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ORIGINAL RESEARCH

A note on stability of non-surjective ε-isometries between the positive cones of L^p-spaces

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Abstract Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be two measure spaces, $1 < p < \infty$, $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$, and let $X_i^+ = \{f \in X_i : f \geq 0\ a.e.\}$ be the positive cone of X_i. In this paper, we first show that for every standard ε-isometry $F : X_1^+ \to X_2^+$, there exists a bounded linear surjective operator $T : X_2 \to X_1$ with $\|T\| = 1$ such that

$$\|TF(u) - u\| \leq 4\varepsilon, \text{ for all } u \in X_1^+.$$

As its application, we prove that if $\liminf_{t \to +\infty} \text{dist}(ty, F(X_1^+))/t < 1/2$ for all $y \in S_{X_2^+}$, then there exists a unique additive surjective isometry $V : X_1^+ \to X_2^+$ so that

$$\|F(u) - V(u)\| \leq 4\varepsilon, \text{ for all } u \in X_1^+.$$

We also unify some results concerning the Hyers-Ulam stability of almost surjective ε-isometries between the positive cones of two L^p-spaces.

Keywords ε-Isometries · Stability · L^p-space

Mathematics Subject Classification Primary 46B04 · 46B20 · Secondary 46A20

1 Introduction

The letters X and Y denote real Banach spaces, and $F : X \to Y$ an ε-isometry for some $\varepsilon \geq 0$, i.e.

$$|\|F(x) - F(y)\| - \|x - y\|| \leq \varepsilon, \quad \text{for all } x, y \in X.$$

The mapping F is called standard if $F(0) = 0$. 0-isometry is simply called isometry.

The classical Mazur-Ulam theorem [15] states that every surjective standard isometry between real Banach spaces is necessarily linear. For general isometries, Figiel [11] extended Mazur-Ulam by showing the remarkable stability result: Every standard isometry $F : X \to Y$ admits a linear left-inverse $T : \overline{span}F(X) \to X$ of norm one, i.e., $TF = Id_X$ with $\|T\| = 1$.

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For standard ε-isometry $F : X \to Y$, the basic question is how close is F to an actual isometry. Making use of a result of Gruber [13], Gevietz [12] proved that for any surjective standard ε-isometry $F : X \to Y$, there exists a linear surjective isometry $U : X \to Y$ such that

$$\|F(x) - U(x)\| \leq 5\varepsilon, \text{ for all } x \in X. \tag{1.1}$$

Omladič and Šemrl [16, 1995] showed that the constant 5 in (1.1) can be improved to a best constant 2 (See, also, [2, pp. 360]). This stability of the ε-isometry is called Hyers-Ulam stability. Sun [19] obtained the Hyers-Ulam stability of the almost surjective ε-isometries between the positive cones of two L^p-spaces.

Since 90s of the last century, the study of stability property of non-surjective ε-isometry has become an active area. A standard ε-isometry $F : X \to Y$ is said to be stable provided there exist a positive number γ and a bounded linear operator $T : \overline{span}F(X) \to X$ such that

$$\|TF(x) - x\| \leq \gamma\varepsilon, \text{ for all } x \in X. \tag{1.2}$$

Motivated by Figiel's theorem, Qian [17] first studied stability of non-surjective ε-isometries, and he showed that every standard ε-isometry between two L^p spaces $(1 < p < \infty)$ is stable with $\gamma = 6$ in (1.2). Furthermore, the constant 6 has been improved to 2 by Šemrl and Väisälä [18]. However, Qian [17] presented a simple counterexample showing that if a separable Banach space Y contains an uncomplemented closed subspace X then for every $\varepsilon > 0$ there is a standard ε-isometry $F : X \to Y$ which is unstable. The counterexample says that the situations in investigating the stability property of non-surjective ε-isometries are rather different and complicated. Recently, Cheng and his collaborators [3, 4] obtained weakly universal stability formulas for all Banach spaces. It is shown that this result is not only a generalized version of Figiel's theorem, but also very useful to the study of stability of ε-isometries (see, for example, [7, 20–23]).

