4) \equiv 0 \pmod 4 \tag{1.3}$$

and

$$ped(9n+7) \equiv 0 \pmod{12}. \tag{1.4}$$

For more information, one can see [6, 7, 9, 14, 15, 17].

By the motivation of the above work, in this paper, we define the partition function $ped_{j,k}(n)$, the number of [j, k]-partitions of a positive integer n into even parts distinct, where none of the parts are congruent to j (mod k) (where $k > j \geq 1$). The generating function for $ped_{j,k}(n)$ is given by

$$\sum_{n=0}^{\infty} ped_{j,k}(n)q^n = \frac{f_4}{f_1}(q^j;q^k)_\infty. \tag{1.5}$$

For example, the [3, 6]-partitions of 7 into even parts distinct $\big(ped_{3,6}(7) = 8\big)$ are

$$7, \quad 6+1, \quad 5+2, \quad 5+1+1, \quad 4+2+1, \quad 4+1+1+1,$$
$$2+1+1+1+1+1, \quad 1+1+1+1+1+1+1.$$

Also, we have establish many infinite families of congruences modulo powers of 2 for $ped_{3,6}(n)$ and congruences modulo powers of 2 and 3 for $ped_{9,18}(n)$. For example, for all $n \geq 0$ and $\alpha, \beta \geq 0$,

$$ped_{9,18}\left(2 \cdot 3^{4\alpha+4} \cdot 7^{2\beta+1}(7n+s) + \frac{11 \cdot 3^{4\alpha+3} \cdot 7^{2\beta+1} + 1}{4}\right) \equiv 0 \pmod{16},$$

where $s = 0, 2, 3, 4, 5, 6$.

2 Preliminary results

In this section, we collect many identities which are useful in proving our main results.

Lemma 2.1 *The following 2-dissections hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8} \tag{2.1}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.2}$$

The identity (2.1) is the 2-dissection of $\phi(q)$ [10, (1.9.4)]. The identity (2.2) is the 2-dissection of $\phi(q)^2$ [10, (1.10.1)]. Also, one can see [2, p.40].

Lemma 2.2 *The following 2-dissections hold:*

$$\frac{f_3}{f_1}=\frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}}+q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}, \tag{2.3}$$

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}} \tag{2.4}$$

and

$$\frac{f_3^3}{f_1}=\frac{f_4^3f_6^2}{f_2^2f_{12}}+q\frac{f_{12}^3}{f_4}. \tag{2.5}$$

The identity (2.3) is the same as (30.10.3) in [10]. The identity (2.4) is essentially (30.10.4) in [10]. The equation (2.5) is the same as (22.1.14) in [10] (after using 22.1.6 and 22.1.7).

Lemma 2.3 *The following 2-dissection holds:*

$$\frac{f_9}{f_1}=\frac{f_{12}^3f_{18}}{f_2^2f_6f_{36}}+q\frac{f_4^2f_6f_{36}}{f_2^3f_{12}}. \tag{2.6}$$

The identity (2.6) was obtained by Xia and Yao [19].

Lemma 2.4 *The following 3-dissection holds:*

$$\frac{f_2}{f_1^2}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}. \tag{2.7}$$

The identity (2.7) was obtained by Hirschhorn and Sellers [11].

Lemma 2.5 *The following 3-dissections hold:*

$$f_1^3=\frac{f_6f_9^6}{f_3f_{18}^3}-3qf_9^3+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^3}, \tag{2.8}$$

$$f_1f_2=\frac{f_6f_9^4}{f_3f_{18}^2}-qf_9f_{18}-2q^2\frac{f_3f_{18}^4}{f_6f_9^2}, \tag{2.9}$$

$$\frac{1}{f_1f_2}=\frac{f_9^9}{f_3^6f_6^2f_{18}^3}+q\frac{f_9^6}{f_3^5f_6^3}+3q^2\frac{f_9^3f_{18}^3}{f_3^4f_6^4}-2q^3\frac{f_{18}^6}{f_3^3f_6^5}+4q^4\frac{f_{18}^9}{f_3^2f_6^6f_9^3}, \tag{2.10}$$

$$\frac{f_4}{f_1}=\frac{f_{12}f_{18}^4}{f_3^3f_{36}^2}+q\frac{f_6^2f_9^3f_{36}}{f_3^4f_{18}^2}-2q^2\frac{f_6f_{18}f_{36}}{f_3^3} \tag{2.11}$$

and

$$\frac{f_2^3}{f_1^3}=\frac{f_6}{f_3}+3q\frac{f_6^4f_9^5}{f_3^8f_{18}}+6q^2\frac{f_6^3f_9^2f_{18}^2}{f_3^7}+12q^3\frac{f_6^2f_{18}^5}{f_3^6f_9}. \tag{2.12}$$

The equation (2.8) is the same as (14.8.5) in [10]. See also [2, p.345]. The equation (2.9) was obtained by Hirschhorn and Sellers [13]. The identity (2.10) was obtained by Chan [5]. The identity (2.11) is nothing but Lemma 2.6 in [3]. The identity (2.12) was proved by Toh [18].

Lemma 2.6 *We require the 5-dissection formula*

$$f_1=f_{25}(R(q^5)^{-1}-q-q^2R(q^5)), \tag{2.13}$$

where

$$R(q)=\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

The identity (2.13) is the same as (8.1.1) in [10].

Lemma 2.7 *We require the 7-dissection formula*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.14}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ and $C(q) = f(-q, -q^6)$.

