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ORIGINAL RESEARCH

Motion of uniformly advancing piston

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Abstract We study, in a unified manner the motion of a uniformly advancing piston in the planar ($m = 0$), cylindrically symmetric ($m = 1$) and spherically symmetric ($m = 2$) isentropic flow governed by Euler's equations. In the three dimensional space the piston is replaced by an expanding cylinder or a sphere which produces a motion in the medium outside. Under the hypothesis of self symmetry pair of ordinary differential equations is derived from Euler's equations of fluid flow. Based on the analysis of critical solutions of this system of ordinary differential equations, unified treatment of motion of uniformly advancing piston is given with different geometries viz. planar, cylindrically symmetric and spherically symmetric flow configuration. In case of nonplanar flows, solutions in the neighbourhood of nontransitional critical points approach these critical points in finite time. Transitional solutions exist in nonplanar cases which doesn't correspond to the motion of a piston. In the planar case, it is found that solutions take infinite time to reach critical line.

Keywords Euler's equations · Piston motion · Critical points · Critical line · Transitional solution · Mach number

Mathematics Subject Classification Primary35L65 · 35J70 · 35R35 · Secondary 35J65

1 Introduction

Solving initial and mixed initial boundary value problems occupies central place in the analysis of partial differential equations. Solving these problems for Euler's system of equations has occupied central place in the analysis of partial differential equations over last few decades. Under the assumption of self symmetry in case of two dimensions, ill posed problems are transformed into well posed system of ordinary differential equations. One of the intriguing difficulty in this analysis is to connect solution from hyperbolic region(-supersonic region) to elliptic region(subsonic region) through sonic surface. Although how solutions are to be connected from supersonic region to subsonic region is always not an easy task, analysis of flow problems using self symmetry has proved to be very useful. Courant and Friedrichs [1] have dealt with the

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case of uniformly expanding sphere preceded by a shock wave. One of the central tool in this discussion is solving Riemann problem. Solving Riemann problem itself has proved to be very difficult task in the case of several space variables. Solving Riemann problems even under the hypothesis of self similarity is largely open because of existence of supersonic and subsonic regions. In this article we will attempt to analyse motion of uniformly advancing piston in case of planar and nonplanar flow configurations. Yuxi Zheng's seminal work [3, 4] is main inspiration behind carrying out the following study. Zheng [3] has dealt with flow with cylindrical symmetry in case of two space variables. Following [3] the present paper deals with planar, cylindrically symmetric and spherically symmetric flows in a unified manner. The basic equations for isentropic Euler systems describing the planar ($m = 0$), cylindrically symmetric ($m = 1$) and spherically symmetric ($m = 2$) adiabatic motion of an ideal fluid having velocity u, density ρ and pressure p defined as $p = \text{constant}\rho^\gamma$, where γ is the adiabatic heat exponent, may be written in conservation form [2]

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho r^m \\ \rho u \end{pmatrix} + \frac{\partial}{\partial r}\begin{pmatrix} \rho u r^m \\ p + \rho u^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -m\rho \frac{u^2}{r} \end{pmatrix}, \tag{1.1}$$

where t denotes time and r denotes the spatial coordinate being either axial in flows with planar geometry or radial in cylindrically symmetric or spherically symmetric flow configurations.

2 Progressive wave solutions

We look for progressive wave solutions of (1.1) when all the quantities u, p and ρ are constant along the rays $\xi = \frac{x}{t} = \text{constant}$. In terms of the variable ξ equations (1.1) for smooth solutions can be written as follows

$$\begin{pmatrix} u - \xi & \frac{(\gamma-1)c}{2} \\ \frac{2c}{(\gamma-1)} & u - \xi \end{pmatrix}\begin{pmatrix} \frac{dc}{d\xi} \\ \frac{du}{d\xi} \end{pmatrix} = \begin{pmatrix} -\frac{m(\gamma-1)uc}{2\xi} \\ 0 \end{pmatrix}, \tag{2.1}$$

where $c = \frac{dp}{d\rho}$ is the sound speed. It is clear that the system (2.1) admits single valued nonsingular solutions provided the matrix on the left is nonsingular. We look for solutions of (2.1) satisfying

