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ORIGINAL RESEARCH

Global analysis of an environmental and death transmission model for Ebola outbreak with perturbation

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Abstract In this paper, we inspect the global asymptotic stability (GAS) of Ebola–epidemic using a modified Susceptible-Infected-Deceased-Pathogen (SIDP) model with perturbation. The feasible region is obtained, and we determine the basic reproduction number using the next-generation matrix method. The GAS of the disease-free and endemic equilibria are discuss and provide parameter conditions for which the stability exists. We show theoretically that the contaminated environmental Ebola human–deceased transmission model is GAS with and without probability. Numerical simulation to support our theoretical findings are presented.

Keywords Global analysis · Environmental transmission · Death transmission · Ebola outbreak · Stochastic differential equation

Mathematics Subject Classification 34D10 · 34D23 · 92B05

1 Introduction

Since the discovery of Ebola virus disease (EVD) in Africa, many mathematical models have been developed to understand the dynamical behaviour of the disease. Among these models is the Susceptible-Infectious-Recovered-Deceased-Pathogen (SIR–DP) model. The SIR-DP model incorporated transmission from a deceased-human through funeral practice and that of the contaminated environment. Berge et al. [1] introduced the model and the disease-free and endemic equilibrium were investigated theoretically, and numerically using nonstandard finite difference approach. Theoretically, the authors showed that at endemic equilibrium of the disease the complete model only exhibits local asymptotic stability (LAS) whereas in the absence of shedding and deceased manipulation in the environment the system is globally asymptotically stable (GAS).

However, scholars have provided an alternative methods to inspect the behaviour of the EVD using the SIR-DP model. Altaf and Atangana [2] used differential operators that possess crossover properties and

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fading memory to inspect the dynamical behaviour of EVD. Also, Yamazaki [3] used reaction-diffusion partial differential equation (PDE) by including diffusive terms to all equations except that of the deceased-human individual. The author showed that the well-posedness and stability analysis does not hold for the Ebola PDE.

The main objective of this work is to continue investigating the endemic equilibrium GAS analysis for the SIR–DP model in [1], where it was shown theoretically that for the model endemic equilibrium to be GAS the environment should not be contaminated by viral particles shed from an infected and deceased individuals. Stochastic perturbation which is also an alternative approach to study stability analysis of a disease model has not yet been considered to the SIR–DP model. Therefore, in this paper, we extend the model from [1] by including perturbation terms to each equation and investigate the GAS of the disease-free and endemic equilibrium of Ebola epidemic. Including perturbation to a deterministic disease model improved the stability analysis of an epidemics population [4].

2 Dynamical analysis of the SIDP deterministic model

We focus our study in section (4) of [1] where the virus recruitment into the environment is zero. We also neglect the dR(t)/dt equation and only Ebola death comes into the deceased compartment. The numbers of individuals in the susceptible, infected,and dead compartments are respectively $S = S(t), I = I(t), D = D(t)$ and the concentration of the virus in the environment is $P = P(t)$ at time t. The transmission rates for an infected and deceased host are respectively given as β_1 and β_2, and λ for the indirect transmission. The recruitment rate due to birth or migration is π, natural death measure at μ and disease induce death rate is measure at δ. The recovered rate with permanent immunity is γ, and viral particles are shed into the environment by an infectious individual at a rate ξ and, from the deceased, at a rate α. The burial time rate is measure at $\boldsymbol{b}$ and the virus decayed rate in the environment is η. The deterministic SIDP model is give as

$$
\begin{aligned}
dS &= (\pi - \beta_1 SI - \beta_2 SD - \lambda SP - \mu S)dt,\\
dI &= (\beta_1 SI + \beta_2 SD + \lambda SP) - (\mu + \delta + \gamma)I)dt,\\
dD &= (\delta I - bD\ )dt,\\
dP &= (\xi I + \alpha D - \eta P)dt,
\end{aligned}
\tag{1}
$$

which is subjected to initial conditions $S(0) = S_0,\ I(0) = I_0,\ D(0) = D_0$ and, $P(0) = P_0$.

2.1 The feasible region and basic reproduction number

Letting $N = S + I$ and adding the first two equations of model (1) gives $dN/dt \le \pi - \mu N(t)$. For $I_0 \ge\ 0$, it implies that $\limsup_{t\to\infty} N(t) \le \pi/\mu$. As a consequence, the third equation of model (1) yields $\limsup_{t\to\infty} D(t) \le \pi\delta/b\mu$. Also, the fourth equation yields $\limsup_{t\to\infty} P(t) \le \pi(b\xi + \alpha\delta)/b\mu\eta$.

Hence the feasible region

$$
A = \left\{ S(t),\ I(t),\ D(t),\ P(t) \in \mathbb{R}^4_+,\ 0 \le N(t) \le \tfrac{\pi}{\mu}; D(t) \le \tfrac{\pi\delta}{b\mu}; P(t) \le \tfrac{\pi(b\xi+\alpha\delta)}{\eta b\mu} \right\},
\tag{2}
$$

for the deterministic model is positively invariant with respect to model (1).

The basic reproduction number (R_0) is a parametric equation use to describe whether disease-free equilibrium (DFE) state exist (i.e., the disease will die out) or an endemic equilibrium (EE) state exists (i.e., the disease continues to persist). The R_0 is derived using the next-generator matrix method [5] defined as $K = FV^{-1}$.