Lemma 2.7 is an exercise in [10], see [10, (10.5.1)]. Also, one can see [2, p.303, Entry 17(v)].

Lemma 2.8 [8, Theorem 2.2] *For any prime* $p \geq 5$,

$$f_1 = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq (\pm p-1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \tag{2.15}$$

Furthermore, for $-(p-1)/2 \leq k \leq (p-1)/2$ and $k \neq (\pm p - 1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.$$

3 Congruences for $ped_{3,6}(n)$

Theorem 3.1 *For all* $n \geq 0$ and $\alpha \geq 0$, *we have*

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(24n+15)\right)q^n \equiv 16f_2^3f_3^3 + 24qf_1f_2f_6^6 \pmod{32}, \tag{3.1}$$

$$ped_{3,6}\left(3^{4\alpha}(648n+135)\right) \equiv ped_{3,6}\left(3^{4\alpha}(72n+15)\right) \pmod{32}, \tag{3.2}$$

$$ped_{3,6}\left(3^{4\alpha}(648n+351)\right) \equiv ped_{3,6}\left(3^{4\alpha}(72n+39)\right) \pmod{32}, \tag{3.3}$$

$$ped_{3,6}\left(3^{4\alpha}(1944n+1863)\right) \equiv ped_{3,6}\left(3^{4\alpha}(216n+207)\right) \pmod{32}, \tag{3.4}$$

$$ped_{3,6}\left(3^{4\alpha}(9720(5n+i)+4455)\right) \equiv ped_{3,6}\left(3^{4\alpha}(1080(5n+i)+495)\right) \pmod{32}, \tag{3.5}$$

where $i = 0, 2, 3, 4$.

Proof Setting $j = 3$ and $k = 6$ in (1.5), we see that

$$\sum_{n=0}^{\infty} ped_{3,6}(n)q^n = \frac{f_3f_4}{f_1f_6}. \tag{3.6}$$

Employing (2.3) in (3.6) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} ped_{3,6}(2n+1)q^n = \frac{f_2f_4^2f_{24}}{f_1^2f_8f_{12}}. \tag{3.7}$$

Substituting (2.1) in (3.7), we obtain

$$\sum_{n=0}^{\infty} ped_{3,6}(4n+1)q^n = \frac{f_2^2f_4^4f_{12}}{f_1^4f_6f_8^2} \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}(4n+3)q^n = 2\frac{f_2^4f_8^2f_{12}}{f_1^4f_4^2f_6}. \tag{3.9}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m} \equiv f_{2k}^{m} \pmod{2}, \tag{3.10}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{4}, \tag{3.11}$$

$$f_k^{8m} \equiv f_{2k}^{4m} \pmod{8}. \tag{3.12}$$

Using (2.2) in (3.9) and invoking (3.11) and (3.12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} ped_{3,6}(8n+3)q^n &= 2\frac{f_2^{12}f_6}{f_1^{10}f_3f_4^2}\\ &\equiv 2\frac{f_4^2f_6}{f_1^2f_3} \pmod{16}\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} ped_{3,6}(8n+7)q^n &= 8\frac{f_4^6f_6}{f_1^6f_3}\\ &\equiv 8\frac{f_1^{18}f_6}{f_3} \pmod{32}.\end{aligned} \tag{3.14}$$

Using (2.8), (3.10) and (3.11) in (3.14), we find that

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+7)q^n \equiv 8f_1^7 + 8q^2f_1f_3^{18} \pmod{16}, \tag{3.15}$$

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+15)q^n \equiv 16f_2^3f_3^3 + 24qf_1f_2f_6^6 \pmod{32} \tag{3.16}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+23)q^n \equiv 24\frac{f_2^3f_3^6}{f_1} + 16qf_2f_3^{15} \pmod{32}. \tag{3.17}$$

The equation (3.16) is $\alpha = 0$ case of (3.1). Suppose that the congruence (3.1) is true for $\alpha \geq 0$. Employing (2.8) and (2.9) in (3.1) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(72n+15)\right)q^n \equiv 16f_1^5 + 16qf_1^{11}f_6^3 \pmod{32}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(72n+39)\right)q^n \equiv 24\frac{f_2^7}{f_1} \pmod{32} \tag{3.19}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(72n+63)\right)q^n \equiv 8f_2^6f_3f_6 + 16f_1^3f_6^3 \pmod{32}. \tag{3.20}$$

Again, utilizing (2.8) in (3.20), we get

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(216n+63)\right)q^n \equiv 16f_1^7 + 8f_1f_2^3 \pmod{32}, \tag{3.21}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(216n+135)\right)q^n \equiv 16f_2^3f_3^3 + 8qf_1f_2f_6^6 \pmod{32} \tag{3.22}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(216n+207)\big)q^n \equiv 16f_1^5f_6^3 \pmod{32}. \tag{3.23}$$

Substituting (2.8) and (2.9) in (3.22), we obtain

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(648n+135)\big)q^n \equiv 16f_1^5 + 16qf_1^{11}f_6^3 \pmod{32}, \tag{3.24}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(648n+351)\big)q^n \equiv 8\frac{f_2^7}{f_1} \pmod{32} \tag{3.25}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(648n+567)\big)q^n \equiv 24f_2^6f_3f_6 + 16f_1^3f_6^3 \pmod{32}. \tag{3.26}$$

In view of the congruences (3.18) and (3.24), we obtain (3.2).

