Indian J Pure Appl Math (2021) 52:1123–1144
https://doi.org/10.1007/s13226-021-00086-9

ORIGINAL RESEARCH

The exact zero-divisor graph of a reduced ring

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Received: 15 July 2020 / Accepted: 12 October 2020 / Published online: 21 June 2021

Abstract The rings considered in this article are commutative with identity which are not integral domains. Let R be a ring. Recall that an element x of R is an exact zero-divisor if there exists a non-zero element y of R such that $Ann(x) = Ry$ and $Ann(y) = Rx$. As in Lalchandani (International J. Science Engineering and Management (IJSEM) 1(6): 14-17, 2016), for a ring R, we denote the set of all exact zero-divisors of R by $EZ(R)$ and $EZ(R)\backslash\{0\}$ by $EZ(R)^*$. Let R be a ring. In the above mentioned article, Lalchandani introduced and studied the properties of a graph denoted by $E\Gamma(R)$, which is an undirected graph whose vertex set is $EZ(R)^*$ and distinct vertices x and y are adjacent in $E\Gamma(R)$ if and only if $Ann(x) = Ry$ and $Ann(y) = Rx$. Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. The aim of this article is to study the interplay between the graph-theoretic properties of $E\Gamma(R)$ and the ring-theoretic properties of R.

Keywords Reduced ring · von Neumann regular ring · Exact zero-divisor · Girth · Clique number · Chromatic number

Mathematics Subject Classification Primary :13A15 and Secondary: 05C25

1 Introduction

The rings considered in this article are commutative with identity and unless otherwise specified, they are not integral domains. The idea of associating a graph with the elements of a ring to investigate the interplay between the ring-theoretic properties of the ring and the graph-theoretic properties of the graph associated with it was initiated by Beck in (1988). In Beck [7], I. Beck was mainly interested in colorings. Let R be a ring. The graph associated with the elements of R in Beck [7] by I. Beck was modified and investigated by Anderson and Livingston in (1999). For a ring R, we denote the set of all zero-divisors of R by $Z(R)$ and $Z(R)\backslash\{0\}$ by $Z(R)^*$. Recall from Anderson and Livingston [3] that the *zero-divisor graph of R*, denoted by $\Gamma(R)$ is an undirected graph whose vertex set is $Z(R)^*$ and distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. Inspired by the research work presented in Anderson and Livingston [3] on the zero-divisor graph of a ring, many algebraists did a lot of work on zero-divisor graphs of rings and proved several

Communicated by Sudhir R Ghorpade, Ph.D.

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interesting and noteworthy theorems on zero-divisor graphs of rings. For a detailed and excellent survey on the zero-divisor graphs of commutative rings, one can refer the survey article by Anderson, Axtell, and Stickles [1].

This article is also motivated by the interesting results proved on the exact zero-divisor graphs of commutative rings in Lalchandani [17], Lalchandani [18]. Let R be a ring. For an element $a \in R$, recall from Atiyah and Macdonald (1969, page 19) that the *annihilator of a in R*, denoted by $Ann_R(a)$ (or simply by $Ann(a)$) is defined as $Ann(a) = \{r \in R \mid ra = 0\}$. Recall from Henriques and Sega [13] that $x \in R$ is said to be an *exact zero-divisor* if there exists $y \in R\backslash\{0\}$ such that $Ann(x) = Ry$ and $Ann(y) = Rx$. As in Lalchandani [17], we denote the set of all exact zero-divisors of a ring R by $EZ(R)$ and $EZ(R)\backslash\{0\}$ by $EZ(R)^*$. The graphs considered in this article are undirected and simple. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. Let R be a ring such that $EZ(R)^* \neq \emptyset$. Inspired by the work on the zero-divisor graphs of rings, in Lalchandani [17], P.T. Lalchandani introduced and investigated an undirected graph associated with R, called the *exact zero-divisor graph of* R, denoted by $E\Gamma(R)$ whose vertex set is $EZ(R)^*$ and distinct vertices x and y are adjacent in $E\Gamma(R)$ if and only if $Ann(x) = Ry$ and $Ann(y) = Rx$. If $x \in EZ(R)$, then there exists a non-zero $y \in R$ such that $Ann(x) = Ry$ and $Ann(y) = Rx$. Hence, $xy = 0$ and so, $x \in Z(R)$. This shows that $EZ(R)^* \subseteq Z(R)^*$. Let $x, y \in EZ(R)^*$. If x and y are adjacent in $E\Gamma(R)$, then $xy = 0$ and so, x and y are adjacent in $\Gamma(R)$. Therefore, for a ring R with $EZ(R)^* \neq \emptyset$, it follows that $E\Gamma(R)$ is a subgraph of $\Gamma(R)$. Several examples of exact zero-divisor graphs of commutative rings were given in Lalchandani [17], Lalchandani [18]. This article is a continuation of the work done on the exact zero-divisor graphs of commutative rings in Lalchandani [17], Lalchandani [18].

For a ring R, we denote the nilradical of R by $nil(R)$. A ring R is said to be *reduced* if $nil(R) = (0)$. Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. The aim of this article is to study the interplay between the graph-theoretic properties of $E\Gamma(R)$ and the ring-theoretic properties of R. For a ring T, we denote the set of all its prime ideals by $Spec(T)$ and the set of all maximal ideals of T by $Max(T)$. We denote the set of all minimal prime ideals of T by $Min(T)$. We denote the Krull dimension of a ring T by $dimT$. A ring T is said to be *quasilocal* if T has a unique maximal ideal. A Noetherian quasilocal ring is referred to as a *local ring*. Let A, B be sets. If A is a subset of B and $A \neq B$, then we denote it symbolically by $A \subset B$. We denote the cardinality of a set A by $|A|$.

Let R be a reduced ring with $EZ(R)^* \neq \emptyset$. In Section 2 of this article, some basic properties of $E\Gamma(R)$ are proved. For any ring T with $Z(T)^* \neq \emptyset$, we know from Anderson and Livingston (1999, Theorem 2.3) that $\Gamma(T)$ is connected and $diam(\Gamma(T)) \leq 3$. First, an example of a local Artinian ring R such that $EZ(R)^* = \emptyset$ is given and it is verified that if T is a non-quasilocal ring with $dimT = 0$, then $EZ(T)^* \neq \emptyset$. For a reduced ring R with $|Min(R)| = 2$, it is shown in Proposition 2.21 that $EZ(R)^* \neq \emptyset$ if and only if each $\mathfrak{p} \in Min(R)$ is principal. It is well-known that a ring R is von Neumann regular if and only if $dimR = 0$ and R is reduced. Motivated by the work presented in Levy and Shapiro [19], we focus our study on investigating $E\Gamma(R)$, where R is a von Neumann regular ring which is not a field. So, to start with we try to investigate $E\Gamma(R)$, where either $dimR = 0$ or R is reduced. For a zero-dimensional ring R with $EZ(R)^* \neq \emptyset$, it is proved in Corollary 2.5 that if $E\Gamma(R)$ is connected, then $|Max(R)| \leq 2$. Let R be a ring with $EZ(R)^* \neq \emptyset$. If $E\Gamma(R)$ is connected, then it is shown in Proposition 2.3 that $diam(E\Gamma(R)) \leq 2$. It is proved in Corollary 2.7 that for a von Neumann regular ring R which is not a field, $EZ(R)^* \neq \emptyset$ and $EZ(R)^* = Z(R)^*$. Among other results, one of the main theorems proved in Section 2 is Theorem 2.9 in which it is shown that if R is a von Neumann regular ring which is not a field, then $E\Gamma(R)$ is connected if and only if $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. It is proved in Lemma 2.14 that each component of $E\Gamma(R)$ is a complete bipartite graph, where R is a reduced ring with $EZ(R)^* \neq \emptyset$ and moreover, necessary and sufficient conditions are determined for a component of $E\Gamma(R)$ to be complete. It is shown in Theorem 2.15 that for a ring R (R can possibly admit non-zero nilpotent elements) with $EZ(R)^* \neq \emptyset$, any component of $E\Gamma(R)$ is either complete or a complete bipartite graph. Let R be a reduced ring which is not an integral domain. With the assumption that $Min(R)$ is finite and each $\mathfrak{p} \in Min(R)$ is principal, in Lemma 2.20, $EZ(R)^*$ is described and in Theorem 2.26, the number of components of $E\Gamma(R)$ is determined. Several corollaries are deduced from Theorem 2.26 and several examples are given in this section to illustrate the results proved. Let R be a ring which is not an integral domain. It was shown in Anderson and Livingston (1999, Theorem 2.2) that $\Gamma(R)$ is finite if and only if R is finite. In Remark 2.33, it is verified that the infinite ring $\mathbb{Z} \times \mathbb{Z}$ is such that $|EZ(\mathbb{Z} \times \mathbb{Z})^*| = 4$. For a ring R with $Z(R)^* \neq \emptyset$, it was proved in Anderson, Levy, and Shapiro (2003, Theorem 2.2) that $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic, where $T(R)$ is the total quotient ring of R. In Example 2.34, with $R = \mathbb{Z} \times \mathbb{Z}$, it is shown that $E\Gamma(R)$ is not isomorphic to $E\Gamma(T(R))$.

Let R be a reduced ring with $EZ(R)^* \neq \emptyset$. Some more properties of $E\Gamma(R)$ are proved in Section 3 of this article. It is proved in Proposition 3.2 that $girth(E\Gamma(R)) \in \{4, \infty\}$. It is shown in Corollary 3.4 that $\omega(E\Gamma(R)) = \chi(E\Gamma(R)) = 2$. It is proved in Proposition 3.5 that for a Boolean ring R which is not a field, $girth(E\Gamma(R)) = \infty$ and moreover, each component of $E\Gamma(R)$ is a complete graph with two vertices. In Theorem 3.7, necessary and sufficient conditions are determined in order that $girth(E\Gamma(R)) = 4$, where R is a von Neumann regular ring which is not a field. Let R be a reduced ring with $EZ(R)^* \neq \emptyset$. It is proved in Corollary 3.9 that $girth(E\Gamma(R)) = 4$ if and only if there exists an edge $a - b$ of $E\Gamma(R)$ such that $|U(\frac{R}{Ra})| > 1$ and $|U(\frac{R}{Rb})| > 1$, where for a ring T, $U(T)$ denotes the group of units of T. Let R be a ring such that $EZ(R)^* \neq \emptyset$, where R is not necessarily reduced. It is proved in Corollary 3.10 that $E\Gamma(R)$ is perfect; that is, $\omega(H) = \chi(H)$ for any induced subgraph H of $E\Gamma(R)$. It is desirable to mention here that in Anderson and Naseer [4], D.D. Anderson and M. Naseer provided an example of a local Artinian ring R with $|R| = 32$ such that $\omega(\Gamma(R)) = 4 < \chi(\Gamma(R)) = 5$.

In Section 4 of this article, some more remarks on $E\Gamma(R)$ are provided, where R is a reduced ring with $EZ(R)^* \neq \emptyset$. It is verified in Corollary 4.1 that $E\Gamma(R)$ is uniquely complemented. It is observed in Corollary 4.2 that no vertex of $E\Gamma(R)$ is an S-vertex. Let $|Min(R)| = n$ and suppose that each $\mathfrak{p} \in Min(R)$ is principal. It is proved in Corollary 4.3 that $2^{n-1} - 1 \leq \gamma(E\Gamma(R)) \leq 2^n - 2$, where $\gamma(E\Gamma(R))$ is the domination number of $E\Gamma(R)$. Let R be a von Neumann regular ring which is not a field. It is shown in Proposition 4.8 that $\gamma(E\Gamma(R)) < \infty$ if and only if $|Max(R)| < \infty$. Let R be a von Neumann regular ring which is not a field. Another main result proved in Section 4 is Theorem 4.9, in which it is shown that $E\Gamma(R)$ has at least one component g such that g is complete if and only if R is a Boolean ring.