$$\lim_{\xi \to \infty}(u, c) = (u_0, c_0). \tag{2.2}$$

Where the initial values u_0 and c_0 are positive constants. Following the notations used in [4] which deals with cylindrically symmetric flows we introduce new variables s, I and K defined as

$$s = 1/\xi, \quad I = us, \quad K = uc. \tag{2.3}$$

It is noteworthy to add a remark about variables $s, I,$ and K. s represents reciprocal of speed. I is a dimensionless quantity which corresponds to flow speed whereas K represents thermodynamical part. Because of assumption of isentropic flow, one thermodynamical quantity K along with mechanical quantity I couple with each other are sufficient to determine the flow. We are dealing with shock with constant speed which keeps the flow effectively isentropic. In terms of these new variables, equations (2.1) and (2.2) can be written as

$$\begin{gathered}((1-I)^2 - K^2)s\frac{d}{ds}\begin{pmatrix} K \\ I \end{pmatrix} = \begin{pmatrix} K\left[(1-I)^2 - K^2 - \frac{m(\gamma-1)}{2}I(1-I)\right] \\ I\left[(1-I)^2 - K^2 - mK^2\right] \end{pmatrix}. \\ (I, K) \to s(u_0, c_0) \text{as } s \to 0_+ \end{gathered} \tag{2.4}$$

Here $s \to 0_+$ means $r \to +\infty$. It may be noticed that $s > 0$ means $t > 0$ for a fixed value of r. Thus the region $K > 0$ corresponds to $s > 0$. Equations (2.1) can be rationalized by introducing a new variable τ, such that

$$\frac{ds}{d\tau} = sD; \quad D^2 = (1-I)^2 - K^2. \tag{2.5}$$

Subsequently, equations (2.1) can be written as

$$\frac{d}{d\tau}\begin{pmatrix} K \\ I \end{pmatrix} = \begin{pmatrix} aK \\ bI \end{pmatrix} \tag{2.6}$$

where $a = D^2 - \frac{m(\gamma-1)I(1-I)}{2}$ and $b = D^2 - mK^2$.

As noted earlier, the directions of increasing s, starting at $s = 0_+$ or that of τ gives the direction of increasing t at any fixed r as τ increases from $-\infty$. This direction is reversed on crossing the critical lines $D = 0$ or $|1 - I| = |K|$ and therefore no integral curve crossing these lines can represent a real flow. Indeed, no solution, starting at $s = 0_+$, enters the region Ω defined underneath, for planar and nonplanar flows. Once K and I are solved from (2.6); s is solved from (2.5), solutions of (2.6) can be obtained by first finding K as a function of I from the differential equation

$$\frac{dK}{dI} = \frac{K}{I}(a/b). \tag{2.7}$$

3 Planar flow (case $m = 0$)

For planar ($m = 0$), the region Ω in the I, K phase plane is the triangular region: $I > 0, K > 0$, and $D > 0$ with vertices (0, 0), (1, 0) and (0, 1) as critical points. The directions of increasing s, starting from $s = 0_+$ is reversed on crossing the critical line $I + K = 1$. In fact, as $s \to \infty$ the solution approaches the line $I + K = 1$ from within the region. We notice that equation (2.1) for $m = 0$ implies that the vector $(\frac{dc}{d\xi}, \frac{du}{d\xi})$is collinear with the eigenvalues $(\frac{\gamma-1}{2}, \mp 1)$ corresponding to the eigenvalues $\xi = u \mp c$, respectively. This means that the system (2.1) for $m = 0$ admits two families of forward (respectively backward) facing rarefaction wave solutions.

$$\begin{aligned} I + \frac{2}{\gamma-1}K &= \text{constant, along} I - K = 1, \\ I - \frac{2}{\gamma-1}K &= \text{constant, along} I + K = 1. \end{aligned} \tag{3.1}$$