Utilizing (2.8) in (3.26), we obtain (3.4),

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(1944n+567)\big)q^n \equiv 16f_1^7 + 24f_1f_2^3 \pmod{32} \tag{3.27}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha+4}(24n+15)\big)q^n \equiv 16f_2^3f_3^3 + 24qf_1f_2f_6^6 \pmod{32}, \tag{3.28}$$

which proves that the result (3.1) is true for $\alpha+1$. By induction, the result (3.1) holds for all integer $\alpha \geq 0$.

In view of the congruences (3.19) and (3.25), we obtain (3.3).

From (3.21) and (3.27), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(1944n+567)\big)q^n\\ &\equiv \sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(216n+63)\big)q^n + 16f_1^7 \pmod{32}.\end{aligned} \tag{3.29}$$

Utilizing (2.13) in (3.29) and then extracting the coefficients of q^{5n+2} from both sides of the resultant equation, we get

$$\begin{aligned}&\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(9720n+4455)\big)q^n\\ &\equiv \sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(1080n+495)\big)q^n + 16qf_5^7 \pmod{32},\end{aligned} \tag{3.30}$$

which implies (3.5). □

Theorem 3.2 *For all $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(72n+45)\big)q^n \equiv 24\frac{f_2f_4f_{12}}{f_3} + 16qf_2^6f_3^9 \pmod{32}, \tag{3.31}$$

$$ped_{3,6}\big(3^{4\alpha}(5832n+5589)\big) \equiv ped_{3,6}\big(3^{4\alpha}(648n+621)\big) \pmod{32}. \tag{3.32}$$

Proof Using (2.2) in (3.8), we find that

$$\sum_{n=0}^{\infty} ped_{3,6}(8n+1)q^n \equiv \frac{f_4^2 f_6}{f_1^4 f_2^2 f_3} \pmod{8} \tag{3.33}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}(8n+5)q^n \equiv 4\frac{f_2^2 f_4^2 f_6}{f_3} \pmod{32}. \tag{3.34}$$

Using (2.9) in (3.34), we find that

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+5)q^n \equiv 4f_1^5 \pmod{8}, \tag{3.35}$$

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+13)q^n \equiv 4qf_1 f_{12}^3 \pmod{8} \tag{3.36}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+21)q^n \equiv 24\frac{f_4 f_6 f_{12}}{f_1} + 16q^2\frac{f_{12}^6}{f_1 f_2} \pmod{32}. \tag{3.37}$$

Employing (2.10) and (2.11) in (3.37), we get

$$\sum_{n=0}^{\infty} ped_{3,6}(72n+21)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.38}$$

$$\sum_{n=0}^{\infty} ped_{3,6}(72n+45)q^n \equiv 24\frac{f_2 f_4 f_{12}}{f_3} + 16qf_2^6 f_3^9 \pmod{32} \tag{3.39}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}(72n+69)q^n \equiv 16f_1^5 f_6^3 + 16f_2^7 f_3^3 \pmod{32}. \tag{3.40}$$

The equation (3.39) is $\alpha = 0$ case of (3.31). Suppose that the congruence (3.31) is true for $\alpha \geq 0$. Using (2.8) and (2.9) in (3.31) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(216n+45)\right)q^n \equiv 8f_1^5 \pmod{16}, \tag{3.41}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(216n+117)\right)q^n \equiv 16f_1^{13} + 16qf_1 f_{12}^3 \pmod{32} \tag{3.42}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(216n+189)\right)q^n \equiv 8\frac{f_4 f_6 f_{12}}{f_1} + 16qf_1^9 f_6^6 \pmod{32}. \tag{3.43}$$

Employing (2.8) and (2.11) in (3.43) we get

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(648n+189)\right)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.44}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(648n+405)\right)q^n \equiv 8\frac{f_2 f_4 f_{12}}{f_3} + 16f_1^{15} + 16qf_2^6 f_3^9 \pmod{32} \tag{3.45}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(648n+621)\big)q^n \equiv 16f_1^5f_6^3 + 16f_2^7f_3^3 \pmod{32}. \tag{3.46}$$

Using (2.8) and (2.9) in (3.45), we get

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(1944n+405)\big)q^n \equiv 8f_1^5 \pmod{16}, \tag{3.47}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(1944n+1053)\big)q^n \equiv 16f_1^{13} + 16f_3^3f_4 \pmod{32} \tag{3.48}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(1944n+1701)\big)q^n \equiv 24\frac{f_4f_6f_{12}}{f_1} + 16qf_3^{15} + 16qf_1^9f_6^6 \pmod{32}. \tag{3.49}$$

Using (2.8) and (2.11) in (3.49), we get

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(5832n+1701)\big)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.50}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(5832n+5589)\big)q^n \equiv 16f_1^5f_6^3 + 16f_2^7f_3^3 \pmod{32} \tag{3.51}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha+4}(72n+45)\big)q^n \equiv 24\frac{f_2f_4f_{12}}{f_3} + 16qf_2^6f_3^9 \pmod{32}, \tag{3.52}$$

which proves that the congruence (3.31) is true for $\alpha+1$. By induction, the congruence (3.31) holds for all integer $\alpha \geq 0$.