There are many situations in which one has to deal with a wedge (or cone) of a Banach space, instead of the whole space (see, [1, 5, 8–10]). For example, every norm semigroup of a Banach space is a wedge. In [19], we showed a weakly stability formula for ε-isometry between the positive cones of two L^p-spaces: Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be two measure spaces, $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, and let $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$, $Y_i = L^q(\Omega_i, \Sigma_i, \mu_i)$. Suppose that $F : X_1^+ \to X_2^+$ is a standard ε-isometry, then for every $x^* \in Y_1^+$, there exists a unique $\phi \in Y_2^+$ with $\|x^*\| = \|\phi\| \equiv r$ such that

$$|\langle x^*, u\rangle - \langle \phi, F(u)\rangle| \leq 2r\varepsilon, \text{ for all } u \in X_1^+. \tag{1.3}$$

In this paper, we mainly study the stability properties of the non-surjective ε-isometries between the positive cones of two L^p-spaces. In section 2, making use of the weakly stability formula we have estabilished, we prove the stability for the ε-isometry $F : X_1^+ \to X_2^+$ (Theorem 2.2): there exists a bounded linear surjective operator $T : X_2 \to X_1$ with $\|T\| = 1$ such that

$$\|TF(u) - u\| \leq 4\varepsilon, \text{ for all } u \in X_1^+.$$

In the section 3, applying the Theorem 2.2, we prove that if $\liminf_{t\to+\infty} \mathrm{dist}(ty, F(X_1^+))/t < 1/2$ for all $y \in S_{X_2^+}$, then there exists a unique additive surjective isometry $V : X_1^+ \to X_2^+$ so that

$$\|F(u) - V(u)\| \leq 4\varepsilon, \text{ for all } u \in X_1^+.$$

Furthermore, we unify some results concerning the Hyers-Ulam stability of almost surjective ε-isometries between the positive cones of two L^p-spaces.

All symbols and notations in this paper are standard. We use X to denote a real Banach space and X^* its dual. B_X (resp. S_X) stands for the closed unit ball (resp. the unit sphere) of X. $B(X, Y)$ denotes for the space of all bounded linear operators from X to Y. For a subspace $M \subset X$, $M^\perp$ presents the annihilator of M, i.e., $M^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{for all } x \in M\}$ Given a topological space Ω, suppose that $\Delta \subset \Omega$ and $\phi : \Omega \to \mathbb{R}$ is a real-valued function, we use the notion $\phi|_\Delta$ to denote the restriction of ϕ to Δ.

2 Stability of non-surjective ε-isometries between the positive cones of L^p-spaces

Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be two measure spaces, $1 < p < \infty$, and let $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$ with X_i^+ be the positive cone of X_i. In this section, we mainly show the stability of the ε-isometries between the positive cones of two L^p-spaces, i.e., for every ε-isometry $F : X_1^+ \to X_2^+$, there exists a bounded linear operator

$T: X_2 \to X_1$ of norm one such that $TF - Id_{X_1^+}$ is uniformly bounded by 4ε on X_1^+. We first recall some preliminaries.

Recall a wedge in a Banach space X is a convex set closed under multiplication by non-negative scalars. The span of a wedge W is simply $W - W$. W is said to be (respectively, almost) reproducing whenever $W - W = X$ (respectively, $W - W$ is dense in X). A wedge W is said to be a cone if $W \cap -W = 0$.

Lemma 2.1. [19, *Lemma* 2.3] *Suppose that* X, Y *are Banach space,* $W \subset X$ *is a closed wedge,* Y *is uniformly convex. Let* $F: W \to Y$ *be a standard* ε*-isometry. Then*

$$V(x) = \lim_{t\to\infty} \frac{F(tx)}{t}, \ \forall x \in W, \tag{2.1}$$

defines an additive isometry $V: W \to Y$.

The Theorem 2.2 below is the main result of this section, and it is essentially inspired by [19, Theorem 3.1]. We give the details for the convenience of the reader.

Theorem 2.2. *Let* $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, *be two measure spaces,* $1 < p < \infty$, *and let* $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$ *with* X_i^+ *be the positive cone of* X_i. *Suppose that* $F: X_1^+ \to X_2^+$ *is a standard* ε*-isometry, then there exists a bounded linear surjective operator* $T: X_2 \to X_1$ *with* $\|T\| = 1$ *such that*

$$\|TF(u) - u\| \le 4\varepsilon, \quad \text{for all } u \in X_1^+. \tag{2.2}$$

Proof. Suppose that $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$, $Y_i = L^q(\Omega_i, \Sigma_i, \mu_i)$ with $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Note that they are both (super) reflexive, and we have $X_i^* = Y_i$, $Y_i^* = X_i$. For the standard ε-isometry $F: X_1^+ \to X_2^+$, we define $V: X_1^+ \to X_2^+$ by