2 Some basic properties of $E\Gamma(R)$, where R is a reduced ring

Let R be a reduced ring which is not an integral domain. The aim of this section is to state and prove some basic properties of $E\Gamma(R)$. Recall from Gilmer (1972, Exercise 16, page 111) that a ring R is said to be *von Neumann regular* if for each element a of R there is an element $b \in R$ such that $a = a^2b$. In this section, we try to characterize von Neumann regular rings R such that $E\Gamma(R)$ has a specific graph-theoretic property. We know from $(a) \Leftrightarrow (d)$ of Gilmer (1972, Exercise 16, page 111) that a ring R is von Neumann regular if and only if $dimR = 0$ and R is reduced. Hence, to begin with we state and prove some basic properties of $E\Gamma(R)$, where R is either a zero-dimensional ring or R is a reduced ring. An element e of a ring T is said to be *idempotent* if $e = e^2$ and an idempotent element e of a ring T is said to be *non-trivial* if $e \notin \{0, 1\}$. For idempotent elements e, f of a ring R, it can be easily verified that $Re = Rf$ if and only if $e = f$. For a ring R, we denote the set of all units of R by $U(R)$ and the set of all non-units of R by $NU(R)$.

Lemma 2.1 *Let R be a von Neumann regular ring which is not a field. Then R admits at least one non-trivial idempotent element.*

Proof Since R is not a field, it is possible to find $a \in R \backslash \{0\}$ such that $a \in NU(R)$. As R is von Neumann regular, there exists $b \in R$ such that $a = a^2b$. This implies that $ab = a^2b^2 = (ab)^2$. Let $e = ab$. It is clear that $e = e^2$ is an idempotent element of R. From $a = ae \neq 0$, it follows that $e \neq 0$. From $a \in NU(R)$, we get that $e = ab \in NU(R)$ and so, $e \notin \{0, 1\}$. □

Lemma 2.2 *Let $n \in \mathbb{N}$ with $n \geq 2$. Let R be a ring such that $dimR = 0$ and $|Max(R)| \geq n$. Then there exist zero-dimensional rings $R_1, R_2, \ldots, R_n$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings.*

Proof We prove this lemma using induction on n. Let $n = 2$. Now, $dimR = 0$ and $|Max(R)| \geq 2$. Let us denote the ring $\frac{R}{nil(R)}$ by T. Note that $dimT = 0$, $|Max(T)| \geq 2$, and we know from Atiyah and Macdonald (1969, Proposition 1.7) that T is reduced. Therefore, T is a von Neumann regular ring and is not a field. Hence, we obtain from Lemma 2.1 that there exists $r \in R$ such that $r + nil(R)$ is a non-trivial idempotent element of T. Since $nil(R)$ is a nil ideal of R, we obtain from Jacobson (1984, Proposition 7.14) that there exists a unique idempotent element e of R such that $r + nil(R) = e + nil(R)$. From $r + nil(R) \notin \{0 + nil(R), 1 + nil(R)\}$, we get that $e \notin \{0, 1\}$. Observe that the mapping $f : R \to Re \times R(1 - e)$ defined by $f(x) = (xe, x(1 - e))$ is an isomorphism of rings. Let us denote the ring Re by R_1 and the ring $R(1 - e)$ by R_2. Note that for each $i \in \{1, 2\}$, R_i is a homomorphic image of R and so, $dimR_i = 0$ and $R \cong R_1 \times R_2$ as rings. Let $n \geq 3$ and assume by induction that the lemma is true for $n - 1$. Now, $|Max(R)| \geq n > n - 1$. Hence, by induction hypothesis, there exist zero-dimensional rings $T_1, T_2, \ldots, T_{n-1}$

such that $R \cong T_1 \times T_2 \times \cdots \times T_{n-1}$ as rings. Since $|Max(R)| \geq n$, it follows that $|Max(T_i)| \geq 2$ for at least one $i \in \{1, 2, \ldots, n-1\}$. Without loss of generality, we can assume that $|Max(T_1)| \geq 2$. Hence, by the case $n = 2$, there exist zero-dimensional rings T_{11}, T_{12} such that $T_1 \cong T_{11} \times T_{12}$ as rings. Therefore, $R \cong T_{11} \times T_{12} \times T_2 \times \cdots \times T_{n-1}$ as rings. Let $R_1 = T_{11}, R_2 = T_{12}, R_3 = T_2, \ldots, R_n = T_{n-1}$. Observe that R_i is a zero-dimensional ring for each $i \in \{1, 2, \ldots, n\}$ and $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$ as rings. □

Let $G = (V, E)$ be a graph. G is said to be *connected* if for each pair of distinct vertices $a, b \in V$, there exists at least one path in G between a and b. Let $G = (V, E)$ be a connected graph. Let $a, b \in V$ with $a \neq b$. Recall from Balakrishnan and Ranganathan [6] that the *distance between a and b*, denoted by $d(a, b)$ is defined as the length of a shortest path in G between a and b. We define $d(a, a) = 0$ and define the *diameter of G*, denoted by $diam(G)$ as $diam(G) = sup\{d(a, b) \mid a, b \in V\}$.

For any ring R with $Z(R)^* \neq \emptyset$, it was proved in Anderson and Livingston (1999, Theorem 2.3) that $\Gamma(R)$ is connected and $diam(\Gamma(R)) \leq 3$. Let R be a von Neumann regular ring which is not a field. In Theorem 2.9, we determine necessary and sufficient conditions for $E\Gamma(R)$ to be connected.

Let $T = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K. Let $\mathfrak{m} = TX + TY$. Let $R = \frac{T}{\mathfrak{m}^2}$. Note that $\mathfrak{m} \in Max(T)$ and $Spec(R) = Max(R) = \{\frac{\mathfrak{m}}{\mathfrak{m}^2}\}$. Observe that $Z(R)^* = \frac{\mathfrak{m}}{\mathfrak{m}^2} \setminus \{0 + \mathfrak{m}^2\}$. We claim that $EZ(R)^* = \emptyset$. For any ring A, it is already noted in the introduction that $EZ(A)^* \subseteq Z(A)^*$. Let $t \in T$ be such that $t + \mathfrak{m}^2 \in Z(R)^*$. Then $t \in \mathfrak{m}$. Observe that for any $s \in \mathfrak{m}$, $(t + \mathfrak{m}^2)(s + \mathfrak{m}^2) = 0 + \mathfrak{m}^2$. Therefore, $Ann(t + \mathfrak{m}^2) = \frac{\mathfrak{m}}{\mathfrak{m}^2}$. As $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ is not principal, there exists no $t_1 \in T$ such that $Ann(t + \mathfrak{m}^2) = R(t_1 + \mathfrak{m}^2)$ and $Ann(t_1 + \mathfrak{m}^2) = R(t + \mathfrak{m}^2)$. This shows that $EZ(R)^* = \emptyset$.

Let R_1, R_2 be rings and let $T = R_1 \times R_2$. Let $x = (1, 0)$ and let $y = (0, 1)$. Observe that $Ann(x) = Ty$ and $Ann(y) = Tx$. Hence, $x, y \in EZ(T)^*$ and so, $EZ(T)^* \neq \emptyset$. Let R be a ring such that $dimR = 0$ and $|Max(R)| \geq 2$. We know from Lemma 2.2 that there exist zero-dimensional rings R_1, R_2 such that $R \cong R_1 \times R_2$ as rings. As $EZ(R_1 \times R_2)^* \neq \emptyset$, we get that $EZ(R)^* \neq \emptyset$. In Corollary 2.5, we determine a necessary condition on $Max(R)$ for $E\Gamma(R)$ to be connected. We use Lemma 2.4 in the proof of Corollary 2.5. Let R be a ring with $EZ(R)^* \neq \emptyset$. If $E\Gamma(R)$ is connected, then we show in Proposition 2.3 that $diam(E\Gamma(R)) \leq 2$.

Proposition 2.3 *Let R be a ring such that $EZ(R)^* \neq \emptyset$. Let $x, y \in EZ(R)^*$ be such that $x \neq y$. If there exists a path in $E\Gamma(R)$ between x and y, then the length of any shortest path in $E\Gamma(R)$ between x and y is at most two. In particular, if $E\Gamma(R)$ is connected, then $diam(E\Gamma(R)) \leq 2$.*

Proof Let m be the length of a shortest path between x and y in $E\Gamma(R)$. Suppose that $m \geq 3$. Let $x_0 = x - x_1 - x_2 - x_3 - \cdots - x_m = y$ be a shortest path between x and y in $E\Gamma(R)$. Hence, $x = x_0$ and x_i are not adjacent in $E\Gamma(R)$ for each $i \in \{2, 3, \ldots, m\}$. Let $j \in \{0, 1, 2, \ldots, m-1\}$. As x_j and x_{j+1} are adjacent in $E\Gamma(R)$, we obtain that $Ann(x_j) = Rx_{j+1}$ and $Ann(x_{j+1}) = Rx_j$. Therefore, $Ann(x_0) = Rx_1 = Ann(x_2) = Rx_3$ and $Ann(x_3) = Rx_2 = Ann(x_1) = Rx_0$. Hence, we obtain that $Ann(x_0) = Rx_3$ and $Ann(x_3) = Rx_0$. This implies that x_0 and x_3 are adjacent in $E\Gamma(R)$. This is a contradiction and so, $m \leq 2$.

Assume that $E\Gamma(R)$ is connected. We know from the previous paragraph that $d(x, y) \leq 2$ in $E\Gamma(R)$ for each $x, y \in EZ(R)^*$. As $diam(E\Gamma(R)) = sup\{d(x, y) \mid x, y \in EZ(R)^*\}$, we obtain that $diam(E\Gamma(R)) \leq 2$. □

Let $G = (V, E)$ be a graph. Recall from Deo (1994, page 21) that a maximal connected subgraph of G is called a *component of G*.

Lemma 2.4 *Let $n \geq 3$ and let $R_1, R_2, R_3, \ldots, R_n$ be rings. Let $R = R_1 \times R_2 \times R_3 \times \cdots \times R_n$. Then $E\Gamma(R)$ is not connected. Moreover, $E\Gamma(R)$ has at least n components.*

Proof Let $i \in \{1, 2, 3, \ldots, n\}$. Let us denote the element of R whose i-th coordinate is 1 and its j-th coordinate equals 0 for all $j \in \{1, 2, 3, \ldots, n\} \setminus \{i\}$ by e_i. Let us denote $(\sum_{k=1}^{n} e_k) - e_i$ by f_i. Observe that $Ann(e_i) = Rf_i$ and $Ann(f_i) = Re_i$. Hence, we obtain that $e_i, f_i \in EZ(R)^*$. Let $i, j \in \{1, 2, 3, \ldots, n\}$ with $i \neq j$. We claim that there exists no path in $E\Gamma(R)$ between e_i and e_j. Suppose that there exists a path in $E\Gamma(R)$ between e_i and e_j. Since $n \geq 3$, we get that $Re_j \neq Rf_i$ and so, it follows that e_i and e_j are not adjacent in $E\Gamma(R)$. Let m be the length of a shortest path in $E\Gamma(R)$ between e_i and e_j. Then it follows from Proposition 2.3 that $m = 2$. Let $e_i - x - e_j$ be a path in $E\Gamma(R)$ of shortest length between e_i and e_j. Now, $x \in EZ(R)^*$ is adjacent to both e_i and e_j in $E\Gamma(R)$. Therefore, $Ann(e_i) = Rx, Ann(x) = Re_i$ and $Ann(x) = Re_j, Ann(e_j) = Rx$. This implies that $Re_i = Re_j$. This is a contradiction. This proves that there exists no path in $E\Gamma(R)$ between e_i and e_j and so, we obtain that $E\Gamma(R)$ is not connected.