The matrices associated with the DFE $\left(E^0 = \left(S^0, I^0,\ D^0,\ \ P^0\right) = (\pi/\mu, 0, 0, 0)\right)$ are

$$
F = \begin{pmatrix} \frac{\pi\beta_1}{\mu} & \frac{\pi\beta_2}{\mu} & \frac{\lambda\pi}{\mu} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} (\mu+\delta+\gamma) & 0 & 0 \\ -\delta & b & 0 \\ -\xi & -\alpha & \eta \end{pmatrix},
$$

and

$$\mathrm{K}=\mathrm{FV}^{-1}=\begin{pmatrix}\frac{\pi\beta_1}{\mu(\mu+\delta+\gamma)}+\frac{\pi\delta\beta_2}{b\mu(\mu+\delta+\gamma)}+\frac{\lambda\pi(b\xi+\alpha\delta)}{\eta b\mu(\mu+\delta+\gamma)} & \frac{\pi\beta_2}{b\mu}+\frac{\lambda\alpha\pi}{\eta b\mu} & \frac{\pi\beta_2}{b\mu}+\frac{\lambda\pi(b+\alpha)}{\eta b\mu}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

The spectral radius of K is

$$\rho(\mathrm{K})=\frac{\pi\beta_1}{\mu(\mu+\delta+\gamma)}+\frac{\pi\delta\beta_2}{b\mu(\mu+\delta+\gamma)}+\frac{\lambda\pi(b\xi+\alpha\delta)}{\eta b\mu(\mu+\delta+\gamma)},$$

hence, the basic reproduction number of the model is

$$\mathrm{R}_0=\frac{\pi\beta_1}{\mu(\mu+\delta+\gamma)}+\frac{\pi\delta\beta_2}{b\mu(\mu+\delta+\gamma)}+\frac{\lambda\pi(b\xi+\alpha\delta)}{\eta b\mu(\mu+\delta+\gamma)}. \quad (3)$$

2.2 Stability analyses of the SIDP deterministic model

Theorem 1 *If $R_0 \leq 1$, the disease-free equilibrium is globally asymptotically stable in* A.

Proof Using the Lyapunov function V: A→ℝ, where,

$$V(S,\ I,\ D,\ P)=I(t).$$

Then the time derivative of V is

$$\frac{dV(S,I,D,P)}{dt}=\beta_1 S^0 I^0+\beta_2 D^0 S^0+\lambda P^0 S^0-(\mu+\delta+\gamma)I^0 \quad (4)$$

where $S^0, I^0,\ D^0,$ *and* P^0, are the solutions of the DFE. That is equating model (1) to zero and let $S = S^0, I = I^0, D = D^0, andP = P^0$, the solutions of the DFE points in terms of I^0 are

$$S^0=\frac{\pi-(\mu+\delta+\gamma)I^0}{\mu},$$
$$D^0=\frac{\delta}{b}I^0,$$
$$P^0=\frac{(b\xi+\alpha\delta)I^0}{b\eta}.$$

By substituting $S^0,\ D^0,$ *and* P^0 into (4) we get

$$\begin{aligned}\frac{dV}{dt} =\ & \beta_1\left(\frac{\pi-(\mu+\delta+\gamma)I^0}{\mu}\right)I^0+\beta_2\left(\frac{\pi-(\mu+\delta+\gamma)I^0(t)}{\mu}\right)\left(\frac{\delta}{b}\right)I^0\\ & -(\mu+\delta+\gamma)\mathrm{I}^0\\ & +\lambda\left(\frac{\pi-(\mu+\delta+\gamma)I^0}{\mu}\right)\left(\frac{(b\xi+\alpha\delta)I^0}{b\eta}\right),\end{aligned} \quad (5)$$

simplifying the right-hand side, we get

$$\begin{aligned}\frac{dV}{dt} =\ & (\mu+\delta+\gamma)I^0\left(\frac{\pi\beta_1}{\mu(\mu+\delta+\gamma)}+\frac{\pi\delta\beta_2}{b\mu(\mu+\delta+\gamma)}+\frac{\lambda\pi(b\xi+\alpha\delta)}{\eta b\mu(\mu+\delta+\gamma)}\right)-(\mu+\delta+\gamma)\mathrm{I}^0\\ & -\frac{((\mu+\delta+\gamma)I^0)^2}{\pi}\left(\frac{\pi\beta_1}{\mu(\mu+\delta+\gamma)}+\frac{\pi\delta\beta_2}{b\mu(\mu+\delta+\gamma)}+\frac{\lambda\pi(b\xi+\alpha\delta)}{\eta b\mu(\mu+\delta+\gamma)}\right).\end{aligned}$$

In terms of R_0,

$$\frac{dV}{dt}=(\mu+\delta+\gamma)I^0\left(\mathrm{R}_0-\frac{(\mu+\delta+\gamma)I^0}{\pi}\mathrm{R}_0-1\right). \quad (6)$$

Thus, if $dV/dt \leq 0$, then for $I^0 = 0$,$\mathrm{R}_0 \leq 1$. Also, if $dV/dt = 0$, then for $I^0 = 0$, R_0= 1. Hence; according to LaSalle's principle [6] the theorem is complete.

Model 1 also has an EE points, $E^* = (S^*,\ I^*,\ D^*,\ P^*)$ where

$$\begin{aligned} S^* &= \frac{\pi}{\mu R_0}\ ,\quad I^* = \frac{\pi}{(\mu+\delta+\gamma)}\left(1-\frac{1}{R_0}\right),\quad D^* = \frac{\pi\delta}{b(\mu+\delta+\gamma)}\left(1-\frac{1}{R_0}\right),\\ P^* &= \frac{\pi(b\xi+\alpha\delta)}{b\eta(\mu+\delta+\gamma)}\left(1-\frac{1}{R_0}\right). \end{aligned} \tag{7}$$

□

Theorem 2 *If $R_0 \geq 1$, the endemic equilibrium is globally asymptotically stable in* A.