$$V(u) = \lim_{t\to\infty} \frac{F(tu)}{t}, \ \forall u \in X_1^+. \tag{2.3}$$

According to Lemma 2.1, $V: X_1^+ \to X_2^+$ is an additive isometry. For any $x \in X_1$, let $u(\omega) = \max\{x(\omega), 0\}$ and $v(\omega) = \max\{-x(\omega), 0\}$ so that $x = u - v \in X_1^+ - X_1^+ = X_1$. Put

$$U(x) = V(u) - V(v), \tag{2.4}$$

then $U: X_1 \to X_2$ is well-defined. The additivity of V implies that U is additive. We assert that U is a linear isometry. For any $x_1, x_2 \in X_1$, there exist $u_1, v_1, u_2, v_2 \in X_1^+$ such that $x_1 = u_1 - v_1$ and $x_2 = u_2 - v_2$. Then, we have

$$\begin{aligned}
\|U(x_1) - U(x_2)\| &= \|(V(u_1) - V(v_1)) - (V(u_2) - V(v_2))\| \\
&= \|(V(u_1) + V(v_2)) - (V(v_1) + V(u_2))\| \\
&= \|V(u_1 + v_2) - V(v_1 + u_2)\| \\
&= \|(u_1 + v_2) - (v_1 + u_2)\| \\
&= \|(u_1 - v_1) - (u_2 - v_2)\| \\
&= \|x_1 - x_2\|.
\end{aligned}$$

This yields that U is an isometry. This and the additivity together imply that U is a linear isometry. Let $M = U(X_1)$, then M is linear isometric to X_1. By [14, p. 162], M is 1-complemented in X_2, i.e., there exist a closed subspace $N \subset X_2$ with $M \cap N = \{0\}$ and a projection $P: X_2 \to M$ along N such that $X_2 = M + N$ and $\|P\| = 1$. We shall show that $T = U^{-1}P$ is the desired mapping. Clearly, T is surjective with $\|T\| = \|P\| = 1$.

For every $x^* \in Y_1^+$, by [19, Theorem 2.5], there exists a unique $\phi \in Y_2^+$ satisfies (1.3). Define $Q: Y_1^+ \to Y_2^+$ by

$$Q(x^*) = \phi. \tag{2.5}$$

Then Q is well-defined and norm preserved. Define $Q|_M: Y_1^+ \to M^*$ by

$$Q|_M(x^*) = Q(x^*)|_M. \tag{2.6}$$

We assert that $Q|_M$ is an additive isometry. It follows from (1.3), for $x_1^*, x_2^* \in Y_1^+$, we have

$$|\langle x_1^* + x_2^*, u\rangle - \langle Q(x_1^*) + Q(x_2^*), F(u)\rangle| \le 2(\|x_1^*\| + \|x_2^*\|)\varepsilon, \text{ for all } u \in X_1^+; \tag{2.7}$$

and

$$|\langle x_1^* + x_2^*, u\rangle - \langle Q(x_1^* + x_2^*), F(u)\rangle| \le 2\|x_1^* + x_2^*\|\varepsilon, \text{ for all } u \in X_1^+. \tag{2.8}$$

(2.7) and (2.8) together imply that

$$|\langle Q(x_1^*) + Q(x_2^*) - Q(x_1^* + x_2^*), F(u)\rangle| \le 4(\|x_1^*\| + \|x_2^*\|)\varepsilon, \text{ for all } u \in X_1^+. \tag{2.9}$$

Substituting nu for u in (2.9), and dividing the both sides by n, we obtain that

$$\langle Q(x_1^*) + Q(x_2^*) - Q(x_1^* + x_2^*), V(u)\rangle = 0.$$

This implies that $(Q(x_1^*) + Q(x_2^*))|_M = Q(x_1^* + x_2^*)|_M$ and $Q|_M$ is additive.