We next verify the moreover part of this lemma. Let $i \in \{1,2,3,\ldots,n\}$. Let g_i be the component of $E\Gamma(R)$ such that $e_i \in V(g_i)$. It is shown in the previous paragraph that for any distinct $i,j \in \{1,2,3,\ldots,n\}$, there exists no path in $E\Gamma(R)$ between e_i and e_j. Hence, it follows that $g_i \neq g_j$ for all distinct $i,j \in \{1,2,3,\ldots,n\}$. This shows that $E\Gamma(R)$ has at least n components. □

Corollary 2.5 *Let R be a ring such that $dimR = 0$ and $EZ(R)^* \neq \emptyset$. If $E\Gamma(R)$ is connected, then $|Max(R)| \leq 2$.*

Proof Assume that $dimR = 0$ and $E\Gamma(R)$ is connected. Suppose that $|Max(R)| \geq 3$. Then it follows from Lemma 2.2 that there exist zero-dimensional rings R_1, R_2, and R_3 such that $R \cong R_1 \times R_2 \times R_3$ as rings. From the assumption that $E\Gamma(R)$ is connected, we obtain that $E\Gamma(R_1 \times R_2 \times R_3)$ is connected. This contradicts Lemma 2.4. Therefore, if $E\Gamma(R)$ is connected, then $|Max(R)| \leq 2$. □

Let R be a von Neumann regular ring which is not a field. We prove in Theorem 2.9 that $E\Gamma(R)$ is connected if and only if $R \cong F_1 \times F_2$ as rings, where F_1 and F_2 are fields. We verify in Corollary 2.7 that for any von Neumann regular ring R which is not a field, $EZ(R)^* = Z(R)^*$. We use Lemma 2.6 in the proof of Corollary 2.7. If $3 \leq |Max(R)| < \infty$, then as a corollary to Theorem 2.26, we are able to determine the number of components of $E\Gamma(R)$ in Corollary 2.28.

Lemma 2.6 *Let R be a von Neumann regular ring. Let $a \in R$. Then there exist $u \in U(R)$ and an idempotent element e of R such that $a = ue$.*

Proof This lemma is $(1) \Rightarrow (3)$ of Gilmer (1972, Exercise 29, page 113). However, we include a proof of this lemma in order to keep the work self-contained.

Since R is von Neumann regular and $a \in R$, there exists $b \in R$ such that $a = a^2b$. Hence, $ab = a^2b^2 = (ab)^2$. Let us denote ab by e. Note that e is an idempotent element of R. Since $e(1-e) = 0$, it follows that $a = a(ab) = ae = (a+1-e)e$. We assert that $a+1-e \in U(R)$. Let $\mathfrak{p} \in Spec(R)$. From $e(1-e) = 0 \in \mathfrak{p}$, it follows that either $e \in \mathfrak{p}$ or $1-e \in \mathfrak{p}$. If $e \in \mathfrak{p}$, then $1-e \notin \mathfrak{p}$. Now, $a = ae \in \mathfrak{p}$ and so, $a+1-e \notin \mathfrak{p}$. If $1-e \in \mathfrak{p}$, then $e = ab \notin \mathfrak{p}$. Hence, $a \notin \mathfrak{p}$ and so, $a+1-e \notin \mathfrak{p}$. As $a+1-e \notin \mathfrak{p}$ for any $\mathfrak{p} \in Spec(R)$, we obtain from Atiyah and Macdonald (1969, Corollary 1.5) that $a+1-e \in U(R)$. Thus $a = ue$ with $u = a+1-e \in U(R)$ and e is an idempotent element of R. □

Corollary 2.7 *Let R be a von Neumann regular ring which is not a field. Then $EZ(R)^* \neq \emptyset$ and $EZ(R)^* = Z(R)^*$.*

Proof Since R is a von Neumann regular ring which is not a field, we know from Lemma 2.1 that there exists an idempotent element e of R such that $e \notin \{0,1\}$. Hence, $1-e \notin \{0,1\}$. Observe that $Ann(e) = R(1-e)$ and $Ann(1-e) = Re$. This shows that $\{e, 1-e\} \subseteq EZ(R)^*$ and indeed, $e - x$ is an edge of $E\Gamma(R)$ with $x = 1-e$. This shows that $EZ(R)^* \neq \emptyset$. For any ring T, we know that $EZ(T)^* \subseteq Z(T)^*$ and so, we get that $EZ(R)^* \subseteq Z(R)^*$. Let $a \in Z(R)^*$. We know from Lemma 2.6 that there exist $u \in U(R)$ and an idempotent element e of R such that $a = ue$. It is clear that $e \notin \{0,1\}$. Observe that $Ann(a) = R(1-e)$ and $Ann(1-e) = Re = R(ue) = Ra$. This proves that $a \in EZ(R)^*$ and so, $Z(R)^* \subseteq EZ(R)^*$. Hence, we obtain that $EZ(R)^* = Z(R)^*$. □

Let $n \geq 2$ and let $R = F_1 \times F_2 \times \cdots \times F_n$, where F_i is a field for each $i \in \{1,2,\ldots,n\}$. Note that if F is any field, then given $\alpha \in F$, there exists $\beta \in F$ such that $\alpha = \alpha^2\beta$. (One can take $\beta = 0$ if $\alpha = 0$ and $\beta = \alpha^{-1}$ if $\alpha \neq 0$.) Hence, it follows that $R = F_1 \times F_2 \times \cdots \times F_n$ is von Neumann regular. Therefore, we obtain from Corollary 2.7 that $EZ(R)^* = Z(R)^*$. For a ring R, we denote $R \backslash \{0\}$ by R^*.

Lemma 2.8 *Let $R = F_1 \times F_2$, where F_1, F_2 are fields. Then $E\Gamma(R) = \Gamma(R)$ is a complete bipartite graph.*

Proof It is already noted in the paragraph which appears just preceding the statement of this lemma that $EZ(R)^* = Z(R)^*$. Note that $EZ(R)^* = Z(R)^* = V_1 \cup V_2$, where $V_1 = \{(\alpha, 0) \mid \alpha \in F_1^*\}$ and $V_2 = \{(0, \beta) \mid \beta \in F_2^*\}$. Observe that $\Gamma(R)$ is a complete bipartite graph with vertex partition $Z(R)^* = V_1 \cup V_2$. For any ring T with $EZ(T)^* \neq \emptyset$, it is already noted in the introduction that $E\Gamma(T)$ is a subgraph of $\Gamma(T)$. For any $\alpha \in F_1^*$ and $\beta \in F_2^*$, $Ann((\alpha,0)) = R(0,1) = R(0,\beta)$ and $Ann((0,\beta)) = R(1,0) = R(\alpha,0)$. This shows that for any $x \in V_1, y \in V_2$, x and y are adjacent in $E\Gamma(R)$. Hence, we get that $\Gamma(R)$ is a subgraph of $E\Gamma(R)$. Therefore, we obtain that $E\Gamma(R) = \Gamma(R)$ is a complete bipartite graph. □

Let R be a von Neumann regular ring which is not a field. In Theorem 2.9, we state and prove some necessary and sufficient conditions for $E\Gamma(R)$ to be connected. Let $n \in \mathbb{N}$ be such that $n \geq 2$. Let $|Max(R)| = n$. Since $dimR = 0$, it follows from Lemma 2.2 that there exist zero-dimensional rings $R_1, R_2, \ldots, R_n$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings. From $|Max(R)| = n$, we get that R_i is quasilocal for each $i \in \{1, 2, \ldots, n\}$. As R is reduced, it follows that R_i is reduced for each $i \in \{1, 2, \ldots, n\}$. Since a quasilocal zero-dimensional reduced ring is a field, we obtain that R_i is a field for each $i \in \{1, 2, \ldots, n\}$. With $F_i = R_i$ for each $i \in \{1, 2, \ldots, n\}$, we get that F_i is a field and $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings.

Theorem 2.9 *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent*:

(1) $E\Gamma(R) = \Gamma(R)$.
(2) $E\Gamma(R)$ *is connected.*
(3) $|Max(R)| = 2$.
(4) $R \cong F_1 \times F_2$ *as rings, where F_1 and F_2 are fields.*
(5) $E\Gamma(R) = \Gamma(R)$ *is a complete bipartite graph.*

Proof We know from $(a) \Leftrightarrow (d)$ of Gilmer (1972, Exercise 16, page 111) that a ring T is von Neumann regular if and only if $dimT = 0$ and T is reduced. We know from Atiyah and Macdonald (1969, Proposition 1.8) that for any ring A, $nil(A)$ is the intersection of all the prime ideals of A. By hypothesis, R is a von Neumann regular ring which is not a field. As every prime ideal of R is maximal and $nil(R) = (0)$, it follows that $|Max(R)| \geq 2$.

$(1) \Rightarrow (2)$ We know from Anderson and Livingston (1999, Theorem 2.3) that $\Gamma(R)$ is connected. Hence, from $E\Gamma(R) = \Gamma(R)$, we obtain that $E\Gamma(R)$ is connected.

$(2) \Rightarrow (3)$ Now, R is a von Neumann regular ring which is not a field. It is already noted in the proof of Corollary 2.7 that there exists an idempotent element e of R such that $e - x$ is an edge of $E\Gamma(R)$ with $x = 1 - e$. As $dimR = 0$ and $E\Gamma(R)$ is connected, we obtain from Corollary 2.5 that $|Max(R)| \leq 2$. It is already noted at the beginning of the proof of this theorem that $|Max(R)| \geq 2$ and so, $|Max(R)| = 2$.

$(3) \Rightarrow (4)$ Now, R is von Neumann regular and $|Max(R)| = 2$. Hence, it follows as is remarked in the paragraph which appears just preceding the statement of this theorem that there exist fields F_1, F_2 such that $R \cong F_1 \times F_2$ as rings.

$(4) \Rightarrow (5)$ We are assuming that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1, 2\}$. We know from Lemma 2.8 that $E\Gamma(F_1 \times F_2) = \Gamma(F_1 \times F_2)$ is a complete bipartite graph. Therefore, it follows that $E\Gamma(R) = \Gamma(R)$ is a complete bipartite graph.

$(5) \Rightarrow (1)$ This is clear. □

Let R be a von Neumann regular ring. If $|Max(R)| \geq 3$, then as $dimR = 0$, it follows from Corollary 2.5 that $E\Gamma(R)$ is not connected. Let T be a reduced ring with $EZ(T)^* \neq \emptyset$. In Theorem 2.15, we prove that each component of $E\Gamma(T)$ is a complete bipartite graph. Let $n \in \mathbb{N}$ with $n \geq 3$. If $|Max(R)| = n$, then we prove in Corollary 2.28 that $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components. We use Lemma 2.11 in the proof of Lemma 2.14.