Proof Let the Lyapunov function defined as

$$V_1(U,t) = \frac{1}{2}\left(U_1^2 + U_2^2 + U_3^2 + U_4^2\right),$$

where $U_1 = S - S^*$; $U_2 = I - I^*$; $U_3 = D - D^*$ and $U_4 = P - P^*$.

$$\frac{dV_1}{dt} = U_1\frac{dU_1}{dt} + U_2\frac{dU_2}{dt} + U_3\frac{dU_3}{dt} + U_4\frac{dU_4}{dt},$$

and

$$\frac{dU_1}{dt} = \frac{dS}{dt};\ \frac{dU_2}{dt} = \frac{dI}{dt}\ ,\ \frac{dU_3}{dt} = \frac{dD}{dt}\ ,\ \frac{dU_4}{dt} = \frac{dP}{dt}.$$

Consider $U_1 = S - S^*$, from the EE equations

$$S^* = \frac{\pi}{(\beta_1 I^* + \beta_2 D^* + \lambda P^* + \mu)},$$

since $V_1(0,0,0,0) = (0,0,0,0)$ and $V_1(U) > 0$, hence $S = S^*,\ I = I^*,\ D = D^*$ and $P = P^*$.
Therefore

$$\begin{aligned} U_1 &= S - \frac{\pi}{(\beta_1 I + \beta_2 D + \lambda P + \mu)};\quad U_2 = I - \frac{(\beta_1 SI + \beta_2 SD + \lambda SP)}{(\mu+\delta+\gamma)};\ U_3 = D - \delta\frac{I}{b}\\ U_4 &= P - \frac{(\xi I + \alpha D)}{\eta}. \end{aligned} \tag{8}$$

then,

$$\begin{aligned} \frac{dV_1}{dt} = &\left(S - \frac{\pi}{(\beta_1 I + \beta_2 D + \lambda P + \mu)}\right)[\pi - S(\mu + \beta_1 I + \beta_2 D + \lambda P)]\\ &+ \left(I - \frac{(\beta_1 SI + \beta_2 SD + \lambda SP)}{(\mu+\delta+\gamma)}\right)[(\beta_1 SI + \beta_2 SD + \lambda SP - (\mu+\delta+\gamma)I)]\\ &+ \left(D - \delta\frac{I}{b}\right)[\delta I - bD] + \left(P - \frac{(\xi I + \alpha D)}{\eta}\right)[\xi I + \alpha D - \eta P]. \end{aligned} \tag{9}$$

Simplifying the right-hand side we get

$$\begin{aligned} \frac{dV_1}{dt} = &-(\beta_1 I + \beta_2 D + \lambda P + \mu)\left(S - \frac{\pi}{(\beta_1 I + \beta_2 D + \lambda P + \mu)}\right)^2 - \eta\left(P - \frac{(\xi I + \alpha D)}{\eta}\right)^2\\ &-(\mu+\delta+\gamma)\left(I - \frac{(\beta_1 SI + \beta_2 SD + \lambda SP)}{(\mu+\delta+\gamma)}\right)^2 - b\left(D - \delta\frac{I}{b}\right)^2 \leq 0, \end{aligned}$$

□

which completes the proof of Theorem 2.

3 Dynamical analysis of the stochastic SIDP

We now modify model (1) by perturbing the model around it equilibrium points (E^0 *and* E^*). We assume the stochastic perturbation is of white noise type[7], which is directly proportional to distance $S(t)$, $I(t)$, $D(t)$, $P(t)$, from the equilibrium values of model (1) and, influence on the time derivative $dS(t)/dt, dI(t)/dt, d(D(t))/dt, dP(t)/dt$, respectively. The SDE-SIDP model can be represented as

$$dX(t) = f(X,\ t)dt + g(X,\ t)dB(t), \tag{10}$$

where f is the deterministic equations of model (1), $g(X,\ t)$ is the diffusion matrix which is also locally Lipschitz function, and $B(t)$ is the 4-dimensional independent standard Brownian motions defined on a complete probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains the null set-$\mathbb{P}$). The differential operator L associated with (10) is define as

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{4} f_i(X,\ t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{4} g^T(X,\ t)\left(\frac{\partial^2}{\partial x_i \partial x_j}\right)_{ij} g(X,t), \tag{11}$$

where T means transpose

$$f_i(X) = \begin{pmatrix} \pi - \beta_1 SI - \beta_2 SD - \lambda SP - \mu S \\ \beta_1 SI + \beta_2 SD + \lambda SP - (\mu + \delta + \gamma)I \\ \delta I - bD \\ \xi I + \alpha D - \eta P \end{pmatrix},$$

and

$$g(X) = \begin{pmatrix} (X_1)\theta_1 & 0 & 0 & 0 \\ 0 & (X_2)\theta_2 & 0 & 0 \\ 0 & 0 & (X_3)\theta_3 & 0 \\ 0 & 0 & 0 & (X_4)\theta_4 \end{pmatrix}.$$

where $\theta_i (1 \leq i)$, is known as the stochastic environmental constant.

If we consider $V(X,t) \in C^{2,1}(\mathbb{R}^d \times (0,\infty), \mathbb{R}_+)$ as Lyapunov function, and the action of L on V is define as

$$\begin{aligned} LV(X,t) = {} & \frac{\partial V(X,t)}{\partial t} + f^T(X,t)\frac{\partial V(X,t)}{\partial x} \\ & + \frac{1}{2} Trace\left(g(X,\ t)\left(\frac{\partial^2 V(X,\ t)}{\partial x_i \partial x_j}\right)_{ij} g^T(X,\ t)\right), \end{aligned} \tag{12}$$

where $1 \leq i,j \leq 4$

$$\frac{\partial V(X,t)}{\partial x} = col\left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_4}\right).$$

Lemma 1 ([8, 9]). *Suppose that there exists a function $V(X,t) \in C^2(\Omega)$ satisfying the following inequalities:*

$$\begin{aligned} \omega_1 |X|^p \leq V(X,\ t) \leq \omega_2 |X|^p, \\ LV(X,\ t) \leq -\omega_3 |X|^p. \end{aligned}$$

where $p > 0$ and $\omega_1, \omega_2, \omega_3$ are positive constants, then the trivial solution of (10) is exponentially p-stable for all $t \geq 0$. Note that, if $p = 2$, then the trivial solution of (10) is globally asymptotically stable in probability.