In view of (3.46) and (3.51), we obtain (3.32). □

Theorem 3.3 For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta}(648n+189)\big)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.53}$$

$$ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+1}(3240n+648i+297)\big) \equiv 0 \pmod{16}, \tag{3.54}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 7^{2\beta}(216n+45)\big)q^n \equiv 8f_1^5 \pmod{16}, \tag{3.55}$$

$$ped_{3,6}\big(3^{4\alpha}\cdot 7^{2\beta+1}(1512n+216s+99)\big) \equiv 0 \pmod{16}, \tag{3.56}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 7^{2\beta}(1944n+405)\big)q^n \equiv 8f_1^5 \pmod{16}, \tag{3.57}$$

$$ped_{3,6}\big(3^{4\alpha}\cdot 7^{2\beta+1}(13608n+1944s+891)\big) \equiv 0 \pmod{16}, \tag{3.58}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta}(5832n+1701)\big)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.59}$$

$$ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+1}(29160n+5832i+2673)\big) \equiv 0 \pmod{16}, \tag{3.60}$$

where $i = 0,2,3,4$ and $s = 0,2,3,4,5,6$.

Proof The equation (3.44) is $\beta = 0$ case of the congruence (3.53). Suppose that the congruence (3.53) is true for $\alpha, \beta \geq 0$. Employing (2.13) in (3.53) and then collecting the coefficients of q^{5n+2} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+1}(648n+297)\big)q^n \equiv 8qf_5^7 \pmod{16}, \tag{3.61}$$

which proves (3.54) and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+2}(648n+189)\big)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.62}$$

which implies that the congruence (3.53) is true for $\beta+1$. By induction, the congruence (3.53) is true for all integers $\alpha, \beta \geq 0$.

The equation (3.41) is $\beta = 0$ case of the congruence (3.55). Suppose that the congruence (3.55) is true for $\alpha, \beta \geq 0$. Employing (2.14) in (3.55) and then collecting the coefficients of q^{7n+3} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 7^{2\beta+1}(216n+99)\big)q^n \equiv 8qf_7^5 \pmod{16}, \tag{3.63}$$

which proves (3.56) and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 7^{2\beta+2}(216n+45)\big)q^n \equiv 8f_1^5 \pmod{16}, \tag{3.64}$$

which implies that the congruence (3.55) is true for $\beta+1$. By induction, the congruence (3.55) is true for all integers $\alpha, \beta \geq 0$.

The proofs of rest of the identities (3.57)-(3.60) are simillar to the proofs of the identities (3.53)-(3.56). So, we omit the details. □

Theorem 3.4 For all $n \geq 0$ and $\alpha, \beta \geq 0$, we have

$$ped_{3,6}\big(3^{4\alpha}(72n+15)\big) \equiv 0 \pmod{16}, \tag{3.65}$$

$$ped_{3,6}(72n+69) \equiv 0 \pmod{16}, \tag{3.66}$$

$$ped_{3,6}\big(3^{4\alpha}(216n+117)\big) \equiv 0 \pmod{16}, \tag{3.67}$$

$$ped_{3,6}\big(3^{4\alpha}(1944n+1053)\big) \equiv 0 \pmod{16}, \tag{3.68}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta}(72n+39)\big)q^n \equiv 8f_1^{13} \pmod{16}, \tag{3.69}$$

$$ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+1}(360n+72i+51)\big) \equiv 0 \pmod{16}, \tag{3.70}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(5^{2\beta}(72n+21)\big)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.71}$$

$$ped_{3,6}\big(5^{2\beta+1}(360n+72s+33)\big) \equiv 0 \pmod{16}, \tag{3.72}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta}(216n+63)\big)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.73}$$

$$ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+1}(1080n+216s+99)\big) \equiv 0 \pmod{16}, \tag{3.74}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta}(648n+351)\big)q^n \equiv 8f_1^{13} \pmod{16}, \tag{3.75}$$

$$ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+1}(3240n+648i+455)\big) \equiv 0 \pmod{16}, \tag{3.76}$$

where $i = 0, 1, 3, 4$ *and* $s = 0, 2, 3, 4$.

Proof From the equations (3.18) and (3.40), we obtain (3.65) and (3.66) respectively.

From the equations (3.42) and (3.48), we obtain (3.67) and (3.68) respectively.
The equation (3.19) becomes

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(72n+39)\big)q^n \equiv 8f_1^{13} \pmod{16}, \tag{3.77}$$

which is $\beta=0$ case of (3.69). Suppose that the congruence (3.69) is true for $\beta\geq 0$. Utilizing (2.13) in (3.69) and then comparing the coefficients of q^{5n+3} on both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+1}(72n+51)\big)q^n \equiv 8q^2f_5^{13} \pmod{16}, \tag{3.78}$$

which proves (3.70) and

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}\cdot 5^{2\beta+2}(72n+39)\big)q^n \equiv 8f_1^{13} \pmod{16}, \tag{3.79}$$

which implies that the congruence (3.69) is true for $\beta+1$. Hence, by induction, the congruence (3.69) holds for all integers $\alpha,\beta\geq 0$.

The rest of the proofs of the identities (3.71)-(3.76) are similar to the proofs of the identities (3.53) and (3.69). So, we omit the details. □