Lemma 2.10 *Let R be a ring such that there exists $a \in EZ(R)^*$ with $Ann(a) \neq Ra$. Then $E\Gamma(R)$ admits at least one edge. In particular, if R is a reduced ring with $EZ(R)^* \neq \emptyset$, then $E\Gamma(R)$ admits at least one edge.*

Proof Let $a \in EZ(R)^*$. Then there exists $b \in Z(R)^*$ such that $Ann(a) = Rb$ and $Ann(b) = Ra$. By hypothesis, $Ann(a) \neq Ra$. Hence, $Ra \neq Rb$ and so, $a \neq b$. Therefore, $a - b$ is an edge of $E\Gamma(R)$.

We next verify the in particular statement of this lemma. Let R be a reduced ring with $EZ(R)^* \neq \emptyset$. Let $a \in EZ(R)^*$. Since R is reduced and $a \neq 0$, it follows that $a^2 \neq 0$ and so, $Ann(a) \neq Ra$. Therefore, we obtain that $E\Gamma(R)$ admits at least one edge. □

Lemma 2.11 *Let R be a ring such that $a - b$ is an edge of $E\Gamma(R)$ with $Ann(a) \neq Ra$. Then for any $c \in EZ(R)^* \backslash \{a, b\}$, either c and a are not adjacent in $E\Gamma(R)$ or c and b are not adjacent in $E\Gamma(R)$. Moreover, if R is reduced, then given any $c \in R^*$, either $ca \neq 0$ or $cb \neq 0$.*

Proof By hypothesis, $a - b$ is an edge of $E\Gamma(R)$ with $Ann(a) \neq Ra$. Now, $Ann(a) = Rb$ and $Ann(b) = Ra$. From $Ann(a) \neq Ra$, it follows that $Ra \neq Rb$. Suppose that c is adjacent to both a and b in $E\Gamma(R)$. Then $Ra = Ann(c) = Rb$. This is in contradiction to the fact that $Ra \neq Rb$. Therefore, either c and a are not adjacent in $E\Gamma(R)$ or c and b are not adjacent in $E\Gamma(R)$.

We next verify the moreover part of this lemma. Since R is reduced and $a \neq 0$, it follows that $a^2 \neq 0$ and so, $Ann(a) \neq Ra$. Now, $Ann(a) = Rb$ and $Ann(b) = Ra$. Let $c \in R^*$. Suppose that $ca = cb = 0$. From $ca = 0$, we get that $c \in Ann(a) = Rb$. Hence, $c = rb$ for some $r \in R$. As $cb = 0$, it follows that $rb^2 = 0$. This implies that $(rb)^2 = 0$. From R is reduced, we obtain that $c = rb = 0$. This is in contradiction to the assumption that $c \neq 0$. Therefore, either $ca \neq 0$ or $cb \neq 0$. □

Recall from Balakrisnan and Ranganathan (2000, Definition 1.1.11) that a simple graph $G = (V, E)$ is said to be *complete* if every pair of distinct vertices are adjacent in G. A complete graph with n vertices is denoted by K_n.

Lemma 2.12 *Let R be a ring such that there exists $a \in EZ(R)^*$ with $Ann(a) = Ra$. Let g be a component of $E\Gamma(R)$ such that $a \in V(g)$. Then g is complete.*

Proof Let $b \in V(g)$ with $b \neq a$. Since g is connected, there exists a path between a and b in g. We claim that a and b are adjacent in g. If $Ra = Rb$, then $Ann(a) = Ann(b)$. Hence, $Ann(a) = Ra = Rb$ and $Ann(a) = Ann(b) = Ra$. This implies that a and b are adjacent in g. Suppose that $Ra \neq Rb$. If a and b are not adjacent in g, then it follows from Proposition 2.3 that there exists a path of length two between a and b in g. Let $c \in V(g)$ be such that $a - c - b$ is a path of length two between a and b in g. Note that $Ra = Ann(a) = Rc$. Therefore, $Ann(a) = Ann(c) = Rb$ and so, $Ra = Rb$. This is in contradiction to the assumption that $Ra \neq Rb$. Therefore, a and b are adjacent in g. This proves that g is complete. □

Corollary 2.13 *Let R be a ring such that there exists $a \in V(g)$ such that $Ann(a) = Ra$, where g is a component of $E\Gamma(R)$. Then for any $x \in V(g)$, $Ann(x) = Rx$.*

Proof We know from Lemma 2.12 that g is complete. Thus for any $x \in V(g)\backslash\{a\}$, $a - x$ is an edge of g. Hence, $Ann(x) = Ra = Ann(a) = Rx$. □

Lemma 2.14 *Let R be a ring such that $EZ(R)^* \neq \emptyset$. Let g be a component of $E\Gamma(R)$ such that $Ann(a) \neq Ra$ for some $a \in V(g)$ (and hence, $Ann(x) \neq Rx$ for each $x \in V(g)$ by Corollary 2.13). Then g is a complete bipartite graph.*

Moreover, if R is reduced, then g is complete if and only if $|U(\frac{R}{Ra})| = 1$ and $|U(\frac{R}{Rb})| = 1$, where $V(g) = \{a, b\}$. If g is complete, then $|U(R)| = 1$.

Proof Let $a \in V(g)$ be such that $Ann(a) \neq Ra$. Hence, we obtain from Lemma 2.10 that there exists $b \in V(g)$ such that $a - b$ is an edge of g. Moreover, it follows from Corollary 2.13 that $Ann(x) \neq Rx$ for each $x \in V(g)$. Now, $Ann(b) = Ra \neq Rb$. Let $c \in V(g)\backslash\{a, b\}$. We know from Lemma 2.11 that either c and a are not adjacent in g or c and b are not adjacent in g.

Case(1): c and a are not adjacent in g.

It follows from Proposition 2.3 that $d(a, c) = 2$ in g. Let $x \in V(g)$ be such that $a - x - c$ is a path of length two between a and c in g. Note that $Rc = Ann(x) = Ra = Ann(b)$ and $Rb = Ann(a) = Rx = Ann(c)$. This proves that $Ra = Rc$ and c and b are adjacent in $E\Gamma(R)$ and so, c and b are adjacent in g.

Case(2): c and b are not adjacent in g.

It can be shown as in Case(1) that $Rb = Rc$ and c and a are adjacent in $E\Gamma(R)$ and so, c and a are adjacent in g.

From the above discussion, it is clear that $V(g) = V_1 \cup V_2$ with $V_1 = \{x \in V(g) \mid Rx = Ra\}$ and $V_2 = \{y \in V(g) \mid Ry = Rb\}$. We verify that g is a complete bipartite graph with vertex partition $V(g) = V_1 \cup V_2$. Observe that $a \in V_1$ and $b \in V_2$. Hence, $V_i \neq \emptyset$ for each $i \in \{1, 2\}$. From $Ra \neq Rb$, we obtain that $V_1 \cap V_2 = \emptyset$. Let $x_1, x_2 \in V_1$ with $x_1 \neq x_2$. Note that $Rx_1 = Ra = Rx_2$. Hence, $Ann(x_1) = Ann(x_2)$. From $Ann(c) \neq Rc$ for each $c \in V(g)$, we get that $Ann(x_1) \neq Rx_2$. Therefore, x_1 and x_2 are not adjacent in g. Let $y_1, y_2 \in V_2$ with $y_1 \neq y_2$. Then $Ry_1 = Rb = Ry_2$. Hence, $Ann(y_1) = Ann(y_2) \neq Ry_2$. Therefore, y_1 and y_2 are not adjacent in g. Let $x \in V_1$ and $y \in V_2$. Now, $Rx = Ra = Ann(b) = Ann(y)$ and $Ry = Rb = Ann(a) = Ann(x)$. Hence, it follows that x and y are adjacent in g. This proves that g is a complete bipartite graph with vertex partition $V(g) = V_1 \cup V_2$.

We next prove the moreover part of this lemma. We are assuming that R is reduced and g is a component of $E\Gamma(R)$. Let $a \in V(g)$. Then there exists $b \in V(g)$ such that $a - b$ is an edge of g. It is already shown in the previous paragraph that g is a complete bipartite graph with vertex partition $V(g) = V_1 \cup V_2$, where $V_1 = \{x \in V(g) \mid Rx = Ra\}$ and $V_2 = \{y \in V(g) \mid Ry = Rb\}$. Let us denote $\{r \in R \mid r + Rb \in U(\frac{R}{Rb})\}$ by U_1 and $\{s \in R \mid s + Ra \in U(\frac{R}{Ra})\}$ by U_2. We claim that $V_1 = \{ra \mid r \in U_1\}$ and $V_2 = \{sb \mid s \in U_2\}$. Let $x \in V_1$. Then $x \in V(g)$ and $Rx = Ra$. Thus there exist $r_1, r_2 \in R$ such that $x = r_1 a$ and $a = r_2 x$. Hence,

$a = r_1r_2a$ and so, $(1 - r_1r_2)a = 0$. This implies that $1 - r_1r_2 \in Ann(a) = Rb$. Therefore, $(r_1 + Rb)(r_2 + Rb) = 1 + Rb$. This proves that $r_1 \in U_1$ and $x = r_1a$. Therefore, $V_1 \subseteq \{ra \mid r \in U_1\}$. Let $r \in U_1$. As $r \in U_1$, there exists $r' \in R$ such that $1 - rr' \in Rb$. Hence, $a - r'(ra) \in Rab = (0)$ and so, $Ra \subseteq R(ra)$. As $R(ra) \subseteq Ra$, we obtain that $R(ra) = Ra$. Thus $Ann(ra) = Ann(a) = Rb$ and $Ann(b) = Ra = Rra$. Hence, ra and b are adjacent in $E\Gamma(R)$ and so, $ra \in V(g)$. This shows that $\{ra \mid r \in U_1\} \subseteq V_1$. Therefore, $V_1 = \{ra \mid r \in U_1\}$. Similarly, it can be shown that $V_2 = \{sb \mid s \in U_2\}$. We next verify that $|V_1| = |U(\frac{R}{Rb})|$. Define a mapping $f : V_1 \to U(\frac{R}{Rb})$ as follows: Let $x \in V_1$. Then $x = ra$ for some $r \in U_1$. Hence, $r + Rb \in U(\frac{R}{Rb})$. Define $f(x) = r + Rb$. This is well-defined. Suppose that $x = ra = r_1a$ for some $r_1 \in U_1$. Then $(r - r_1)a = 0$ and so, $r - r_1 \in Ann(a) = Rb$. Therefore, $r + Rb = r_1 + Rb$. We claim that f is one-one and onto. Let $x_1, x_2 \in V_1$ be such that $f(x_1) = f(x_2)$. Let $x_1 = r_1a$ and $x_2 = r_2a$ for some $r_1, r_2 \in U_1$. Now, $r_1 + Rb = f(x_1) = f(x_2) = r_2 + Rb$. This implies that $r_1 - r_2 \in Rb$ and so, $x_1 - x_2 = (r_1 - r_2)a \in Rab = (0)$. Hence, $x_1 = x_2$. This shows that f is one-one. Let $r \in R$ be such that $r + Rb \in U(\frac{R}{Rb})$. Then $r \in U_1$ and $x = ra \in V_1$. Note that $f(x) = r + Rb$. This proves that f is onto. Therefore, the mapping $f : V_1 \to U(\frac{R}{Rb})$ is a bijection and so, $|V_1| = |U(\frac{R}{Rb})|$. Similarly, it can be shown that $|V_2| = |U(\frac{R}{Ra})|$.

Now, g is a complete bipartite graph with vertex partition $V(g) = V_1 \cup V_2$. Thus g is complete if and only if $|V_1| = 1 = |V_2|$. Therefore, we obtain that g is complete if and only if $|U(\frac{R}{Ra})| = 1 = |U(\frac{R}{Rb})|$, where $V(g) = \{a, b\}$.