The stability definition in [8] *is considered throughout this section.*

3.1 Stochastic stability of the disease-free equilibrium

Theorem 3 *Assume that* $\theta_1^2 < 2\mu$ *and* $R_0 < 1$ *holds, then the DFE* $(\pi/\mu\ ,\ 0,\ 0,\ 0,)$ *is globally asymptotically stable in probability.*

Proof Let define the Lyapunov function as

$$V_2(\ \mathrm{U}\ ) = \frac{1}{2}C_1U_1^2 + C_2U_2 + C_3U_3 + C_4U_4, \tag{13}$$

where $U_1 = S - \frac{\pi}{\mu}, U_2 = I \quad U_3 = D, \quad U_4 = P$ and $C_i\ (i = 1, 2, 3, 4)$ are real positive constants to be chosen later. The linearization of this system around E^0 takes the form

$$dU(t) = f(U(t))dt + g(U(t))dB(t),$$

where $U(t) = col(U_1(t), \quad U_2(t), U_3(t),\ U_4(t))$

$$f(U(t)) = \begin{pmatrix} -U_1\mu - \frac{\pi}{\mu}\beta_1U_2 - \frac{\pi}{\mu}\beta_2U_3 - \frac{\pi}{\mu}\lambda U_4 \\ -U_2\left(\mu + \delta + \gamma - \frac{\pi}{\mu}\beta_1\right) + \frac{\pi}{\mu}\beta_2U_3 + \frac{\pi}{\mu}\lambda U_4 \\ \delta U_2 - bU_3 \\ U_2\xi + \alpha U_3 - \eta U_4 \end{pmatrix},$$

and

$$g(U(t)) = \begin{pmatrix} U_1\theta_1 & 0 & 0 & 0 \\ 0 & U_2\theta_2 & 0 & 0 \\ 0 & 0 & U_3\theta_3 & 0 \\ 0 & 0 & 0 & U_4\theta_4 \end{pmatrix},$$

hence

$$\left(g(U)\left(\frac{\partial^2 V_2(U)}{\partial U_i \partial U_j}\right)_{ij} g^T(U)\right) = \begin{pmatrix} C_1\theta_1^2U_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using (12), we get

$$\begin{aligned} LV_2 =& C_1U_1\left(-U_1\mu - \frac{\pi}{\mu}\beta_1U_2 - \frac{\pi}{\mu}\beta_2U_3 - \frac{\pi}{\mu}\lambda U_4\right) + C_2\left(U_2\left(\frac{\pi}{\mu}\beta_1 - (\mu + \delta + \gamma)\right)\right. \\ &\left.+\frac{\pi}{\mu}\beta_2U_3 + \frac{\pi}{\mu}\lambda U_4\right) \\ &+C_3(\delta U_2 - bU_3) + C_4(\xi U_2 + \alpha U_3 - \eta U_4) + \frac{1}{2}\theta_1^2U_1^2C_1 \\ =& -C_1\left(\mu - \frac{1}{2}\theta_1^2\right)U_1^2 - \left(\left(\mu + \delta + \gamma - \frac{\pi\beta_1}{\mu}\right)C_2 - (\delta C_3 + \xi C_4)\right)U_2 \\ &- \left(bC_3 - \left(\frac{\pi\beta_2}{\mu}C_2 + \alpha C_4\right)\right)U_3 \\ &- \left(\eta C_4 - \frac{\lambda\pi}{\mu}C_2\right)U_4 - \frac{\pi}{\mu}C_1(\beta_1U_2 + \beta_2U_3 + \lambda U_4)U_1. \end{aligned} \tag{14}$$

Since $S + I \le \pi/\mu$ in A [10], we have $U_1 \le 0, 0 < U_2 \le \frac{\pi}{\mu}, 0 < U_3 \le \frac{\pi}{\mu}, 0 < U_4 \le \frac{\pi}{\mu}$, hence,

$$
\begin{aligned}
LV_2 \leq & -C_1\left(\mu-\frac{1}{2}\theta_1^2\right)U_1^2-\left(\left(\mu+\delta+\gamma-\frac{\pi\beta_1}{\mu}\right)C_2-(\delta C_3+\xi C_4)\right)U_2 \\
& -\left(bC_3-\left(\frac{\pi\beta_2}{\mu}C_2+\alpha C_4\right)\right)U_3-\left(\eta C_4-\frac{\lambda\pi}{\mu}C_2\right)U_4.
\end{aligned} \tag{15}
$$

We choose C_2, C_3 and C_4 such that

$$\delta C_3+\xi C_4<\left(\mu+\delta+\gamma-\frac{\pi\beta_1}{\mu}\right)C_2$$

and

$$\alpha C_4 + \frac{\pi\beta_2}{\mu}C_2<bC_3,$$

which equivalent to

$$\frac{\alpha\delta}{b}C_4+\frac{\pi\delta\beta_2}{b\mu}C_2<\delta C_3<\left(\mu+\delta+\gamma-\frac{\pi\beta_1}{\mu}\right)C_2-\xi C_4,$$

we can see that

$$\left(\frac{\alpha\delta}{b}+\xi\right)C_4<\left((\mu+\delta+\gamma)-\frac{\pi}{\mu}\left(\beta_1+\frac{\delta\beta_2}{b}\right)\right)C_2.$$