Theorem 3.5 For all $n\geq 0$ and $\alpha\geq 0$, we have

$$ped_{3,6}(24n+7) \equiv 0 \pmod 8, \tag{3.80}$$

$$ped_{3,6}(24n+23) \equiv 0 \pmod 8, \tag{3.81}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(3^{4\alpha}(24n+3)\big)q^n \equiv 2f_1f_2 \pmod 8, \tag{3.82}$$

$$ped_{3,6}\big(3^{4\alpha+1}(72n+24i+9)\big) \equiv 0 \pmod 8, \tag{3.83}$$

$$ped_{3,6}\big(3^{4\alpha+3}(72n+24i+9)\big) \equiv 0 \pmod 8, \tag{3.84}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(5^{2\alpha}(24n+11)\big)q^n \equiv 4f_2f_3^3 \pmod 8, \tag{3.85}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(5^{2\alpha+1}(24n+7)\big)q^n \equiv 4q^2f_{10}f_{15}^3 \pmod 8, \tag{3.86}$$

$$ped_{3,6}\big(5^{2\alpha+1}(120n+24s+7)\big) \equiv 0 \pmod 8, \tag{3.87}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(7^{2\alpha}(24n+5)\big)q^n \equiv 4f_1^5 \pmod 8, \tag{3.88}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\big(7^{2\alpha+1}(24n+11)\big)q^n \equiv 4qf_7^5 \pmod 8, \tag{3.89}$$

$$ped_{3,6}\big(7^{2\alpha+1}(168n+24t+11)\big) \equiv 0 \pmod 8, \tag{3.90}$$

where $i=1,2$, $s=0,1,3,4$ and $t=0,2,3,4,5,6$.

Proof From the equations (3.15) and (3.17), we obtain (3.80) and (3.81) respectively.
The equation (3.13) becomes

$$\sum_{n=0}^{\infty} ped_{3,6}(8n+3)q^n \equiv 2\frac{f_1^2f_2^2f_6}{f_3} \pmod 8. \tag{3.91}$$

Using (2.9) in (3.91), we find that

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+3)q^n \equiv 2f_1f_2 \pmod 8, \tag{3.92}$$

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+11)q^n \equiv 4f_2f_3^3 \pmod 8 \tag{3.93}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}(24n+19)q^n \equiv 2f_1f_6^3 \pmod 4. \tag{3.94}$$

The equation (3.92) is $\alpha = 0$ case of (3.82). Suppose that the congruence (3.82) is true for $\alpha \geq 0$. Utilizing (2.9) in (3.82) and then comparing the coefficients of q^{3n}, q^{3n+1} and q^{3n+2} on both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(72n+3)\right)q^n \equiv 2f_1 \pmod 4, \tag{3.95}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(72n+27)\right)q^n \equiv 6f_3f_6 \pmod 8 \tag{3.96}$$

and

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(72n+51)\right)q^n \equiv 4\frac{f_6^3}{f_1} \pmod 8. \tag{3.97}$$

The equation (3.96) implies (3.83) and

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(216n+27)\right)q^n \equiv 6f_1f_2 \pmod 8. \tag{3.98}$$

Again, using (2.9) in (3.98), we obtain

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha}(648n+243)\right)q^n \equiv 2f_3f_6 \pmod 8, \tag{3.99}$$

which implies (3.84) and

$$\sum_{n=0}^{\infty} ped_{3,6}\left(3^{4\alpha+4}(24n+3)\right)q^n \equiv 2f_1f_2 \pmod 8, \tag{3.100}$$

which proves that the congruence (3.82) is true for $\alpha + 1$. By induction, the congruence (3.82) holds for all integer $\alpha \geq 0$.

The rest of the proofs of the identities are similar. So, we omit the details. □

Theorem 3.6 *For all* $n \geq 0$ and $\alpha \geq 0$, *we have*

$$ped_{3,6}(24n+9) \equiv 0 \pmod 4, \tag{3.101}$$

$$ped_{3,6}(24n+17) \equiv 0 \pmod 4, \tag{3.102}$$

$$ped_{3,6}(24n+13) \equiv 0 \pmod 4, \tag{3.103}$$

$$ped_{3,6}(72n+3) \equiv \begin{cases} 2 \pmod 4 & \text{if n is a pentagonal number,} \\ 0 \pmod 4 & \text{otherwise,} \end{cases} \tag{3.104}$$

$$ped_{3,6}\left(3^{4\alpha}(72n+51)\right) \equiv 0 \pmod 4, \tag{3.105}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\left(5^{2\alpha}(24n+19)\right)q^n \equiv 2f_1f_6^3 \pmod 4, \tag{3.106}$$

$$\sum_{n=0}^{\infty} ped_{3,6}\left(5^{2\alpha+1}(24n+23)\right)q^n \equiv 2q^3f_5f_{30}^3 \pmod 4, \tag{3.107}$$

$$ped_{3,6}\left(5^{2\alpha+1}(120n+24i+23)\right) \equiv 0 \pmod 4, \tag{3.108}$$

where $i = 0, 1, 2, 4$.

Proof The equation (3.33) becomes

$$\sum_{n=0}^{\infty} ped_{3,6}(8n+1)q^n \equiv \frac{f_6}{f_3} \pmod 4. \tag{3.109}$$

Equating the coefficients of q^{3n+1} and q^{3n+2} on both sides of the above equation, we arrive at (3.101) and (3.102) respectively.

From the equation (3.36), we get (3.103).

From the congruences (3.95) and (3.97), we obtain (3.104) and (3.105) respectively.