Assume that g is complete. Let $u \in U(R)$. From $|U(\frac{R}{Ra})| = 1 = |U(\frac{R}{Rb})|$, we get that $u - 1 \in Ra \cap Rb$. Since $Rab = (0)$ and R is reduced, it follows that $Ra \cap Rb = (0)$. Therefore, $u - 1 = 0$ and so, $u = 1$. This proves that $|U(R)| = 1$. □

Theorem 2.15 *Let R be a ring such that $EZ(R)^* \neq \emptyset$. If g is any component of $E\Gamma(R)$, then g is either complete or a complete bipartite graph. If R is reduced, then each component of $E\Gamma(R)$ is a complete bipartite graph.*

Proof Let g be a component of $E\Gamma(R)$. If there exists $a \in V(g)$ such that $Ann(a) = Ra$, then we know from Lemma 2.12 that g is complete. Suppose that $Ann(a) \neq Ra$ for each $a \in V(g)$. Then we know from Lemma 2.14 that g is a complete bipartite graph. If R is reduced, then $Ann(a) \neq Ra$ for each $a \in V(g)$, where g is any component of $E\Gamma(R)$. Therefore, g is a complete bipartite graph for each component g of $E\Gamma(R)$. □

Corollary 2.16 *Let R be a ring such that $EZ(R)^* \neq \emptyset$. Then $E\Gamma(R)$ is connected if and only if $E\Gamma(R)$ is either complete or a complete bipartite graph. If R is reduced, then $E\Gamma(R)$ is connected if and only if $E\Gamma(R)$ is a complete bipartite graph.*

Proof Note that $E\Gamma(R)$ is connected if and only $E\Gamma(R)$ is the only component of $E\Gamma(R)$. Hence, the proof of this corollary follows immediately from Theorem 2.15. □

Corollary 2.17 *Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. The following statements are equivalent*:

(1) *$E\Gamma(R)$ is complete.*
(2) $|EZ(R)^*| = 2$.

Proof We are assuming that R is a reduced ring with $EZ(R)^* \neq \emptyset$. We know from Lemma 2.10 that $E\Gamma(R)$ admits at least one edge. Let $a - b$ be an edge of $E\Gamma(R)$. Hence, $|EZ(R)^*| \geq 2$.

(1) $\Rightarrow$ (2) We are assuming that $E\Gamma(R)$ is complete. Let $c \in EZ(R)^*$. Then we claim that $c \in \{a, b\}$. If $c \notin \{a, b\}$, then c is adjacent to both a and b in $E\Gamma(R)$. We know from Lemma 2.11 that either $ca \neq 0$ or $cb \neq 0$. This implies that either c and a or c and b are not adjacent in $\Gamma(R)$ and hence, either c and a or c and b are not adjacent in $E\Gamma(R)$. This is a contradiction. Therefore, $c \in \{a, b\}$. This proves that $EZ(R)^* = \{a, b\}$ and so, $|EZ(R)^*| = 2$. One can also apply the following argument to conclude that $|EZ(R)^*| = 2$. We know from Corollary 2.16 that $E\Gamma(R)$ is a complete bipartite graph. Thus $E\Gamma(R)$ is complete as well as a complete bipartite graph. Therefore, $|EZ(R)^*| = |V(E\Gamma(R)| = 2$.

(2) $\Rightarrow$ (1) We are assuming that $|EZ(R)^*| = 2$. It is already noted in the beginning of the proof of this corollary that $a - b$ is an edge of $E\Gamma(R)$. Hence, $V(E\Gamma(R))| = EZ(R)^* = \{a, b\}$. As a and b are adjacent in $E\Gamma(R)$, it follows that $E\Gamma(R)$ is complete. □

For any $n \geq 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$.

Corollary 2.18 *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent*:

(1) $E\Gamma(R)$ *is complete.*
(2) $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *as rings.*

Proof (1) $\Rightarrow$ (2) We are assuming that $E\Gamma(R)$ is complete, where R is a von Neumann regular ring which is not a field. We know from (2) $\Rightarrow$ (4) of Theorem 2.9 that $R \cong F_1 \times F_2$ as rings, where F_i is a field for each $i \in \{1,2\}$. As R is reduced, we obtain from (1) $\Rightarrow$ (2) of Corollary 2.17 that $|EZ(R)^*| = 2$. Therefore, $|EZ(F_1 \times F_2)^*| = 2$. From $EZ(F_1 \times F_2)^* = \{(\alpha, 0) \mid \alpha \in F_1^*\} \cup \{(0,\beta) \mid \beta \in F_2^*\}$, we get that $|F_1| = |F_2| = 2$. Therefore, $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as rings.

(2) $\Rightarrow$ (1). Let us denote the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ by T. Note that $EZ(T)^* = \{(1,0),(0,1)\}$. It follows from (2) $\Rightarrow$ (1) of Corollary 2.17 that $E\Gamma(T)$ is complete. From $R \cong T$ as rings, we get that $E\Gamma(R)$ is complete. □

In Example 2.19, we provide a quasilocal reduced ring R with an infinite number of minimal prime ideals such that $EZ(R)^* = \emptyset$. The reduced ring R given in Example 2.19 is due to R. Gilmer and W. Heinzer and it appeared in Gilmer and Heinzer (1980, page 16).

Example 2.19 Let $\{X_i\}_{i=1}^{\infty}$ be a set of indeterminates over a field K. Let $D = \bigcup_{n=1}^{\infty} K[[X_1,\ldots,X_n]]$, where for each $n \in \mathbb{N}$, $K[[X_1,\ldots,X_n]]$ is the power series ring in $X_1,\ldots,X_n$ over K. Let I be the ideal of D generated by $\{X_iX_j \mid i,j \in \mathbb{N}, i \neq j\}$. Let $R = \frac{D}{I}$. Then R is a quasilocal reduced ring, $Min(R)$ is infinite, $Z(R)^* \neq \emptyset$ but $EZ(R)^* = \emptyset$.

Proof For each $n \in \mathbb{N}$, let us denote $X_n + I$ by x_n. It was noted in Gilmer and Heinzer (1980, page 16) that R is quasilocal with $\mathfrak{m} = \sum_{n=1}^{\infty} Rx_n$ as its unique maximal ideal and R is reduced. It is clear that $Z(R) = \mathfrak{m}$ and so, $Z(R)^* \neq \emptyset$. It was also noted in Gilmer and Heinzer (1980, page 16) that $Min(R) = \{\mathfrak{p}_{\mathfrak{i}} \mid \mathfrak{i} \in \mathbb{N}\}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_{\mathfrak{i}}$ is the ideal of generated by $\{x_j \mid j \in \mathbb{N}\backslash\{i\}\}$. Hence, $Min(R)$ is infinite. Let $r \in Z(R)^*$. Then it is possible to find $n,k \in \mathbb{N}$ such that $r \in \sum_{t=1}^{k}(Kx_1^t + \cdots + Kx_n^t)$. Observe that $x_j \in Ann(r)$ for all $j \geq n+1$. Hence, $Ann(r)$ cannot be principal. Therefore, it is impossible to find a non-zero $s \in R$ such that $Ann(r) = Rs$ and $Ann(s) = Rr$. This shows that $EZ(R)^* = \emptyset$. □

Let R be a reduced ring which is not an integral domain. It follows from Atiyah and Macdonald (1969, Proposition 1.8) and Kaplansky (1974, Theorem 10) that $\bigcap_{\mathfrak{p} \in Min(R)} \mathfrak{p} = (0)$. Suppose that there exists $n \in \mathbb{N}$ such that $|Min(R)| = n$ and each $\mathfrak{p} \in Min(R)$ is principal. Observe that $n \geq 2$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1,2,\ldots,n\}\}$. For each $i \in \{1,2,\ldots,n\}$, let $p_i \in \mathfrak{p}_{\mathfrak{i}}$ be such that $\mathfrak{p}_i = Rp_i$. In Lemma 2.20, we determine $EZ(R)^*$.

Lemma 2.20 *Let R be a reduced ring and let $n \in \mathbb{N}$ with $n \geq 2$ be such that $Min(R) = \{\mathfrak{p}_i = Rp_i \mid i \in \{1,2,\ldots,n\}\}$. Let $\{A_j \mid j \in \{1,2,\ldots,2^n-2\}\}$ denote the collection of all proper non-empty subsets of $\{1,2,\ldots,n\}$. For each $j \in \{1,2,\ldots,2^n-2\}$, let $B_j = A_j^c$, where A_j^c is the complement of A_j in $\{1,2,\ldots,n\}$ and let $U_j = \{r \in R \mid r + R(\prod_{t \in B_j} p_t) \in U(\frac{R}{R(\prod_{t \in B_j} p_t)})\}$. Then*

$$EZ(R)^* = \bigcup_{j \in \{1,2,\ldots,2^n-2\}} \{(\prod_{k \in A_j} p_k)r \mid r \in U_j\}.$$

Proof Let $j \in \{1,2,\ldots,2^n-2\}$ and let $r \in U_j$. Let us denote $\prod_{k \in A_j} p_k$ by a_j and $\prod_{t \in B_j} p_t$ by b_j. Observe that $a_j, b_j \in R^*$ and from $\bigcap_{i=1}^{n} Rp_i = (0)$, it follows that $a_jb_j = 0$. Since distinct minimal prime ideals are not comparable under the inclusion relation, it follows that $a_j \notin Rp_t$ for each $t \in B_j$ and $b_j \notin Rp_k$ for each $k \in A_j$. As $\bigcap_{k \in A_j} Rp_k = R(\prod_{k \in A_j} p_k)$ and $\bigcap_{t \in B_j} Rp_t = R(\prod_{t \in B_j} p_t)$, we get that $Ann(a_j) = Rb_j$ and $Ann(b_j) = Ra_j$. Hence, $a_j - b_j$ is an edge of $E\Gamma(R)$. Let us denote $(\prod_{k \in A_j} p_k)r$ by x_j. That is, a_jr by x_j. Since $r + Rb_j \in U(\frac{R}{Rb_j})$, we obtain from the proof of moreover part of Lemma 2.14 that $Ann(x_j) = Rb_j$ and $Ann(b_j) = Ra_j = Rx_j$ and so, $x_j \in EZ(R)^*$. This shows that $\bigcup_{j\in\{1,2,\ldots,2^n-2\}} \{(\prod_{k\in A_j} p_k)r \mid r \in U_j\} \subseteq EZ(R)^*$.

Let $z \in EZ(R)^*$. For any ring T, $EZ(T)^* \subseteq Z(T)^*$. Therefore, $z \in Z(R)^* = (\bigcup_{i=1}^{n} Rp_i)\backslash\{0\}$. From $\bigcap_{i=1}^{n} Rp_i = (0)$, it follows that there exists $j \in \{1, 2, \ldots, 2^n - 2\}$ such that $z = (\prod_{k\in A_j} p_k)w$ for some $w \in R$ such that $w \notin Rp_t$ for each $t \in B_j$. Since $z \in EZ(R)^*$, there exists $v \in Z(R)^*$ such that $Ann(z) = Rv$ and $Ann(v) = Rz$. Observe that $Ann(z) = R(\prod_{t\in B_j} p_t)$ and $Ann(v) = Rz$. From $zv = 0$, it follows that $v \in R(\prod_{t\in B_j} p_t)$ and so, $\prod_{k\in A_j} p_k \in Ann(v) = Rz$. This implies that $\prod_{k\in A_j} p_k = (\prod_{k\in A_j} p_k)ww_1$ for some $w_1 \in R$. Hence, $(\prod_{k\in A_j} p_k)(1 - ww_1) = 0$. Therefore, $1 - ww_1 \in R(\prod_{t\in B_j} p_t)$ and so, $w \in U_j$. Therefore, $z = (\prod_{k\in A_j} p_k)w$ for some $j \in \{1, 2, \ldots, 2^n - 2\}$ and $w \in U_j$. This proves that $EZ(R)^* \subseteq \bigcup_{j\in\{1,2,\ldots,2^n-2\}} \{(\prod_{k\in A_j} p_k)r \mid r \in U_j\}$.