We observe that the integral curves in the neighbourhood of (1, 0) approach along the lines $K = \alpha(1 - I)$. This, in view of (3.1) and the fact that $D > 0$, implies that $\frac{\gamma-1}{2} \leq \alpha < 1$. Further since I is bounded near 1 it follows from (2.5) that s is finite at the end (1, 0). In order to discuss the behaviour of the solution near (1, 0) one can linearise equation (2.7) about $(I, K) = (0, 1)$ to yield the system

$$\begin{aligned} \frac{dI}{d\tau} &= 0, \\ \frac{dK}{d\tau} &= -2I - 2(K-1). \end{aligned} \tag{3.2}$$

Since the coefficient matrix of the system (3.2), has one of the eigenvalue is 0; the nontrivial solution would be determined only upto a multiplicative constant and would lie on the line $I + K = 1$ passing through (0, 1); as a result other critical points would be located arbitrarily close to the point (0, 1), violating the assumption that (0, 1) is an isolated critical point. In other words $D \to 0$ exponentially as $\tau \to \infty$ and hence it follows from (2.5) that s approaches a finite number as $\tau \to \infty$ Simillarly, one can see that linearization of (2.7) about (0, 0) yields

$$\begin{aligned} \frac{dI}{d\tau} &= I, \\ \frac{dK}{d\tau} &= K. \end{aligned}$$

Showing thereby that the solution in the neighbourhood of (0, 0) approaches (0, 0) along the line $K = \beta I$ which by virtue of (3.1) lies in the interval $(0, \frac{\gamma-1}{2}]$.

4 Invariant region for cases $m = 1$ and $m = 2$

The following domains define invariant region for the system of ordinary differential equations (2.6)

$$\Omega : \begin{cases} I > 0, K > 0 \\ a := -\frac{K}{2}\left[m(\gamma-1)I(1-I) + 2K^2 - 2(1-I)^2\right], \text{ for } 0 \leq I \leq \frac{2}{(1+\gamma-m+m\gamma)}, \\ b := -I\left(K^2 - (1-I)^2 + mK^2\right), \text{ for } \frac{2}{(1+\gamma-m+m\gamma)} < I \leq 1. \\ a \geq 0, \quad b \leq 0 \end{cases}$$

We get that some solutions in Ω may move to the point (1, 0), while others move to the point (0, 1), with exactly one integral curve moving into the following point which we denote by Q in this article maintaining the same notation as in [4]

$$Q = \left(\frac{2}{(1+\gamma-m+m\gamma)}, \frac{(\gamma-1)\sqrt{1+m}}{(1+\gamma-m+m\gamma)}\right).$$

5 Transitional solution

Transitional solutions exist only in case of cylindrically symmetric ($m = 1$) and spherically symmetric ($m = 2$) flows. For planar case transition solution doesn't exist and this case ($m = 0$) displays behaviour singular in some sense and requires special treatment. We will deal with this case separately. There is no explicit formula to determine initial state which connects to transitional solution. In order to determine the initial state (u_0, c_0) that connects to the transitional solution we introduce the variable L defined as $L = \frac{K^2}{I^2}$ which represents reciprocal of a square of Mach number $M = u/c$. The equation governing L can be obtained from (2.6). Simple manipulation yields the following

$$\begin{gathered} \frac{dL}{dI} = mL\left[\frac{(\gamma-1)(1-I) - 2IL}{(1+m)LI^2 - (1-I)^2}\right], \\ L = (u_0/c_0)^{-2} \text{at} I = 0. \end{gathered} \tag{5.1}$$

The transitional solution starting from the point $(0, M_0^{-2})$ to the point (I, L) in the (I, L) plane, with

$$I = \frac{2}{1+\gamma-m+m\gamma},$$
$$L = \frac{(\gamma-1)^2(1+m)}{4}.$$

Like the case for $m = 1$ described in [4] we found in the case $m = 2$ that for a specific initial data (I_0, K_0) we get integral curve of (2.6) starting at (I_0, K_0) reach transitional solution and takes infinite time to reach there. Transitional solution gives rise to the following asymptotic form of the self similar solution of Euler's system with cylindrical ($m = 1$) and spherical($m = 2$) flow configurations as $\xi \to \infty$. (It may be noticed that this discussion doesn't extend to the planar case).

$$\begin{aligned} u(r,t) &= \frac{2}{1+\gamma-m+m\gamma}\frac{r}{t}, \\ p'(x,t) &= \frac{(\gamma-1)\sqrt{1+m}}{1+\gamma-m+m\gamma}\frac{r}{t}. \end{aligned} \tag{5.2}$$

Following Zheng[4] we have drawn a diagram Figure 1 to represents transitional solution as a point of intersection of numerator and denominator of (5.1).Explicit formula for initial condition which leads to transitional solution is not known and we can argue on the lines of Zheng[4] to give an upper bound for initial Mach number that finally gives rise to transitional Mach number. It appears that initial condition leading to transitional solution cannot spiral around transitional solution.