The congruence (3.94) is $\alpha = 0$ case of (3.106). The rest of the proofs of the identities are similar. So, we omit the details. □

4 Congruences for $ped_{9,18}(n)$

Theorem 4.1 *For all* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2 \cdot 3^{4\alpha+2}n + \frac{7 \cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 24f_1f_2f_3^6 \pmod{32}, \tag{4.1}$$

$$ped_{9,18}\left(2 \cdot 3^{4\alpha+5}n + \frac{23 \cdot 3^{4\alpha+4}+1}{4}\right) \equiv ped_{9,18}\left(2 \cdot 3^{4\alpha+3}n + \frac{23 \cdot 3^{4\alpha+2}+1}{4}\right) \pmod{32}, \tag{4.2}$$

$$ped_{9,18}\left(2 \cdot 3^{4\alpha+6}n + \frac{13 \cdot 3^{4\alpha+5}+1}{4}\right) \equiv ped_{9,18}\left(2 \cdot 3^{4\alpha+4}n + \frac{13 \cdot 3^{4\alpha+3}+1}{4}\right) \pmod{32}. \tag{4.3}$$

Proof Setting $j = 9$ and $k = 18$ in (1.5), we see that

$$\sum_{n=0}^{\infty} ped_{9,18}(n)q^n = \frac{f_4f_9}{f_1f_{18}}. \tag{4.4}$$

Using (2.11) in (4.4) and then collecting the coefficients of q^{3n+1} from both sides of the resultant equation, we get

$$\sum_{n=0}^{\infty} ped_{9,18}(3n+1)q^n = \frac{f_2^2f_3^4f_{12}}{f_1^4f_6^3}. \tag{4.5}$$

Utilizing (2.4) in (4.5), we find that

$$\sum_{n=0}^{\infty} ped_{9,18}(6n+4)q^n \equiv 4f_1f_2f_6^2 \pmod{32}. \tag{4.6}$$

Using (2.9) in (4.6), we get

$$\sum_{n=0}^{\infty} ped_{9,18}(18n+4)q^n \equiv 4f_1^5 \pmod 8, \tag{4.7}$$

$$\sum_{n=0}^{\infty} ped_{9,18}(18n+10)q^n \equiv 4f_2^2f_3^3 \pmod{8} \tag{4.8}$$

and

$$\sum_{n=0}^{\infty} ped_{9,18}(18n+16)q^n \equiv 24f_1f_2f_3^6 \pmod{32}, \tag{4.9}$$

which is $\alpha = 0$ case of (4.1). Suppose that the congruence (4.1) is true for $\alpha \geq 0$. Employing (2.9) in the equation (4.1) and then collecting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we get

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+3}n + \frac{7\cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 8f_1^7 \pmod{16}, \tag{4.10}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+3}n + \frac{5\cdot 3^{4\alpha+3}+1}{4}\right)q^n \equiv 8f_1^6f_3f_6 \pmod{32} \tag{4.11}$$

and

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+3}n + \frac{23\cdot 3^{4\alpha+2}+1}{4}\right)q^n \equiv 16f_1^5f_6^3 \pmod{32}. \tag{4.12}$$

Using (2.8) in (4.11), we get

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+4}n + \frac{5\cdot 3^{4\alpha+3}+1}{4}\right)q^n \equiv 8f_1^5 \pmod{16}, \tag{4.13}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+4}n + \frac{13\cdot 3^{4\alpha+3}+1}{4}\right)q^n \equiv 16f_2^2f_3^3 \pmod{32} \tag{4.14}$$

and

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+4}n + \frac{7\cdot 3^{4\alpha+4}+1}{4}\right)q^n \equiv 8f_1f_2f_3^6 \pmod{32}. \tag{4.15}$$

Utilizing (2.9) in (4.15), we get

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+5}n + \frac{7\cdot 3^{4\alpha+4}+1}{4}\right)q^n \equiv 8f_1^7 \pmod{16}, \tag{4.16}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+5}n + \frac{5\cdot 3^{4\alpha+5}+1}{4}\right)q^n \equiv 24f_1^6f_3f_6 \pmod{32} \tag{4.17}$$

and

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+5}n + \frac{23\cdot 3^{4\alpha+4}+1}{4}\right)q^n \equiv 16f_1^5f_3^6 \pmod{32}. \tag{4.18}$$

In view of the congruences (4.12) and (4.18), we obtain (4.2).

Using (2.8) in (4.17), we find that

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+6}n + \frac{5\cdot 3^{4\alpha+5}+1}{4}\right)q^n \equiv 8f_1^5 \pmod{16}, \tag{4.19}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+6}n + \frac{13\cdot 3^{4\alpha+5}+1}{4}\right)q^n \equiv 16f_2^2f_3^3 \pmod{32} \tag{4.20}$$

and

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+6}n+\frac{7\cdot 3^{4\alpha+6}+1}{4}\right)q^n \equiv 24f_1f_2f_3^6 \pmod{32}, \tag{4.21}$$

which implies that the congruence (4.1) is true for $\alpha+1$. By induction, the congruence (4.1) is true for all integer $\alpha \geq 0$.

In view of the congruences (4.14) and (4.20), we obtain (4.3). □

Theorem 4.2 *For all* $n \geq 0$ and $\alpha, \beta \geq 0$, *we have*

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta}n+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta}+1}{4}\right)q^n \equiv 8f_1^7 \pmod{16}, \tag{4.22}$$

$$ped_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}(5n+i)+\frac{11\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}+1}{4}\right) \equiv 0 \pmod{16}, \tag{4.23}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+4}\cdot 7^{2\beta}n+\frac{5\cdot 3^{4\alpha+3}\cdot 7^{2\beta}+1}{4}\right)q^n \equiv 8f_1^5 \pmod{16}, \tag{4.24}$$

$$ped_{9,18}\left(2\cdot 3^{4\alpha+4}\cdot 7^{2\beta+1}(7n+s)+\frac{11\cdot 3^{4\alpha+3}\cdot 7^{2\beta+1}+1}{4}\right) \equiv 0 \pmod{16}, \tag{4.25}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+5}\cdot 5^{2\beta}n+\frac{7\cdot 3^{4\alpha+4}\cdot 5^{2\beta}+1}{4}\right)q^n \equiv 8f_1^7 \pmod{16}, \tag{4.26}$$