Therefore, we obtain that $EZ(R)^* = \bigcup_{j\in\{1,2,\ldots,2^n-2\}} \{(\prod_{k\in A_j} p_k)r \mid r \in U_j\}$. □

Let R be a reduced ring with $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. We prove in Proposition 2.21 that $EZ(R)^* \neq \emptyset$ if and only if $\mathfrak{p}_i$ is principal for each $i \in \{1, 2\}$. Moreover, we prove in Proposition 2.21 that if $EZ(R)^* \neq \emptyset$, then $E\Gamma(R)$ is a complete bipartite graph.

Proposition 2.21 *Let R be a reduced ring with $|Min(R)| = 2$. Let $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2\}\}$. The following statements are equivalent:*

(1) $EZ(R)^* \neq \emptyset$.
(2) $\mathfrak{p}_i$ *is principal for each* $i \in \{1, 2\}$.

Moreover if either (1) or (2) holds (and hence, both (1) and (2) hold), then $E\Gamma(R)$ is a complete bipartite graph with vertex partition $EZ(R)^* = V_1 \cup V_2$, where $V_1 = \{x \in EZ(R)^* \mid Rx = \mathfrak{p}_1\}$ and $V_2 = \{y \in EZ(R)^* \mid Ry = \mathfrak{p}_2\}$.

Proof Note that $\bigcap_{i=1}^{2} \mathfrak{p}_i = (0)$ and $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$. For any $x \in \mathfrak{p}_1\backslash\mathfrak{p}_2$, $Ann(x) = \mathfrak{p}_2$ and for any $y \in \mathfrak{p}_2\backslash\mathfrak{p}_1$, $Ann(y) = \mathfrak{p}_1$.

(1) $\Rightarrow$ (2) Let $a \in EZ(R)^*$. Then we know from the proof of Lemma 2.10 that there exists $b \in EZ(R)^*$ such that $a - b$ is an edge of $E\Gamma(R)$. Hence, $Ann(a) = Rb$ and $Ann(b) = Ra$. We can assume without loss of generality that $a \in \mathfrak{p}_1\backslash\mathfrak{p}_2$. In such a case, it follows that $b \in \mathfrak{p}_2\backslash\mathfrak{p}_1$. Thus $\mathfrak{p}_2 = Ann(a) = Rb$ is principal and $\mathfrak{p}_1 = Ann(b) = Ra$ is principal. This proves that $\mathfrak{p}_i$ is principal for each $i \in \{1, 2\}$.

(2) $\Rightarrow$ (1) Let $i \in \{1, 2\}$ and let $p_i \in \mathfrak{p}_i$ be such that $\mathfrak{p}_i = Rp_i$. We know from Lemma 2.20 that $V(E\Gamma(R)) = EZ(R)^* = \{p_1 r \mid r \in U_1\} \cup \{p_2 s \mid s \in U_2\}$, where $U_1 = \{r \in R \mid r + \mathfrak{p}_2 \in U(\frac{R}{\mathfrak{p}_2})\}$ and $U_2 = \{s \in R \mid s + \mathfrak{p}_1 \in U(\frac{R}{\mathfrak{p}_1})\}$. This proves that $EZ(R)^* \neq \emptyset$.

Let us denote $\{p_1 r \mid r \in U_1\}$ by V_1 and $\{p_2 s \mid s \in U_2\}$ by V_2. Note that $p_i \in V_i$ for each $i \in \{1, 2\}$ and from $Ann(p_1) = \mathfrak{p}_2 = Rp_2$ and $Ann(p_2) = \mathfrak{p}_1 = Rp_1$, we obtain that $p_1 - p_2$ is an edge of $E\Gamma(R)$. Let $x \in V_1$ and $y \in V_2$. We know from the proof of moreover part of Lemma 2.14 that $Rx = Rp_1 = \mathfrak{p}_1$ and $Ry = Rp_2 = \mathfrak{p}_2$. Also, we know from the proof of Lemma 2.14 that the subgraph g of $E\Gamma(R)$ induced by $V_1 \cup V_2$ is a complete bipartite graph with vertex partition $V(g) = V_1 \cup V_2$. As $V(g) = EZ(R)^*$, it follows that $E\Gamma(R)$ is a complete bipartite graph with vertex partition $EZ(R)^* = V_1 \cup V_2$.

The moreover part of this proposition is already verified in the proof of (2) $\Rightarrow$ (1) of this proposition. □

Corollary 2.22 *Let $R = D_1 \times D_2$, where D_i is an integral domain for each $i \in \{1, 2\}$. Then $EZ(R)^* \neq \emptyset$ and $E\Gamma(R)$ is a complete bipartite graph with vertex partition $EZ(R)^* = V_1 \cup V_2$, where $V_1 = \{(u, 0) \mid u \in U(D_1)\}$ and $V_2 = \{(0, v) \mid v \in U(D_2)\}$.*

Proof Observe that R is a reduced ring with $Min(R) = \{\mathfrak{p}_1 = D_1 \times (0), \mathfrak{p}_2 = (0) \times D_2\}$. Note that $\mathfrak{p}_1 = R(1, 0)$ and $\mathfrak{p}_2 = R(0, 1)$. We know from the proof of (2) $\Rightarrow$ (1) of Proposition 2.21 that $EZ(R)^* \neq \emptyset$ and $E\Gamma(R)$ is a complete bipartite graph with vertex partition $EZ(R)^* = V_1 \cup V_2$, where $V_1 = \{x \in EZ(R)^* \mid Rx = R(1, 0)\}$ and $V_2 = \{y \in EZ(R)^* \mid Ry = R(0, 1)\}$. It is clear that $V_1 = \{(u, 0) \mid u \in U(D_1)\}$ and $V_2 = \{(0, v) \mid v \in U(D_2)\}$. □

Let R be a reduced ring which is not an integral domain. Let $n \in \mathbb{N}$ be such that $|Min(R)| = n$. Suppose that $\mathfrak{p}$ is principal for each $\mathfrak{p} \in Min(R)$. We prove in Proposition 2.23 that $E\Gamma(R)$ is connected if and only if $n = 2$.

Proposition 2.23 *Let R be a reduced ring which is not an integral domain. Let $n \in \mathbb{N}$ be such that $Min(R) = \{\mathfrak{p}_i = Rp_i \mid i \in \{1, 2, \ldots, n\}\}$. The following statements are equivalent:*

(1) *$E\Gamma(R)$ is a complete bipartite graph.*
(2) *$E\Gamma(R)$ is connected.*
(3) *$n = 2$.*

Proof We know from Lemma 2.20 that $EZ(R)^* \neq \emptyset$ and $EZ(R)^*$ is already determined in Lemma 2.20.

(1) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (3) Suppose that $n \geq 3$. We know from Lemma 2.20 that $p_1, p_2 \in EZ(R)^*$. It is clear that $p_1 \neq p_2$. By assumption, $E\Gamma(R)$ is connected. Since $p_1p_2 \notin \mathfrak{p}_3$, it follows that $p_1p_2 \neq 0$. Hence, p_1 and p_2 are not adjacent in $\Gamma(R)$ and so, they are not adjacent in $E\Gamma(R)$. It follows from Proposition 2.3 that $d(p_1, p_2) = 2$ in $E\Gamma(R)$. Let $z \in EZ(R)^*$ be such that $p_1 - z - p_2$ is a path of shortest length between p_1 and p_2 in $E\Gamma(R)$. Observe that $Rp_1 = Ann(z) = Rp_2$. This is impossible. Therefore, $n \leq 2$. Since R is not an integral domain, we obtain that $n \geq 2$ and so, $n = 2$.

(3) $\Rightarrow$ (1) We know from the moreover part of Proposition 2.21 that $E\Gamma(R)$ is a complete bipartite graph with vertex partition $EZ(R)^* = V_1 \cup V_2$, where $V_1 = \{x \in EZ(R)^* \mid Rx = Rp_1\}$ and $V_2 = \{y \in EZ(R)^* \mid Ry = Rp_2\}$. □

We provide Example 2.24 to illustrate Proposition 2.23.

Example 2.24 Let T be a unique factorization domain. Let p_1, p_2 be non-associate prime elements of T. Let $R = \frac{T}{Tp_1p_2}$. Then $E\Gamma(R)$ is a complete bipartite graph and $E\Gamma(R)$ is complete if and only if $|U(\frac{R}{\mathfrak{p}_i})| = 1$ for each $i \in \{1, 2\}$, where $\mathfrak{p}_i = \frac{Tp_i}{Tp_1p_2}$.

Proof Note that R is a reduced ring and $Min(R) = \{\frac{Tp_1}{Tp_1p_2}, \frac{Tp_2}{Tp_1p_2}\}$. As $|Min(R)| = 2$ and each $\mathfrak{p} \in Min(R)$ is principal, it follows from (3) $\Rightarrow$ (1) of Proposition 2.23 that $E\Gamma(R)$ is a complete bipartite graph. Let $i \in \{1, 2\}$. Let us denote $\frac{Tp_i}{Tp_1p_2}$ by $\mathfrak{p}_i$. Let us denote $p_1 + Tp_1p_2$ by a and $p_2 + Tp_1p_2$ by b. Note that $a - b$ is an edge of $E\Gamma(R)$, $\mathfrak{p}_1 = Ra$ and $\mathfrak{p}_2 = Rb$. Therefore, it follows from the proof of moreover part of Lemma 2.14 that $E\Gamma(R)$ is complete if and only if $|U(\frac{R}{\mathfrak{p}_i})| = 1$ for each $i \in \{1, 2\}$.

In particular with $T = K[X, Y]$, the polynomial ring in two variables X, Y over a field K, $p_1 = X$, $p_2 = Y$, and $R = \frac{T}{Tp_1p_2}$, it follows that $E\Gamma(R)$ is a complete bipartite graph. We know from Lemma 2.20 that $EZ(R)^* = \{xr \mid r + \mathfrak{p}_2 \in U(\frac{R}{\mathfrak{p}_2})\} \cup \{ys \mid s + \mathfrak{p}_1 \in U(\frac{R}{\mathfrak{p}_1})\}$, where $x = X + XYK[X, Y], y = Y + XYK[X, Y]$, $\mathfrak{p}_1 = \frac{Tp_1}{Tp_1p_2}$, and $\mathfrak{p}_2 = \frac{Tp_2}{Tp_1p_2}$. Observe that $Z(R) = \bigcup_{i=1}^{2} \mathfrak{p}_i$ and from $\frac{R}{\mathfrak{p}_1} \cong K[Y]$ as rings, $\frac{R}{\mathfrak{p}_2} \cong K[X]$ as rings, it follows that $U(\frac{R}{\mathfrak{p}_i}) \cong K^*$ as groups for each $i \in \{1, 2\}$. Note that $x(x-1) \in Z(R)^*$. We assert that $x(x-1) \notin EZ(R)^*$. If $x(x-1) \in EZ(R)^*$, then from the fact that $x(x-1) \notin \mathfrak{p}_2$, it follows that $x(x-1) = xr$ for some $r \in R$ with $r + \mathfrak{p}_2 = \alpha + \mathfrak{p}_2$ for some $\alpha \in K^*$. Hence, $r = \alpha + ya$ for some $a \in R$. This implies that $x(x-1) = x(\alpha + ya) = \alpha x$ and so, $X(X-1) - \alpha X \in XYK[X, Y]$. This is impossible and so, $x(x-1) \notin EZ(R)^*$. This shows that $Z(R)^* \neq EZ(R)^*$. With $K = \mathbb{Z}_2$, we get that $|U(\frac{R}{\mathfrak{p}_i})| = 1$ for each $i \in \{1, 2\}$. Therefore, we obtain that $E\Gamma(R)$ is complete and we know from (1) $\Rightarrow$ (2) of Corollary 2.17 that $E\Gamma(R)$ is a complete graph with two vertices. □

In Example 2.25, we provide non-isomorphic reduced rings R and T such that both the graphs $E\Gamma(R)$ and $E\Gamma(T)$ are complete graphs with two vertices.