We choose C_2 and C_4 such that

$$\frac{\lambda\pi}{\mu}C_2<\eta C_4,$$

which equivalent to

$$\frac{\lambda\pi}{\eta\mu}\left(\frac{\alpha\delta}{b}+\xi\right)C_2<\left(\frac{\alpha\delta}{b}+\xi\right)C_4<\left((\mu+\delta+\gamma)-\frac{\pi}{\mu}\left(\beta_1+\frac{\delta\beta_2}{b}\right)\right)C_2,$$

we can see that

$$\frac{\pi}{\mu}\left(\beta_1+\frac{\delta\beta_2}{b}+\frac{\lambda(b\xi+\alpha\delta)}{\eta b}\right)<(\mu+\delta+\gamma),$$

hence

$$R_0<1.$$

Since the conditions of the DFE of Theorem 3 are satisfy it signify that the solution of (15) is less than zero. Therefore, from (15) we get

$$
\begin{aligned}
LV_2 \leq & -C_1\left(\mu-\frac{1}{2}\theta_1^2\right)U_1^2-\frac{\pi}{\mu}\left(\left(\mu+\delta+\gamma-\frac{\pi\beta_1}{\mu}\right)C_2-(\delta C_3+\xi C_4)\right)U_2^2 \\
& -\frac{\pi}{\mu}\left(bC_3-\left(\frac{\pi\beta_2}{\mu}C_2+\alpha C_4\right)\right)U_3^2-\frac{\pi}{\mu}C_4\left(\eta C_4-\frac{\lambda\pi}{\mu}C_2\right)U_4^2,
\end{aligned} \tag{16}
$$

which according to Lemma 1 complete the proof of Theorem 3. □

3.2 Stochastic stability of the endemic equilibrium

The SDE of the SIDP model can be at its interior EE by using a change of variable technique given as:

$$u_1=S-S^* \;;\quad u_2=I-I^* \;;\quad u_3=D-D^* \quad;\quad u_4=P-P^*.$$

The linearization of this system around the endemic system takes the form

$$du(t) = f(u(t))dt + g(u(t))dB(t), \tag{17}$$

where $u(t) = col(u_1(t),\ \ u_2(t), u_3(t),\ u_4(t))$

$$g(u(t)) = \begin{pmatrix} u_1\theta_1 & 0 & 0 & 0 \\ 0 & u_2\theta_2 & 0 & 0 \\ 0 & 0 & u_3\theta_3 & 0 \\ 0 & 0 & 0 & u_4\theta_4 \end{pmatrix},$$

$$f(u(t)) = \begin{pmatrix} -u_1(\mu + \beta_1 I^* + \beta_2 D^* + \lambda P^*) - \beta_1 S^* u_2 - \beta_2 S^* u_3 - \lambda S^* u_4 \\ u_1(\beta_1 I^* + \beta_2 D^* + \lambda P^*) - u_2(\mu + \delta + \gamma - \beta_1 S^*) + \beta_2 S^* u_3 + \lambda S^* u_4 \\ \delta u_2 - b u_3 \\ u_2 \xi + \alpha u_3 - \eta u_4 \end{pmatrix},$$

From the first EE equation of (1), that is

$$\pi - S^*(\beta_1 I^* + \beta_2 D^* + \lambda P + \mu) = 0 \Rightarrow \beta_1 I^* + \beta_2 D^* + \lambda P^* = \frac{\pi}{S^*} - \mu,$$

Using Lemma 1, we obtain the following theorem.

Theorem 4 *Suppose that* $\theta_1^2 < 2\pi/S^*, \theta_2^2 < 2(\mu + \gamma + \delta),\ \theta_3^2 < 2b,\ \theta_4^2 < 2\eta,$

$$\begin{aligned} &\frac{\alpha^2 S^* \beta_1}{\xi^2} + \frac{S^*(\varphi \beta_2 S^*)^2}{2\pi - S^* \theta_1^2} < \frac{(\delta + \beta_2 S^*)^2}{2(\mu + \gamma + \delta) - \theta_2^2}, \\ &S^* \varphi^2 > \frac{2\pi - S^* \theta_1^2}{2(\mu + \gamma + \delta) - \theta_2^2}, \end{aligned} \tag{18}$$

where $\varphi = (\pi - \mu S^*)/\beta_1 S^{*^2}$, then the EE is globally asymptotically stable in probability.

Proof Let consider a Lyapunov function

$$V_3(u,t) = \frac{1}{2}\left(Z_1 u_1^2 + Z_2 u_2^2 + Z_3 u_1^2 + Z_4 u_4^2\right),$$

where $Z_i (i = 1,2,3,4)$ are positive constants to be chosen later.

$$g^T(u,\ t)\frac{\partial^2 V_3(u,t)}{\partial u^2} g(u,\ t) = \begin{pmatrix} Z_1(u_1\theta_1)^2 & 0 & 0 & 0 \\ 0 & Z_2(u_2\theta_2)^2 & 0 & 0 \\ 0 & 0 & Z_3(u_3\theta_3)^2 & 0 \\ 0 & 0 & 0 & Z_4(u_4\theta_4)^2 \end{pmatrix},$$

$$\frac{1}{2} Trace\left(g^T(u,\ t)\frac{\partial^2 V_3(u,t)}{\partial u^2} g(u,\ t)\right) = \frac{1}{2}\left(\theta_1^2 u_1^2 Z_1 + \theta_2^2 u_2^2 Z_2 + \theta_3^2 u_3^2 Z_3 + \theta_4^2 u_4^2 Z_4\right),$$

By using (12), we get

$$\begin{aligned} LV_3 = &- Z_1\left(\frac{\pi}{S^*} - \frac{1}{2}\theta_1^2\right)u_1^2 - Z_2\left((\mu+\delta+\gamma) - \beta_1 S^*(t) - \frac{1}{2}\theta_2^2\right)u_2^2 \\ &- Z_3\left(b - \frac{1}{2}\theta_3^2\right)u_3^2 - Z_4\left(\eta - \frac{1}{2}\theta_4^2\right)u_4^2 - \left(Z_1\beta_1 S^* - Z_2\left(\frac{\pi - \mu S^*}{S^*}\right)\right)u_1 u_2 \\ &- Z_1\ (u_1 u_3 \beta_2 S^* + u_1 u_4 \lambda S^*) + (Z_3\delta + Z_2\beta_2 S^*)u_2 u_3 \\ &+ (Z_2 \lambda S^* + Z_4\ \xi)u_2 u_4 + Z_4 \alpha u_3 u_4. \end{aligned} \tag{19}$$