For $x_1^*, x_2^* \in Y_1^+$, by (1.3) again, we have

$$|\langle x_1^* - x_2^*, u\rangle - \langle Q(x_1^*) - Q(x_2^*), F(u)\rangle| \le 2(\|x_1^*\| + \|x_2^*\|)\varepsilon, \text{ for all } u \in X_1^+. \tag{2.10}$$

Thus, for every $x = u - v \in X_1^+ - X_1^+ = X_1$, we have

$$|\langle x_1^* - x_2^*, u - v\rangle - \langle Q(x_1^*) - Q(x_2^*), F(u) - F(v)\rangle| \le 4(\|x_1^*\| + \|x_2^*\|)\varepsilon. \tag{2.11}$$

Since Y_1 is uniformly smooth, there exists a unique $x = u_1 - v_1 \in X_1^+ - X_1^+ = X_1$ with $\|x\| = 1$ so that $x = d\|x_1^* - x_2^*\|$, i.e., $\langle x, x_1^* - x_2^*\rangle = \|x_1^* - x_2^*\|$. Substituting nu_1 and nv_1 for u and v in (2.11), and dividing the both sides by n, we have

$$\langle x_1^* - x_2^*, u_1 - v_1\rangle = \langle Q(x_1^*) - Q(x_2^*), V(u_1) - V(v_1)\rangle. \tag{2.12}$$

Note that $\|V(u_1) - V(v_1)\| = \|u_1 - v_1\| = 1$ and $V(u_1) - V(v_1) \in M$, we obtain that $\|Q|_M(x_1^*) - Q|_M(x_2^*)\| \ge \|x_1^* - x_2^*\|$.

On the other hand, note that $M \subset X_2$ is reflexive, strictly convex and Gateaux smooth, M^* is also Gateaux smooth. There exists $g \in M$ with $\|g\| = 1$ such that $g = d\|Q|_M(x_1^*) - Q|_M(x_2^*)\|$. Since $U : X_1 \to M$ is a linear surjective isometry, there is a $x = u_1 - v_1 \in X_1^+ - X_1^+ = X_1$ so that $U(x) = g$. Substituting nu_1 and nv_1 for u and v in (2.11), and dividing the both sides by n, we have

$$\langle x_1^* - x_2^*, u_1 - v_1\rangle = \langle Q(x_1^*) - Q(x_2^*), V(u_1) - V(v_1)\rangle = \langle Q|_M(x_1^*) - Q|_M(x_2^*), g\rangle. \tag{2.13}$$

Note that $\|u_1 - v_1\| = \|U(u_1 - v_1)\| = 1$, we obtain that $\|x_1^* - x_2^*\| \ge \|Q|_M(x_1^*) - Q|_M(x_2^*)\|$. Consequently, we have $\|x_1^* - x_2^*\| = \|Q|_M(x_1^*) - Q|_M(x_2^*)\|$ and $Q|_M$ is an isometry. Therefore, for any $x^* \in Y_1^+$, we have

$$\|x^*\| = \|Q(x^*)\| = \|Q|_M(x^*)\| = \|Q|_M(x^*)P\|.$$

Since X_2 is uniformly convex and uniformly smooth, we have $Q(x^*) = Q|_M(x^*)P$.

For any $x^* \in Y_1$, we define that $x_1^*(\omega) = \max\{x^*(\omega), 0\}$ and $x_2^*(\omega) = \max\{-x^*(\omega), 0\}$ so that $x_1^*, x_2^* \in Y_1^+$ and $x^* = x_1^* - x_2^*$. Put

$$\bar{Q}|_M(x^*) = Q|_M(x_1^*) - Q|_M(x_2^*), \tag{2.14}$$

then $\bar{Q}|_M : Y_1 \to M^*$ is well-defined. Since $Q|_M$ is an additive isometry, $\bar{Q}|_M$ is a linear isometry. For every $x = u - v \in X_1^+ - X_1^+ = X_1$ and $x^* = x_1^* - x_2^* \in Y_1^+ - Y_1^+ = Y_1$, by substituting nu_1 and nv_1 for u and v in (2.11), and dividing the both sides by n, we have

$$\begin{aligned}\langle x^*, x\rangle &= \langle x_1^* - x_2^*, u - v\rangle \\ &= \langle Q(x_1^*) - Q(x_2^*), V(u_1) - V(v_1)\rangle \\ &= \langle Q(x_1^*)|_M - Q(x_2^*)|_M, U(u_1 - v_1)\rangle \\ &= \langle \bar{Q}|_M(x^*), U(x)\rangle.\end{aligned} \tag{2.15}$$

This implies that $x^* = \bar{Q}|_M(x^*)U$.

Note that $U : X_1 \to M$ is a linear surjective isometry. Let $U^* : M^* \to Y_1$ be the conjugate operator of U, then U^* is also a linear surjective isometry.