Example 2.25 Let $R = \frac{\mathbb{Z}_2[X,Y]}{XY\mathbb{Z}_2[X,Y]}$, where $\mathbb{Z}_2[X, Y]$ is the polynomial ring in two variables X, Y over $\mathbb{Z}_2$. Let $T = \mathbb{Z}_2 \times \mathbb{Z}_2$. It is noted in the verification of Example 2.24 that $E\Gamma(R)$ is a complete graph with two vertices. We know from the proof of (2) $\Rightarrow$ (1) of Corollary 2.18 that $E\Gamma(T)$ is a complete graph with two vertices. Observe that $dimR = 1$ but $dimT = 0$. Therefore, R and T are not isomorphic as rings. □

Let R be a reduced ring. Let $n \in \mathbb{N}$ with $n \geq 3$ be such that $Min(R) = \{\mathfrak{p}_i = Rp_i \mid i \in \{1, 2, 3, \ldots, n\}\}$. We prove in Theorem 2.26 that $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph.

Theorem 2.26 *Let R be a reduced ring. Let $n \in \mathbb{N}$ be such that $n \geq 3$. Let $Min(R) = \{\mathfrak{p}_i = Rp_i \mid i \in \{1, 2, 3, \ldots, n\}\}$. Then $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph.*

Proof Let $\mathcal{C} = \{A_j \mid j \in \{1, 2, 3, \ldots, 2^n - 2\}\}$ denote the collection of all non-empty proper subsets of $\{1, 2, 3, \ldots, n\}$. We know from Lemma 2.20 that $EZ(R)^* = \bigcup_{j \in \{1,2,3,\ldots,2^n-2\}} \{(\prod_{k \in A_j} p_k) r \mid r \in U_j\}$, where for each $j \in \{1, 2, 3, \ldots, 2^n - 2\}$, $B_j = \{1, 2, 3, \ldots, n\} \backslash A_j$, and $U_j = \{r \in R \mid r + R(\prod_{t \in B_j} p_t) \in U(\frac{R}{R(\prod_{t \in B_j} p_t)})\}$. It is possible to arrange the members of $\mathcal{C}$ in such a way that $\mathcal{C} = \{A_1, A_2, A_3, \ldots, A_{2^{n-1}-1}, B_1 = A_{2^{n-1}}, B_2 = A_{2^{n-1}+1}, B_3 = A_{2^{n-1}+2}, \ldots, B_{2^{n-1}-1} = A_{2^n-2}\}$. Let $j \in \{1, 2, 3, \ldots, 2^{n-1} - 1\}$. Let g_j be the subgraph of $E\Gamma(R)$ induced by $V(g_j)^{(1)} \cup V(g_j)^{(2)}$ with $V(g_j)^{(1)} = \{(\prod_{k \in A_j} p_k) r \mid r \in U_j\}$ and $V(g_j)^{(2)} = \{(\prod_{t \in B_j} p_t) s \mid s \in V_j\}$, where $V_j = \{s \in R \mid s + R(\prod_{k \in A_j} p_k) \in U(\frac{R}{R(\prod_{k \in A_j} p_k)})\}$. Observe that $V(g_j) = V(g_j)^{(1)} \cup V(g_j)^{(2)}$. Note that it follows from the proof of moreover part of Lemma 2.14 that for any $x \in V(g_j)^{(1)}$, $Rx = R(\prod_{k \in A_j} p_k)$ and for any $y \in V(g_j)^{(2)}$, $Ry = R(\prod_{t \in B_j} p_t)$, $Ann(x) = Ry$, and $Ann(y) = Rx$. Hence, g_j is a complete bipartite graph. Let $j_1, j_2 \in \{1, 2, 3, \ldots, 2^{n-1} - 1\}$ be such that $j_1 \neq j_2$. We claim that $V(g_{j_1}) \cap V(g_{j_2}) = \emptyset$. If $z \in V(g_{j_1})$, then $Rz \in \{R(\prod_{k \in A_{j_1}} p_k), R(\prod_{t \in B_{j_1}} p_t)\}$. If $w \in V(g_{j_2})$, then $Rw \in \{R(\prod_{k' \in A_{j_2}} p_{k'}), R(\prod_{t' \in B_{j_2}} p_{t'})\}$.

From $A_{j_1} \notin \{A_{j_2}, B_{j_2}\}$ and $B_{j_1} \notin \{A_{j_2}, B_{j_2}\}$, we get that $R(\prod_{k \in A_{j_1}} p_k) \notin \{R(\prod_{k' \in A_{j_2}} p_{k'}), R(\prod_{t' \in B_{j_2}} p_{t'})\}$ and $R(\prod_{t \in B_{j_1}} p_t) \notin \{R(\prod_{k' \in A_{j_2}} p_{k'}), R(\prod_{t' \in B_{j_2}} p_{t'})\}$. Therefore, it follows that $V(g_{j_1}) \cap V(g_{j_2}) = \emptyset$. Observe that $V(E\Gamma(R)) = EZ(R)^* = \bigcup_{j=1}^{2^{n-1}-1} V(g_j)$. If $x - y$ is an edge of $E\Gamma(R)$ with $x \in V(g_j)$ for some $j \in \{1, 2, 3, \ldots, 2^{n-1} - 1\}$, then y must be in $V(g_j)$. Let g be any component of $E\Gamma(R)$. We know from Theorem 2.15 that g is a complete bipartite graph. Let $a - b$ be an edge of g. Now, $a, b \in EZ(R)^*$. From Lemma 2.20 and from the arrangement of members of $\mathcal{C}$, we can assume without loss of generality that $a = (\prod_{k \in A_j} p_k) r$ for some $j \in \{1, 2, 3, \ldots, 2^{n-1} - 1\}$, $r \in U_j$ and $b = (\prod_{t \in B_j} p_t) s$ with $s \in V_j$. It follows from the proof of Lemma 2.14 that $g = g_j$. From the above given arguments, it is clear that $\{g_j \mid j \in \{1, 2, 3, \ldots, 2^{n-1} - 1\}\}$ is the set of all components of $E\Gamma(R)$. This proves that $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph. $\square$

In Corollaries 2.27, 2.28, and 2.29, we provide reduced rings R satisfying the hypotheses of Theorem 2.26.

Corollary 2.27 *Let $n \geq 3$ and let $R = D_1 \times D_2 \times D_3 \times \cdots \times D_n$, where D_i is an integral domain for each $i \in \{1, 2, 3, \ldots, n\}$. Then $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph.*

Proof It is clear that R is a reduced ring and $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, 3, \ldots, n\}\}$, where for each $i \in \{1, 2, 3, \ldots, n\}$, $\mathfrak{p}_i = I_1 \times I_2 \times I_3 \times \cdots \times I_n$, with $I_i = (0)$ and $I_j = D_j$ for all $j \in \{1, 2, 3, \ldots, n\} \backslash \{i\}$. For each $i \in \{1, 2, 3, \ldots, n\}$, let e_i denote the element of R whose i-th coordinate is 1 and whose j-th coordinate equals 0 for all $j \in \{1, 2, 3, \ldots, n\} \backslash \{i\}$. Observe that for each $i \in \{1, 2, 3, \ldots, n\}$, $\mathfrak{p}_i = Rf_i$ with $f_i = (1, 1, 1, \ldots, 1) - e_i$. It now follows from Theorem 2.26 that $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph. $\square$

Corollary 2.28 *Let R be a von Neumann regular ring and let $n \in \mathbb{N}$ with $n \geq 3$. If $|Max(R)| = n$, then $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph.*

Proof Now, R is von Neumann regular with $|Max(R)| = n \geq 3$. Hence, it follows from the remark mentioned in the paragraph which appears just preceding the statement of Theorem 2.9 that there exist fields $F_1, F_2, F_3, \ldots, F_n$ such that $R \cong F_1 \times F_2 \times F_3 \times \cdots \times F_n$ as rings. It now follows from Corollary 2.27 that $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph. □

Corollary 2.29 *Let T be a unique factorization domain. Let $n \geq 3$ and let $p_1, p_2, p_3, \ldots, p_n$ be pairwise non-associate prime elements of T. Let $R = \frac{T}{T(\prod_{i=1}^{n} p_i)}$. Then $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph.*

Proof Note that R is a reduced ring and $Min(R) = \{\mathfrak{p}_i \mid i \in \{1, 2, 3, \ldots, n\}\}$ with $\mathfrak{p}_k = \frac{Tp_k}{T(\prod_{i=1}^{n} p_i)}$ is principal for each $k \in \{1, 2, 3, \ldots, n\}$. Hence, we obtain from Theorem 2.26 that $E\Gamma(R)$ has exactly $2^{n-1} - 1$ components and each component is a complete bipartite graph. □

In Example 2.30, we provide a reduced ring T with an infinite number of minimal prime ideals such that $E\Gamma(T)$ is a complete bipartite graph.

Example 2.30 Let $T = R \times R$, where R is as in Example 2.19. Then $Min(T)$ is infinite and $E\Gamma(T)$ is a complete bipartite graph.