$$ped_{9,18}\left(2\cdot 3^{4\alpha+5}\cdot 5^{2\beta+1}(5n+i)+\frac{11\cdot 3^{4\alpha+4}\cdot 5^{2\beta+1}+1}{4}\right) \equiv 0 \pmod{16}, \tag{4.27}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+6}\cdot 7^{2\beta}n+\frac{5\cdot 3^{4\alpha+5}\cdot 7^{2\beta}+1}{4}\right)q^n \equiv 8f_1^5 \pmod{16}, \tag{4.28}$$

$$ped_{9,18}\left(2\cdot 3^{4\alpha+6}\cdot 7^{2\beta+1}(7n+s)+\frac{11\cdot 3^{4\alpha+5}\cdot 7^{2\beta+1}+1}{4}\right) \equiv 0 \pmod{16}, \tag{4.29}$$

where $i = 0, 2, 3, 4$ and $s = 0, 2, 3, 4, 5, 6$.

Proof The congruence (4.10) is $\beta = 0$ case of (4.22). Suppose that the congruence (4.22) is true for $\alpha, \beta \geq 0$. Employing (2.13) in the equation (4.22) and then collecting the coefficients of q^{5n+2}, we get

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta+1}n+\frac{11\cdot 3^{4\alpha+2}\cdot 5^{2\beta+1}+1}{4}\right)q^n \equiv 8qf_5^7 \pmod{16}, \tag{4.30}$$

which proves (4.23) and

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+3}\cdot 5^{2\beta+2}n+\frac{7\cdot 3^{4\alpha+2}\cdot 5^{2\beta+2}+1}{4}\right)q^n \equiv 8f_1^7 \pmod{16}, \tag{4.31}$$

which proves that the congruence (4.22) is true for $\beta+1$. By induction, the congruence (4.22) holds for all integers $\alpha, \beta \geq 0$.

The congruence (4.13) is $\beta = 0$ case of (4.24). Suppose that the congruence (4.24) is true for $\beta \geq 0$. Employing (2.14) in the equation (4.24) and then collecting the coefficients of q^{7n+3}, we get

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+4}\cdot 7^{2\beta+1}n+\frac{11\cdot 3^{4\alpha+3}\cdot 7^{2\beta+1}+1}{4}\right)q^n \equiv 8qf_7^5 \pmod{16}, \tag{4.32}$$

which proves (4.25) and

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2\cdot 3^{4\alpha+4}\cdot 7^{2\beta+2}n+\frac{5\cdot 3^{4\alpha+3}\cdot 7^{2\beta+2}+1}{4}\right)q^n \equiv 8f_1^5 \pmod{16}, \tag{4.33}$$

which implies that the congruence (4.24) is true for $\beta+1$. By induction, the congruence (4.24) is true for all integers $\alpha, \beta \geq 0$.

The proofs of the congruences (4.26)-(4.29) are similar to the proofs of the congruences (4.22)-(4.25). So, we omit the details. □

Theorem 4.3 *For all* $n \geq 0$ and $\beta \geq 0$, *we have*

$$ped_{9,18}(18n+10) \equiv 0 \pmod 4, \tag{4.34}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(18 \cdot 7^{2\beta} n + \frac{15 \cdot 7^{2\beta}+1}{4}\right) q^n \equiv 4 f_1^5 \pmod 8, \tag{4.35}$$

$$ped_{9,18}\left(18 \cdot 7^{2\beta+1}(7n+s) + \frac{33 \cdot 7^{2\beta+1}+1}{4}\right) \equiv 0 \pmod 8, \tag{4.36}$$

where $s = 0, 2, 3, 4, 5, 6$.

Proof From the equation (4.8), we obtain (4.34).

The equation (4.7) is $\beta = 0$ case of (4.35). The rest of the proofs of the identities (4.35) and (4.36) are similar to the proofs of the identities (4.24) and (4.25). So, we omit the details. □

Theorem 4.4 *For all* $n \geq 0$, *we have*

$$ped_{9,18}(54n+52) \equiv 0 \pmod{27}, \tag{4.37}$$

$$ped_{9,18}(162n+88) \equiv 0 \pmod{27}, \tag{4.38}$$

$$ped_{9,18}(162n+142) \equiv 0 \pmod{27}. \tag{4.39}$$

Proof Using (2.7) in (4.5) and then collecting the coefficients of q^{3n+2} from both sides of the resultant equation, we find that

$$\sum_{n=0}^{\infty} ped_{9,18}(9n+7) q^n = 12 \frac{f_2^3 f_3^6 f_4}{f_1^{10}}. \tag{4.40}$$

From the binomial theorem, it is easy to see that for any positive integers ℓ and k,

$$f_\ell^{3k} \equiv f_{3\ell}^k \pmod 3, \tag{4.41}$$

$$f_\ell^{9k} \equiv f_{3\ell}^{3k} \pmod 9. \tag{4.42}$$

The equation (4.40) along with (4.42) becomes

$$\sum_{n=0}^{\infty} ped_{9,18}(9n+7) q^n \equiv 12 \frac{f_2^3 f_3^3 f_4}{f_1} \pmod{27}. \tag{4.43}$$