We choose $Z_2 = Z_3$ and

$$Z_1 = Z_2 \frac{(\pi - \mu S^*)}{\beta_1 S^{*^2}} = \varphi Z_2.$$

Then we get

$$\begin{aligned} LV_3 = & - \ Z_1\left(\frac{\pi}{S^*} - \frac{1}{2}\theta_1^2\right)u_1^2 - Z_2\left((\mu+\gamma+\delta) - \frac{1}{2}\theta_2^2\right)u_2^2 - Z_3\left(b - \frac{1}{2}\theta_3^2\right)u_3^2 \& + Z_2 S^*\beta_1 u_2^2 \\ & - \ Z_4\left(\eta - \frac{1}{2}\theta_4^2\right)u_4^2 - Z_2\ (\varphi\beta_2 S^* u_1 u_3 + \varphi\lambda S^* u_1 u_4) + Z_3(\delta + \beta_2 S^*)u_2 u_3 \\ & + Z_3\lambda S^* u_2 u_4 + Z_4\ \xi u_2 u_4 + Z_4 \alpha u_3 u_4. \end{aligned} \tag{20}$$

Using Cauchy inequality [11] for $\varphi\beta_2 S^* u_1 u_3$, $(\delta + \beta_2 S^*)u_2 u_3$, $\varphi\lambda S^* u_1 u_4$, $\xi u_2 u_4$, $\alpha u_3 u_4$ and $\lambda S^* u_2 u_4$ we get

$$\begin{aligned} \varphi\beta_2 S^* u_1 u_3 &\le \frac{2S^*(\varphi\beta_2 S^*)^2 u_3^2}{2\pi - S^*\theta_1^2} + \frac{1}{4}\left(\frac{\pi}{S^*} - \frac{1}{2}\theta_1^2\right)u_1^2, \\ \varphi\lambda S^* u_1 u_4 &\le \frac{2S^*(\varphi\lambda S^*)^2 u_4^2}{2\pi - S^*\theta_1^2} + \frac{1}{4}\left(\frac{\pi}{S^*} - \frac{1}{2}\theta_1^2\right)u_1^2, \\ (\delta + \beta_2 S^*)u_2 u_3 &\le \frac{2(\delta + \beta_2 S^*)^2 u_3^2}{2(\mu+\gamma+\delta) - \theta_2^2} + \frac{1}{4}\left((\mu+\gamma+\delta) - \frac{1}{2}\theta_2^2\right)u_2^2, \\ \lambda S^* u_2 u_4 &\le \frac{2(\lambda S^*)^2 u_4^2}{2(\mu+\gamma+\delta) - \theta_2^2} + \frac{1}{4}\left((\mu+\gamma+\delta) - \frac{1}{2}\theta_2^2\right)u_2^2, \\ \xi u_2 u_4 &\le \frac{2\xi^2 u_2^2}{2\eta - \theta_4^2} + \frac{1}{4}\left(\eta - \frac{1}{2}\theta_4^2\right)u_4^2, \\ \alpha u_3 u_4 &\le \frac{2\alpha^2 u_3^2}{2\eta - \theta_4^2} + \frac{1}{4}\left(\eta - \frac{1}{2}\theta_4^2\right)u_4^2. \end{aligned} \tag{21}$$

Substituting (21) into (20), we get

$$\begin{aligned} LV_3 \le & - \ Z_1\left(\frac{2\varphi+1}{2\varphi}\right)\left(\frac{\pi}{S^*} - \frac{1}{2}\theta_1^2\right)u_1^2 - Z_2\frac{1}{2}\left(\mu+\gamma+\delta - \frac{1}{2}\theta_2^2\right)u_2^2 \\ & - \left(Z_4\frac{2\xi^2}{\theta_4^2 - 2\eta} - Z_3 S^*\beta_1\right)u_2^2 \\ & - Z_3\left(b - \frac{1}{2}\theta_3^2\right)u_3^2 - \left(Z_3\left(\frac{2S^*(\varphi\beta_2 S^*)^2}{2\pi - S^*\theta_1^2} - \frac{2(\ \delta + \beta_2 S^*)^2}{2(\mu+\gamma+\delta) - \theta_2^2}\right)\right. \\ & \left. - Z_4\frac{2\alpha^2}{2\eta - \theta_4^2}\right)u_3^2 \\ & - Z_4\frac{1}{2}\left(\eta - \frac{1}{2}\theta_4^2\right)u_4^2 - \ Z_3\left(\frac{2S^*(\varphi\lambda S^*)^2}{2\pi - S^*\theta_1^2} - \frac{2(\lambda S^*)^2}{2(\mu+\gamma+\delta) - \theta_2^2}\right)u_4^2. \end{aligned} \tag{22}$$