For any $x^* \in Y_1$ and $x \in X_1$, by (2.15) we have

$$\langle U^* \bar{Q}|_M(x^*), x\rangle = \langle \bar{Q}|_M(x^*), Ux\rangle = \langle x^*, x\rangle. \tag{2.16}$$

Thus, $U^* \bar{Q}|_M = Id_{Y_1}$. This implies that $\bar{Q}|_M = (U^*)^{-1}$ and $\bar{Q}|_M$ is a linear surjective isometry.

For any $\varphi \in M^*$, there exists $x^* \in Y_1$ so that $\bar{Q}|_M(x^*) = \varphi$. Let $x_1^* = \max\{x^*(w), 0\}$ and $x_2^* = \max\{-x^*(w), 0\}$. Then we have $x^* = x_1^* - x_2^* \in Y_1^+ - Y_1^+ = Y_1$ and $\|x_1^*\| \le \|x^*\|$ and $\|x_2^*\| \le \|x^*\|$. By (2.15), for any $u \in X_1^+$, we have

$$\begin{aligned}
|\langle \varphi, U(u) - PF(u)\rangle| &= |\langle \varphi, U(u)\rangle - \langle \varphi, PF(u)\rangle| \\
&= |\langle \bar{Q}|_M(x^*), U(u)\rangle - \langle \bar{Q}|_M(x^*), PF(u)\rangle| \\
&= |\langle x^*, u\rangle - \langle Q|_M(x_1^*) - Q|_M(x_2^*), PF(u)\rangle| \\
&= |\langle x^*, u\rangle - \langle Q|_M(x_1^*)P - Q|_M(x_2^*)P, F(u)\rangle| \\
&= |\langle x^*, u\rangle - \langle Q(x_1^*) - Q(x_2^*), F(u)\rangle| \\
&= |\langle x_1^* - x_2^*, u\rangle - \langle Q(x_1^*) - Q(x_2^*), F(u)\rangle| \\
&\le 2(\|x_1^*\| + \|x_2^*\|)\varepsilon \le 4\|x^*\|\varepsilon \\
&= 4\|\varphi\|\varepsilon.
\end{aligned}$$

This yields that

$$\|U(u) - PF(u)\| \le 4\varepsilon, \text{ for all } u \in X_1^+. \tag{2.17}$$

Let $T = U^{-1}P$, and we complete the proof. □

3 Some applications

Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be two measure spaces, $1 < p < \infty$, and let $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$ with X_i^+ be the positive cone of X_i. Suppose that $F : X_1^+ \to X_2^+$ is a standard ε-isometry. In this section, we will present some applications of our main result Theorem 2.2. We first present a sufficient and necessary condition to guarantee that F can be uniformly approximated by an additive surjective isometry $V : X_1^+ \to X_2^+$ (Theorem 3.1). Then we unify some results concerning the Hyers-Ulam stability of almost surjective ε-isometries between the positive cones of two L^p-spaces. We first recall some notations which are used in Vestfrid [21].

Suppose that Y is a Banach space, for a nonempty $A \subset Y$, and $y \in S_Y$, we denote

$$\varrho(y, A) = \liminf_{|t| \to \infty} \operatorname{dist}(ty, A)/|t|, \text{ and } \tau(A) = \sup_{y \in S_Y} \varrho(y, A).$$

Vestfrid [21] first proved that for any Banach spaces X, Y, and for any standard ε-isometry $F : X \to Y$, if $\tau(F(X)) < 1/2$, then there is a surjective linear isometry $U : X \to Y$ such that

$$\|F(x) - U(x)\| \le 2\varepsilon, \text{ for all } x \in X.$$

Furthermore, Cheng et al. [6] showed that the result is still true if the condition $\tau(F(X)) < 1/2$ was replaced by $\varrho(y, F(X)) < 1/2$ for all $y \in S_Y$ (see also, [23]).