Using (2.5) in (4.43), we get

$$\begin{aligned} \sum_{n=0}^{\infty} ped_{9,18}(18n+7) q^n &\equiv 12 \frac{f_1 f_2^4 f_3^2}{f_6} \pmod{27} \\ &\equiv 3 f_1 f_2 f_3^2 \pmod 9 \end{aligned} \tag{4.44}$$

and

$$\sum_{n=0}^{\infty} ped_{9,18}(18n+16) q^n \equiv 12 f_1^3 f_6^3 \pmod{27}. \tag{4.45}$$

Employing (2.8) in (4.45), we obtain (4.37) and

$$\sum_{n=0}^{\infty} ped_{9,18}(54n+34) q^n \equiv 18 f_6 f_9 \pmod{27}, \tag{4.46}$$

which implies (4.38) and (4.39). □

Theorem 4.5 *For any prime* $p > 5$ with $\left(\frac{-6}{p}\right) = -1$, $1 \le j \le p-1$ and for all $n \ge 0$, $\alpha \ge 0$, *we have*

$$\sum_{n=0}^{\infty} ped_{9,18}\left(162 \cdot p^{2\alpha} n + \frac{135 \cdot p^{2\alpha} + 1}{4}\right) q^n \equiv 18 f_2 f_3 \pmod{27}, \tag{4.47}$$

$$ped_{9,18}\left(162 \cdot p^{2\alpha+1}(pn+i) + \frac{135 \cdot p^{2\alpha+2} + 1}{4}\right) \equiv 0 \pmod{27}, \tag{4.48}$$

where $i = 1, 2, 3, \cdots p-1$.

Proof From the equation (4.46), we have

$$\sum_{n=0}^{\infty} ped_{9,18}(162n+34) q^n \equiv 18 f_2 f_3 \pmod{27}, \tag{4.49}$$

which is $\alpha = 0$ case of the congruence (4.47). Suppose that the congruence (4.47) is true for all $\alpha \ge 0$. Employing (2.15) in (4.47), we get

$$\sum_{n=0}^{\infty} ped_{9,18}\left(162 \cdot p^{2\alpha+1} n + \frac{135 \cdot p^{2\alpha+2} + 1}{4}\right) q^n \equiv 18 f_{2p} f_{3p} \pmod{27}, \tag{4.50}$$

which proves (4.48) and

$$\sum_{n=0}^{\infty} ped_{9,18}\left(162 \cdot p^{2\alpha+2} n + \frac{135 \cdot p^{2\alpha+2} + 1}{4}\right) q^n \equiv 18 f_2 f_3 \pmod{27}, \tag{4.51}$$

which proves that the congruence (4.47) is true for $\alpha + 1$. By induction, the congruence (4.47) holds for all integer $\alpha \ge 0$. □

Theorem 4.6 *For all* $n \ge 0$ and $\alpha \ge 0$, *we have*

$$ped_{9,18}(18n+7) \equiv 0 \pmod{12}, \tag{4.52}$$

$$ped_{9,18}(18n+13) \equiv 0 \pmod{12}, \tag{4.53}$$

$$ped_{9,18}(6n+3) \equiv 0 \pmod{3}, \tag{4.54}$$

$$ped_{9,18}(6n+5) \equiv 0 \pmod{6}, \tag{4.55}$$

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2 \cdot 3^{4\alpha+2} n + \frac{3^{4\alpha+3} + 1}{4}\right) q^n \equiv 3 f_1 f_2 f_3^2 \pmod{9}. \tag{4.56}$$

Proof Employing (2.6) in (4.4) and then collecting the coefficients of q^{2n+1} from both sides of the resultant equation, we obtain

$$\sum_{n=0}^{\infty} ped_{9,18}(2n+1) q^n = \frac{f_2^3 f_3 f_{18}}{f_1^3 f_6 f_9}. \tag{4.57}$$

Using (2.12) in (4.57), we have

$$\sum_{n=0}^{\infty} ped_{9,18}(6n+1) q^n = \frac{f_6}{f_3} + 12q \frac{f_2 f_6^6}{f_1^5 f_3^2}, \tag{4.58}$$

$$\sum_{n=0}^{\infty} ped_{9,18}(6n+3) q^n = 3 \frac{f_2^3 f_3^4}{f_1^7} \tag{4.59}$$

and

$$\sum_{n=0}^{\infty} ped_{9,18}(6n+5)q^n = 6\frac{f_2^2 f_3 f_6^3}{f_1^6}. \tag{4.60}$$

From the equations (4.58)-(4.60), we obtain (4.52)-(4.55) respectively.

The congruence (4.44) is $\alpha = 0$ case of (4.56). Suppose that the congruence (4.56) is true for $\alpha \geq 0$. Employing (2.9) in the equation (4.56) and then collecting the coefficients of q^{3n}, we get

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2 \cdot 3^{4\alpha+3}n + \frac{3^{4\alpha+3}+1}{4}\right)q^n \equiv 3\frac{f_1 f_2 f_3^4}{f_6^2} \pmod 9. \tag{4.61}$$

Using (2.9) in (4.61) and then collecting the coefficients of q^{3n} and q^{3n+1} alternatively from the resultant equations, we obtain

$$\sum_{n=0}^{\infty} ped_{9,18}\left(2 \cdot 3^{4\alpha+6}n + \frac{3^{4\alpha+7}+1}{4}\right)q^n \equiv 3f_1 f_2 f_3^2 \pmod 9, \tag{4.62}$$

which implies that the congruence (4.56) is true for $\alpha + 1$. By induction, the congruence (4.56) is true for all integer $\alpha \geq 0$. □

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