Let choose Z_3 and Z_4 such that

$$Z_3 S^*\beta_1 < Z_4\frac{2\xi^2}{\theta_4^2 - 2\eta},$$

and

$$Z_4 \frac{2\alpha^2}{2\eta - \theta_4^2} < Z_3 \left(\frac{2S^*(\varphi \beta_2 S^*)^2}{2\pi - S^* \theta_1^2} - \frac{2(\delta + \beta_2 S^*)^2}{2(\mu + \gamma + \delta) - \theta_2^2} = f \right),$$

which is equivalent to

$$Z_4 \frac{2\alpha^2 S^* \beta_1}{(2\eta - \theta_4^2) f} < Z_3 S^* \beta_1 < Z_4 \frac{2\xi^2}{\theta_4^2 - 2\eta},$$

we can see that

$$\frac{\alpha^2 S^* \beta_1}{(2\eta - \theta_4^2) f} < \frac{\xi^2}{\theta_4^2 - 2\eta},$$

therefore

$$\frac{\alpha^2 S^* \beta_1}{\xi^2} < \frac{(\delta + \beta_2 S^*)^2}{2(\mu + \gamma + \delta) - \theta_2^2} - \frac{S^*(\varphi \beta_2 S^*)^2}{2\pi - S^* \theta_1^2},$$

hence

$$\frac{\alpha^2 S^* \beta_1}{\xi^2} + \frac{S^*(\varphi \beta_2 S^*)^2}{2\pi - S^* \theta_1^2} < \frac{(\delta + \beta_2 S^*)^2}{2(\mu + \gamma + \delta) - \theta_2^2}. \tag{23}$$

Also, we choose Z_3 such that

$$\frac{S^*(\varphi \lambda S^*)^2}{2\pi - S^* \theta_1^2} > \frac{(\lambda S^*)^2}{2(\mu + \gamma + \delta) - \theta_2^2},$$

and

$$S^* \varphi^2 > \frac{2\pi - S^* \theta_1^2}{2(\mu + \gamma + \delta) - \theta_2^2}. \tag{24}$$

Therefore, since the conditions of (18) are satisfy we can rewrite (22) for some positive constants M, P, Q and R, as

$$LV_3 \leq -(Mu_1^2 + Pu_2^2 + Qu_3^3 + Ru_4^2), \tag{25}$$

where

$$\begin{aligned}
M &= Z_1 \left(\frac{2\varphi + 1}{2\varphi} \right) \left(\frac{\pi}{S^*} - \frac{1}{2} \theta_1^2 \right), \\
P &= Z_2 \frac{1}{2} \left(\mu + \gamma + \delta - \frac{1}{2} \theta_2^2 \right) - S^* \beta_1 Z_3 - \frac{2\xi^2}{2\eta - \theta_4^2} Z_4 \\
Q &= Z_3 \left(b + \frac{2(\delta + \beta_2 S^*)^2}{2(\mu + \gamma + \delta) - \theta_2^2} - \frac{2S^*(\varphi \beta_2 S^*)^2}{2\pi - S^* \theta_1^2} - \frac{1}{2} \theta_3^2 \right) - Z_4 \frac{2\alpha^2}{2\eta - \theta_4^2}, \\
R &= Z_4 \frac{1}{2} \left(\eta - \frac{1}{2} \theta_4^2 \right) + Z_3 \left(\frac{2S^*(\varphi \lambda S^*)^2}{2\pi - S^* \theta_1^2} - \frac{2(\lambda S^*)^2}{2(\mu + \gamma + \delta) - \theta_2^2} \right)
\end{aligned}$$

If we let $\vartheta = \min\{M, P, Q, R\}$; then $\vartheta > 0$, from (22)we can see that

$$LV_3(u(t)) \leq -\vartheta |u(t)|^2.$$

According to Lemma 1, we conclude that the SIDP-SDE model EE is asymptotically stable in probability. □

4 Numerical simulation

In this section, we give some numerical analysis to support our theorems using the Euler-Maruyama method as described in [12]. Since the DFE always exist, we only analyse the EE state of the stochastic SIDP model. Therefore, the SIDP-SDEs model can be rewritten as the following discretisation equations

$$\begin{aligned}
S_{n+1} =& S_n + ((\pi - \mu S_n - S_n(\beta_1 I_n + \beta_2 D_n + \lambda P_n)))\Delta t + \theta_1(S_n - S^*)\sqrt{\Delta t}\rho_n,\\
I_{n+1} =& I_n + ((S_n(\beta_1 I_n + \beta_2 D_n + \lambda P_n) - (\mu + \gamma + \delta)I_n))\Delta t + \theta_2(I_n - I^*)\sqrt{\Delta t}\rho_n,\\
D_{n+1} =& D_n + ((\delta I_n - bD_n))\Delta t + \theta_3(D_n - D^*)\sqrt{\Delta t}\rho_n,\\
P_{n+1} =& P_n + ((\xi I_n + \alpha D_n - \eta P_n))\Delta t + \theta_4(P_n - P^*)\sqrt{\Delta t}\rho_n,
\end{aligned}$$

where $\rho_n \sim N(0,1), (1 \le n \le N)$ with time partition interval [0,T] into $N = 1000$ equal subintervals of width $\Delta t = T/N$, where T = 200.

Using parameter values,

$$\beta_1{=}0.006, \beta_2{=}0.012, \lambda{=}0.01, \mu{=}0.02, \delta{=}0.9, \gamma{=}0.06, b{=}0.8, \xi{=}0.04, \alpha{=}0.04,\\ \eta{=}0.03,$$

as in [1], and by assuming $\pi{=}5$, we get, $R_0 = 12.2$, and $E^* = (20.49,\ 4.68,\ 5.27,\ 13.27)$. With initial values of $S(0) = 80,\ I(0) = 10,\ D(0) = 10,\ P(0) = 0$, in Figure 1,Theorem 2 is confirmed to be GAS at the endemic state.