The following theorem can be regarded as a generalization of Cheng [6, Theorem 2.5]. For a standard ε-isometry $F : X_1^+ \to X_2^+$, and for $y \in S_{X_2^+}$, we denote

$$\overline{\varrho}(y, F(X_1^+)) = \liminf_{t \to +\infty} \operatorname{dist}(ty, F(X_1^+))/t, \text{ and } \overline{\tau}(F(X_1^+)) = \sup_{y \in S_{X_2^+}} \overline{\varrho}(y, F(X_1^+)).$$

Theorem 3.1. *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be two measure spaces, $1 < p < \infty$, and let $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$ with X_i^+ be the positive cone of X_i. Suppose that $F : X_1^+ \to X_2^+$ is a standard ε-isometry, then there is a unique additive surjective isometry $V : X_1^+ \to X_2^+$ such that*

$$\|F(u) - V(u)\| \leq 4\varepsilon, \text{ for all } u \in X_1^+, \tag{3.1}$$

if and only if

$$\overline{\varrho}(y, F(X_1^+)) = \liminf_{t \to +\infty} \operatorname{dist}(ty, F(X_1^+))/t < 1/2 \text{ for all } y \in S_{X_2^+}. \tag{3.2}$$

Proof. Necessity: Suppose that there exists a unique additive surjective isometry $V : X_1^+ \to X_2^+$ such that (3.1) holds, then for any $y \in S_{X_2^+}$ and $t \in \mathbb{R}^+$, there exist $\{u_t\} \in X_1^+$ such that $V(u_t) = ty$. This implies that

$$\operatorname{dist}(ty, F(X_1^+)) \leq \|V(u_t) - F(u_t)\| \leq 4\varepsilon.$$

Thus,

$$\overline{\varrho}(y, F(X_1^+)) = \liminf_{t \to +\infty} \operatorname{dist}(ty, F(X_1^+))/t \leq \liminf_{t \to +\infty} 4\varepsilon/t = 0 < 1/2.$$

(3.2) is shown.

Sufficiency: Suppose that $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)$, $Y_i = L^q(\Omega_i, \Sigma_i, \mu_i)$ with $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $F : X_1^+ \to X_2^+$ is a standard ε-isometry. Let $V : X_1^+ \to X_2^+$ be as in (2.3), then V is an additive isometry. We shall show that V is surjective, and it is the desired mapping.

Let $U : X_1 \to X_2$ be as in (2.4), then U is a linear isometry with $U|_{X_1^+} = V$. Note that $M = U(X_1)$ is 1-complemented in X_2. Then, there exists a closed subspace $N \subset X_2$ such that $X_2 = M + N$ with $M \cap N = \{0\}$. $P : X_2 \to M$ is the projection along N with $\|P\| = 1$. We assert that U is surjective. Suppose, to the contrary, that $M = U(X_1)$ is a proper closed subspace of X_2. There exists a $g \in N \cap X_2^+$ with $\|g\| = 1$. Hypothesis (3.2) implies that there exist a sequence $(t_n) \subset \mathbb{R}^+$ with $t_n \to \infty$ and a sequence $\{u_n\}_{n \in \mathbb{N}} \subset X_1^+$ such that

$$\lim_n \frac{\|t_n g - F(u_n)\|}{t_n} \equiv \beta < 1/2. \tag{3.3}$$

According to Theorem 2.2, there exists a bounded linear operator $T : X_2 \to X_1$ with $\|T\| = 1$ such that

$$\|TF(u) - u\| \leq 4\varepsilon, \quad \text{for all } u \in X_1^+. \tag{3.4}$$

Since $\|T\| = 1$, $\|T^*\| = 1$. For any $x^* \in S_{Y_1}$, we have $\|T^*(x^*)\| \leq \|x^*\|$. Note that Y_1 is uniformly smooth, there exists a $x = u - v \in X_1^+ - X_1^+$ with $\|x\| = 1$ such that $\langle x^*, x \rangle = \|x^*\|$. Substituting nu and nv for u in (3.4) respectively, and dividing the both sides by n, we have

$$\left\| T\left(\frac{F(nu)}{n} - \frac{F(nv)}{n}\right) - (u - v) \right\| \leq \frac{8\varepsilon}{n}. \tag{3.5}$$

Therefore,

$$\begin{aligned} &|\langle T^* x^*, \frac{F(nu)}{n} - \frac{F(nv)}{n} \rangle - \langle x^*, u - v \rangle| \\ &\quad = |\langle x^*, T(\frac{F(nu)}{n} - \frac{F(nv)}{n}) - (u - v) \rangle| \to 0. \end{aligned} \tag{3.6}$$

This implies that $\langle T^* x^*, V(u) - V(v) \rangle = \langle T^* x^*, U(x) \rangle = \langle x^*, x \rangle$, and $\|T^*(x^*)\| \geq \|x^*\|$. This shows that $T^* : Y_1 \to Y_2$ is a linear isometry. Note that $T^* x^* = d\|U(x)\|$. Since $\|T^* x^* P\| \leq 1$ and $\langle T^* x^* P, U(x) \rangle = \langle T^* x^*, U(x) \rangle = \|U(x)\|$, we have $T^* x^* P = T^* x^*$. This entails that $T^* x^* \in N^{\perp}$.