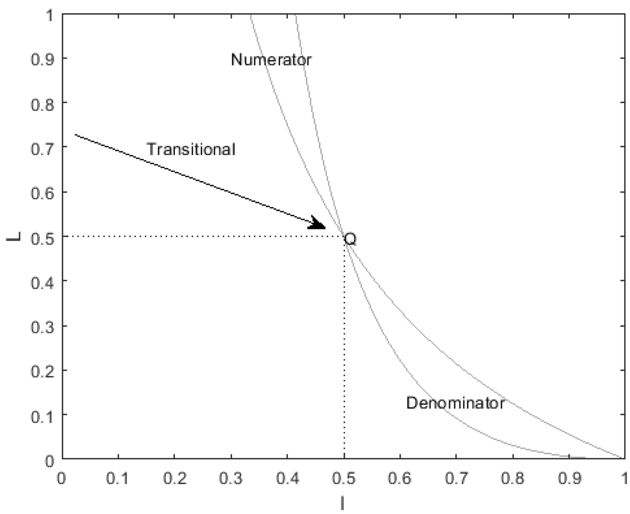


Fig. 1 Transitional Solution

6 Analysis of stationary points for the case $m = 2$

We can linearise the equations (2.6) around a point $(I, K) = (0, 1)$ and get the following-

$$\begin{aligned} \frac{dK}{d\tau} &= -2(K-1) - (\gamma+1)I, \\ \frac{dI}{d\tau} &= -2I. \end{aligned} \tag{6.1}$$

The eigenvalues of (6.1) are $\lambda_1 = -2$ and $\lambda_1 = -2$. So solutions of (2.6) near (0, 1) approach (0, 1) exponentially as $\tau \to +\infty$. Equation (2.5) implies that, since D approaches 0 exponentially, s approaches a finite number as $\tau \to +\infty$. As Zheng[4] has found, in the case $m = 2$ we find that linearisation of (2.6) at (1, 0) yields trivial system of zero right hand side. We observe that solutions near (1, 0) will enter (1, 0) in the sector bounded by $K = 0$ and the line $K = \alpha(1 - I)$ for some $0 < \alpha < 1/\sqrt{3}$ such that $\alpha^2 + \frac{(\gamma-1)}{3} > 0$. In this way every solution that ends at (1, 0) will be such that $K < \alpha(1 - I)$ near $I = 1$ for some $\alpha < 1/\sqrt{(3)}$ since $\gamma > 1$, thus when I is close to 1 (2.4) yields

$$s\frac{dI}{ds} > I\frac{1 - 3\alpha^2}{1 - \alpha^2} = \text{a positive constant.}$$

This means that s is finite at the end (1, 0) since I is bounded.