In Figure 2, for

$$\begin{aligned}
\varphi &= \frac{\pi - \mu S*}{S^{*2}\beta_1} = \frac{4.5902}{2.5190} = 1.8222,\\
&\frac{S^*\alpha^2\beta_1}{\xi^2} + \frac{S^*(\varphi S^*\beta_2)^2}{2\pi - S^*\theta_1^2} = \frac{0.0002}{0.016} + \frac{4.1137}{9.2604} = 0.5672\\
&< \frac{S^*(\delta + S^*\beta_2)^2}{2(\mu+\gamma+\delta) - \theta_2^2} = \frac{1.3130}{1.8975} = 0.6920,\\
S^*\varphi^2 &= 68.0203 > \frac{2\pi - S^*\theta_1^2}{2(\mu+\gamma+\delta) - \theta_2^2} = \frac{9.2624}{1.8975} = 4.8814,\\
\theta_1^2 &= (0.19)^2 = 0.0361 < \frac{2\pi}{S^*} = 0.4880, \quad \theta_4^2 = (0.2)^2 = 0.04 < 2(0.03) = 0.06,\\
\theta_2^2 &= (0.25)^2 = 0.0625 < 2(\mu+\gamma+\delta) = 1.96, \ \theta_3^2 = (0.15)^2 = 0.0225 < 2b = 1.6,
\end{aligned}$$

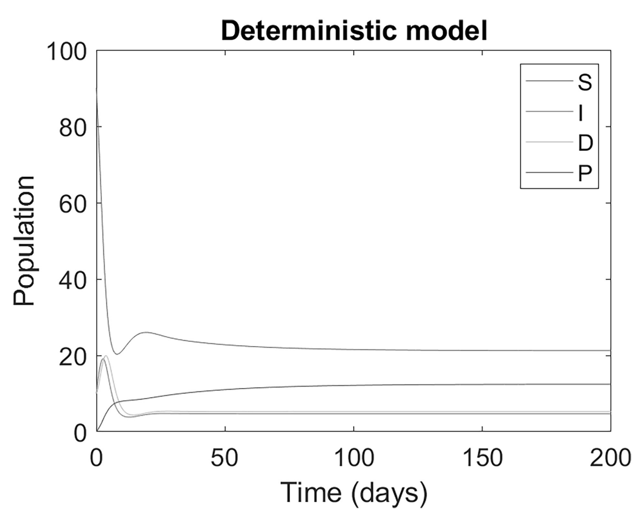

Fig. 1 Global analysis for the deterministic SIDP model when $R_0 > 1$ and $\Delta t = 0.2$

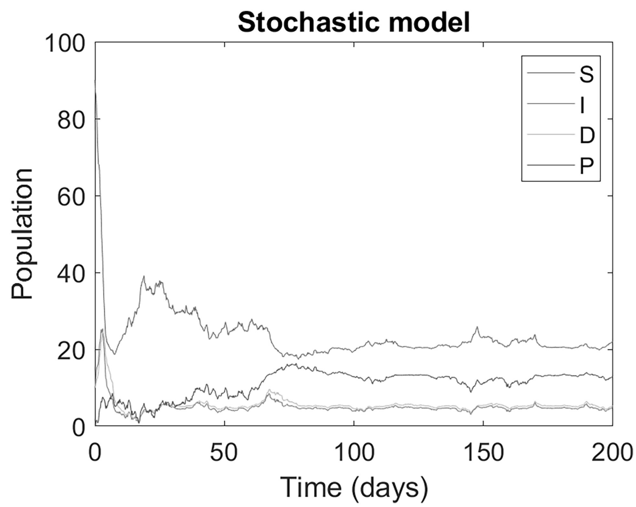

Fig. 2 Global analysis for the stochastic SIDP model when $R_0 > 1$, using $\theta_1 = 0.19, \theta_2 = 0.25, \theta_3 = 0.15, \theta_4 = 0.22, \Delta t = 0.2$

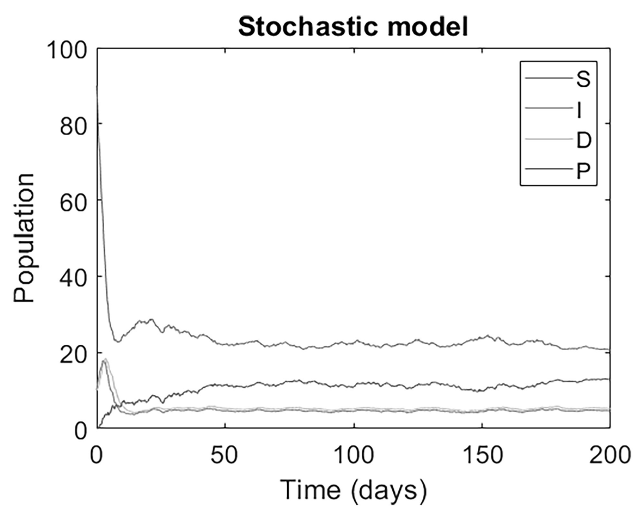

Fig. 3 Global analysis for the stochastic SIDP model when $R_0 > 1$, using $\theta_1 = 0.0475, \theta_2 = 0.0625, \theta_3 = 0.0.0375, \theta_4 = 0.0550, \Delta t = 0.2$

we observe Theorem 4 is confirmed to be GAS at the endemic state and the fluctuation of the disease dynamics is high since the noise intensities are large.

In Figure 3, we notice that the stochastic model's stability properties are similar to the deterministic when the noise intensity is small. This signifies that in the case of the large noise intensity, the stochastic model is more appropriate to explain the Ebola epidemic dynamics and the deterministic for small intensity noise.

5 Conclusion

In this paper, we use stochastic perturbation to investigate the global stability of the Susceptible-Infected-Deceased-Pathogen (SIDP) model for Ebola epidemics. Theoretically, our study shows that the contaminated environmental Ebola human–deceased transmission model is globally asymptotically stable with and without probability at the model endemic equilibrium states. From the numerical results, we notice that when the noise density is too small, the stochastic and deterministic models stability property are equal. In this situation we can neglect the noise and use the deterministic model to describe the disease dynamics. The stochastic model shows to be more appropriate to understand the dynamic of the host-deceased-pathogen transmission model of EVD when the noise intensities are large.

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