For any u_n in (3.3), let $u_n^* \in S_{Y_1}$ with $u_n^* = d\|u_n\|$. Therefore, by (3.3), we have

$$
\begin{aligned}
4\varepsilon &\geq \|TF(u_n) - u_n\| \geq |\langle u_n^*, TF(u_n) - u_n\rangle| \\
&= |\langle u_n^*, u_n\rangle - \langle T^*u_n^*, F(u_n)\rangle| \\
&= |\langle u_n^*, u_n\rangle - \langle T^*u_n^*, F(u_n) - t_n g\rangle| \\
&\geq \|u_n\| - \|F(u_n) - t_n g\| \\
&\geq \|F(u_n)\| - \varepsilon - \|F(u_n) - t_n g\| \\
&\geq \|t_n g\| - 2\|F(u_n) - t_n g\| - \varepsilon \\
&= t_n\left(1 - 2\frac{\|F(u_n) - t_n g\|}{t_n}\right) - \varepsilon \to \infty.
\end{aligned}
$$

This contradiction says that U is surjective.

Substituting nu for u in (3.4), and dividing the both sides by n, we have

$$\|TV(u) - u\| = \lim_n \left\|T\left(\frac{F(nu)}{n}\right) - u\right\| \leq \lim_n \frac{4\varepsilon}{n} = 0. \tag{3.7}$$

This yields that $TV = Id_{X_1^+}$, and $TU = Id_{X_1}$. Note that U is a linear surjective isometry, T is also a linear surjective isometry with $T = U^{-1}$. Therefore, by (3.4), for any $u \in X_1^+$, we have

$$\|F(u) - V(u)\| = \|U^{-1}(F(u) - V(u))\| = \|TF(u) - u\| \leq 4\varepsilon. \tag{3.8}$$

The proof is complete. □

Let (M_1, d_1) and (M_2, d_2) be two metric spaces and δ be a nonnegative real number. Recall that a map $F : M_1 \longrightarrow M_2$ is called δ-surjective, if for every $y \in M_2$ there is $x \in M_1$ with $d_2(y, F(x)) \leq \delta$ (see [18]). (see, [3]) A subset N in a metric space (M, d) is called a sublinear growth net in metric d provided for any fixed $w_0 \in M$,

$$\lim_{d(w,w_0)\to\infty} \frac{d(w,N)}{d(w,w_0)} = 0.$$

According to Theorem 3.1 , we present the following results concerning the Hyers-Ulam stability of almost surjective ε-isometries between the positive cones of two L^p-spaces.

Theorem 3.2. *Let X_i, $F : X_1^+ \to X_2^+$ be as in Theorem 3.1. Then the following statements are equivalent:*

(1) *There is a unique additive surjective isometry $V : X_1^+ \to X_2^+$ such that $\|F(u) - V(u)\| \leq 4\varepsilon$ for all $u \in X_1^+$,*
(2) *F is δ-surjective for some $\delta \geq 0$,*
(3) *$F(X_1^+)$ contains a sublinear growth net of X_2^+,*
(4) $\overline{\varrho}(y, F(X_1^+)) = \liminf_{t\to+\infty} \mathrm{dist}(ty, F(X_1^+))/t = 0$ for all $y \in S_{X_2^+}$,
(5) $\overline{\varrho}(y, F(X_1^+)) = \liminf_{t\to+\infty} \mathrm{dist}(ty, F(X_1^+))/t < 1/2$ for all $y \in S_{X_2^+}$,
(6) $\overline{\tau}(F(X_1^+)) < 1/2$,
(7) $\overline{\tau}(F(X_1^+)) = 0$.

Proof. [19, Theorem 3.1] yields that (1) ⇔ (2) ⇔ (3). It is trivial that (1) ⇒ (4) ⇒ (7) ⇒ (6) ⇒ (5). Theorem 3.1 implies that (5) ⇒ (1). Therefore, all of them are equivalent. □

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