7 Motion of Piston for cases $m = 1$ and $m = 2$

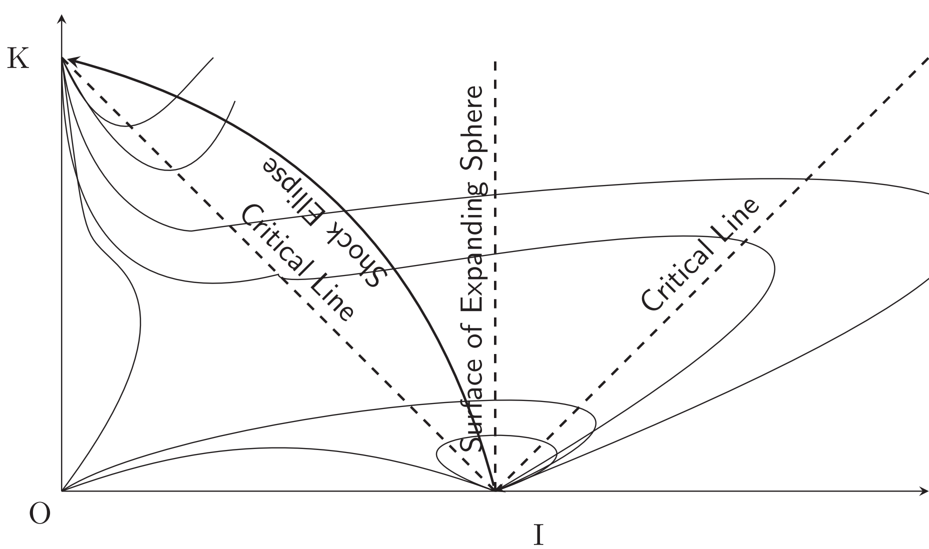
Problem we are addressing in this article is that of uniformly advancing piston, cylindrical ($m = 1$) or spherical ($m = 2$) pushed at constant velocity into an infinite surrounding medium. Here we consider the shock wave $r = \Sigma t$ racing ahead of the piston has constant speed Σ so that shock conditions are compatible with the assumption of isentropic flow on both side of discontinuity surface. At the surface of piston the velocity of gas is equal to that of the surface $\xi = \xi_0$. Hence $u = \xi = \xi_0$ there. i.e $I = 1$ there. Following Courant and Freidrichs [1] gas and sound velocities behind the shock front, namely I_s and K_s satisfy the relation

$$
\begin{aligned}
(1-\mu^2)K_s^2 &= (1-I_s) - \mu^2(1-I_s)^2 \\
&= (1-\mu^2)(1-I_s)^2 + I_s(1-I_s) \\
I_s &= (1-\mu^2)(1-\frac{c_0^2}{\Sigma^2}).
\end{aligned} \tag{7.1}
$$

Where $\mu^2 = \frac{\gamma-1}{\gamma+1}$. and c_0 is the speed of sound ahead of the shock. Second equation in (7.1) gives value of I_s; since $\mu < 1$, $I_s < (1-\mu^2)$ represents physical states behind shock front. We notice that states in the (I, K) plane are integral curves emanating from (I_s, K_s). Note that $I_s < 1$ always if only physically possible states are considered. Since $\frac{dI}{ds} > 0$ at (I_s, K_s) we can extend integral curve with $s > 0$ till the critical lines $|1-I| = |K|$. From the convexity of integral curve we have $I = 1$ reached by this integral curve before it reaches the critical line at which s reverses its sign and flow ceases from remaining physical. Let this value of s be s_0 when integral cure emanating from (I_s, K_s) reaches the state $I = 1$. Let us briefly state how this s_0 may be calculated. From equations(2.6) we get the following differential equation for K in terms of I.

$$
\frac{dK}{dI} = \frac{Km\Big((\gamma-1)I(1-I) + 2K^2 - 2(1-I)^2\Big)}{2I\Big(K^2 - (1-I)^2 + mK^2\Big)}. \tag{7.2}
$$

For $m = 1$ and $m = 2$ we can solve this equation for K in terms of I for any given state (I_s, K_s). With this value of K we can solve second equation in (2.4) along with the third equation there to get s_0 when the integral curve reaches $I = 1$. Since both $K(s)$ and $I(s)$ remain monotonic/convex till the point $I = 1$ is reached as a function of s the same is true for s as a function of I. In other words, s is a monotonic function of shock speed from which (I_s, K_s) are determined using shock ellipse. Thus s_0 is a monotonic function of shock speed Σ. In other words for given speed of leading shock front produced by advancing piston, for geometries that correspond to $m = 1$ and $m = 2$, we have determined s_0 and hence value of K as well, at that point and we can then extend solution till the critical line on the lines of the method indicated above in the section on construction of global solutions out of stationary points. Argument presented in this section is represented in the following diagram.



Analysis which we have carried out is applicable for the analysis of spherical or cylindrical wavefront propagating away from the central axis in case of cylindrical fronts and away from the origin in case of spherical fronts. By front we mean any information which is approximately cylindrically or spherically symmetric.

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