Proof In the notation of Example 2.19, R is a quasilocal reduced ring with $\mathfrak{m} = \sum_{n=1}^{\infty} Rx_n$ as its unique maximal ideal and $Min(R)$ is infinite with $Min(R) = \{\mathfrak{p}_{\mathfrak{i}} \mid \mathfrak{i} \in \mathbb{N}\}$, where for each $i \in \mathbb{N}$, $\mathfrak{p}_{\mathfrak{i}}$ is the ideal of R generated by $\{x_j \mid j \in \mathbb{N} \backslash \{i\}\}$. It is noted in the proof of Example 2.19 that $Z(R) = \mathfrak{m}$ and $EZ(R)^* = \emptyset$. It is clear that $T = R \times R$ is a reduced ring and $Min(T) = \{\mathfrak{p}_i \times R \mid i \in \mathbb{N}\} \cup \{R \times \mathfrak{p}_i \mid i \in \mathbb{N}\}$. Hence, T has an infinite number of minimal prime ideals. Note that in the notation of Example 2.19, $R = \frac{D}{I}$, where $D = \bigcup_{n=1}^{\infty} K[[X_1, \ldots, X_n]]$ and I is the ideal of D generated by $\{X_iX_j \mid i, j \in \mathbb{N}, i \neq j\}$ and $x_i = X_i + I$ for each $i \in \mathbb{N}$. Observe that $Z(T) = (Z(R) \times R) \cup (R \times Z(R)) = (\mathfrak{m} \times R) \cup (R \times \mathfrak{m})$. We claim that $EZ(T)^* = \{(u, 0 + I) \mid u \in U(R)\} \cup \{(0 + I, v) \mid v \in U(R)\}$. For any $u, v \in U(R)$, note that $Ann((u, 0 + I)) = (0 + I) \times R = T(0 + I, 1 + I) = T(0 + I, v)$ and $Ann((0 + I, 1 + I)) = Ann((0 + I, v)) = R \times (0 + I) = T(u, 0 + I)$. This shows that $(u, 0 + I), (0 + I, v) \in EZ(T)^*$. This proves that $\{(u, 0 + I) \mid u \in U(R)\} \cup \{(0 + I, v) \mid v \in U(R)\} \subseteq EZ(R)^*$ and moreover, $(u, 0 + I)$ and $(0 + I, v)$ are adjacent in $E\Gamma(T)$. Let $t \in EZ(T)^*$. Then either $t = (m, r)$ for some $m \in \mathfrak{m}$ and $r \in R$ or $t = (s, m')$ for some $s \in R$ and $m' \in \mathfrak{m}$. Suppose that $t = (m, r)$ for some $r \in R$. Observe that there exists $(r_1, r_2) \in Z(T)^*$ such that $Ann(t) = Ann(m) \times Ann(r) = T(r_1, r_2) = Rr_1 \times Rr_2$ and $Ann(r_1) \times Ann(r_2) = Ann((r_1, r_2)) = T(m, r) = Rm \times Rr$. This implies that $Ann(m) = Rr_1$ and $Ann(r_1) = Rm$. If $m \neq 0 + I$, then we get that $m \in EZ(R)^*$. This is impossible, since it is already verified in the proof of Example 2.19 that $EZ(R)^* = \emptyset$. Hence, $m = 0 + I$. Since $t \in EZ(T)^*$, it follows that $r \neq 0 + I$. If $r \in \mathfrak{m}$, then from $Ann(r) = Rr_2$ and $Ann(r_2) = Rr$, we get that $r \in EZ(R)^*$. This is in contradiction to the fact that $EZ(R)^* = \emptyset$. Therefore, $r \in U(R)$. and so, $t = (0 + I, v)$ for some $v \in U(R)$. Similarly, if $t = (s, m')$ for some $s \in R$ and $m' \in \mathfrak{m}$, then it can be shown that $m' = 0 + I$ and $s \in U(R)$. Thus in this case, $t = (u, 0 + I)$ for some $u \in U(R)$. This shows that $EZ(T)^* \subseteq \{(u, 0 + I) \mid u \in U(R)\} \cup \{(0 + I, v) \mid v \in U(R)\}$ and so, $EZ(T)^* = \{(u, 0 + I) \mid u \in U(R)\} \cup \{(0 + I, v) \mid v \in U(R)\}$. Let $V_1 = \{(u, 0 + I) \mid u \in U(R)\}$ and let $V_2 = \{(0 + I, v) \mid v \in U(R)\}$. It is already verified in this proof that each element of V_1 is adjacent to every element of V_2 in $E\Gamma(T)$. Let $u_1, u_2 \in U(R)$ with $u_1 \neq u_2$. As $u_1u_2 \in U(R)$, it follows that $(u_1, 0 + I)(u_2, 0 + I) \neq (0 + I, 0 + I)$. Therefore, $(u_1, 0 + I)$ and $(u_2, 0 + I)$ are not adjacent in $\Gamma(T)$ and so, they are not adjacent in $E\Gamma(T)$. Similarly, we obtain that no two distinct elements of V_2 are adjacent in $E\Gamma(T)$. Therefore, $E\Gamma(T)$ is a complete bipartite graph with vertex partition $EZ(T)^* = V_1 \cup V_2$. □

In Example 2.31, we provide a reduced ring R with $|Min(R)| = 2$ such that $EZ(R)^* = \emptyset$.

Example 2.31 Let $T = K[X_1, X_2, X_3]$ be the polynomial ring in three variables X_1, X_2, X_3 over a field K. Let I be the ideal of T given by $I = TX_1X_2 + TX_1X_3$. Let $R = \frac{T}{I}$. Then R is reduced, $|Min(R)| = 2$, and $EZ(R)^* = \emptyset$.

Proof Observe that $TX_1, TX_2 + TX_3 \in Spec(T)$ and $I = TX_1 \cap (TX_2 + TX_3)$. Therefore, it follows that $R = \frac{T}{I}$ is reduced. Let $\mathfrak{p}_1 = \frac{TX_1}{I}$ and let $\mathfrak{p}_2 = \frac{TX_2+TX_3}{I}$. It is not hard to verify that $Min(R) = \{\mathfrak{p}_i \mid i \in \{1,2\}\}$. Therefore, $|Min(R)| = 2$. Note that $\mathfrak{p}_1 = R(X_1 + I)$ is principal. We claim that $\mathfrak{p}_2$ is not principal. Suppose that $\mathfrak{p}_2$ is principal. Then there exist $f, g \in T$ such that $\mathfrak{p}_2 = Rr$ with $r = (X_2f + X_3g) + I$. Now, $X_2 + I, X_3 + I \in \mathfrak{p}_2$. Hence, $X_2 + I = r(h_1 + I)$ for some $h_1 \in T$ and $X_3 + I = r(h_2 + I)$ for some $h_2 \in T$. Therefore, we get that $(X_2 + I)(h_2 + I) = (X_3 + I)(h_1 + I)$ and so, $X_2h_2 - X_3h_1 \in I = TX_1X_2 + TX_1X_3$. This implies that $h_2 \in TX_1 + TX_3$, since $TX_1 + TX_3 \in Spec(T)$ and $X_2 \notin TX_1 + TX_3$. Hence, $h_2 = h_{21}X_1 + h_{22}X_3$ for some $h_{21}, h_{22} \in T$. From $X_3 + I = r(h_2 + I)$, it follows that $X_3 + I = (X_2f + X_3g + I)(h_{22}X_3 + I)$, since $X_1X_2, X_1X_3 \in I$. Therefore, $X_3 - fh_{22}X_2X_3 - h_{22}gX_3^2 \in I$. Since $I \subset TX_1$, we get that $X_3 - h_{22}gX_3^2 \in TX_1 + TX_2$. As $TX_1 + TX_2 \in Spec(T)$, from $X_3(1 - h_{22}gX_3) \in TX_1 + TX_2$, it follows that either $X_3 \in TX_1 + TX_2$ or $1 - h_{22}gX_3 \in TX_1 + TX_2$. But both are impossible. Therefore, $\mathfrak{p}_2$ is not principal and so, we obtain from (1) $\Rightarrow$ (2) of Proposition 2.21 that $EZ(R)^* = \emptyset$. □

In Example 2.32, we provide a reduced ring S with $|Min(S)| = 3$ such that $EZ(S)^* \neq \emptyset$, S admits a non-principal minimal prime ideal of S, and $E\Gamma(S)$ is a complete bipartite graph. Example 2.32 illustrates that (1) $\Rightarrow$ (2) of Proposition 2.21 can fail to hold if the assumption on the number of minimal prime ideals of the reduced ring is omitted. Moreover, Example 2.32 also illustrates that the conclusion of Theorem 2.26 can fail to hold if the assumption that each minimal prime ideal is principal is dropped.

Example 2.32 Let $S = R \times F$, where R is as in Example 2.31 and F is a field. Then S is reduced, $|Min(S)| = 3$, $EZ(S)^* \neq \emptyset$, S admits a non-principal minimal prime ideal, and $E\Gamma(S)$ is a complete bipartite graph.

Proof As R is reduced, it is clear that S is reduced. In the notation of Example 2.31, $R = \frac{K[X_1,X_2,X_3]}{I}$, where $I = TX_1X_2 + TX_1X_3$, $Min(R) = \{\mathfrak{p}_i \mid i \in \{1,2\}\}$ and so, $Min(S) = \{\mathfrak{p}_i \times F \mid i \in \{1,2\}\} \cup \{R \times (0)\}$. Therefore, $|Min(S)| = 3$. Observe that for any $u \in U(R)$ and any $\alpha \in F^*$, $Ann_S((u,0)) = (0 + I) \times F = S(0 + I, \alpha)$ and $Ann_S((0 + I, \alpha)) = R \times (0) = S(u, 0)$. Hence, $\{(u,0), (0 + I, \alpha) \mid u \in U(R), \alpha \in F^*\} \subseteq EZ(S)^*$ and so, $EZ(S)^* \neq \emptyset$. It is already verified in Example 2.31 that $\mathfrak{p}_2$ is not principal. Hence, the minimal prime ideal $\mathfrak{p}_2 \times F$ of S is not principal. Note that $Z(S) = (Z(R) \times F) \cup (R \times (0))$. Let $r \in R, \alpha \in F$ be such that $(r, \alpha) \in EZ(S)^*$. As $EZ(A)^* \subseteq Z(A)^*$ for any ring A, we get that either $(r, \alpha) \in Z(R) \times F$ or $(r, \alpha) \in R \times (0)$. We consider the following cases.

Case(1): $(r, \alpha) \in Z(R) \times F$.

As $(r, \alpha) \in EZ(S)^*$, there exists $(r_1, \alpha_1) \in S = R \times F$ such that $Ann_S((r, \alpha)) = S(r_1, \alpha_1)$ and $Ann_S((r_1, \alpha_1)) = S(r, \alpha)$. If $r \neq 0 + I$, then it can be shown as in the proof of Example 2.30 that $r \in EZ(R)^*$. This is in contradiction to the fact that $EZ(R)^* = \emptyset$. Therefore, $r = 0 + I$ and so, $\alpha \neq 0$.

Case(2): $(r, \alpha) \in R \times (0)$.

Now, $\alpha = 0$ and $r \neq 0 + I$. Note that there exists $(r_1, \alpha_1) \in R \times F$ such that $Ann_R(r) \times Ann_F(0) = S(r_1, \alpha_1)$ and $Ann_R(r_1) \times Ann_F(\alpha_1) = S(r, 0)$. Hence, $Ann_R(r) = Rr_1$, $F = F\alpha_1$, and $Ann_R(r_1) = Rr$. Since $r \neq 0 + I$ and $EZ(R)^* = \emptyset$, we obtain that $r \notin Z(R)$ and so, $Ann_R(r) = (0 + I)$. This implies that $r_1 = 0 + I$. Hence, from $Rr = Ann_R(r_1) = Ann_R(0 + I) = R$, it follows that $r \in U(R)$.

From the above discussion, it is clear that $EZ(S)^* = \{(u, 0) \mid u \in U(R)\} \cup \{(0 + I, \alpha) \mid \alpha \in F^*\}$. Note that $E\Gamma(S)$ is a complete bipartite graph with vertex partition $EZ(S)^* = V_1 \cup V_2$, where $V_1 = \{(u, 0) \mid u \in U(R)\}$ and $V_2 = \{(0 + I, \alpha) \mid \alpha \in F^*\}$. □

Remark 2.33 Recall from Deo (1994, page 7) that a graph $G = (V, E)$ is called a *finite graph* if $|V| < \infty$ and $|E| < \infty$. Let R be a ring such that $Z(R)^* \neq \emptyset$. We know from Anderson and Livingston (1999, Theorem 2.2) (see also Ganesan (1964, Theorem 1)) that $\Gamma(R)$ is finite if and only if R is finite. In this Remark, we provide an example of an infinite ring R such that $EZ(R)^* \neq \emptyset$ and $E\Gamma(R)$ is finite. Let $R = \mathbb{Z} \times \mathbb{Z}$. We know that $U(\mathbb{Z}) = \{1, -1\}$. We know from Corollary 2.22 that $E\Gamma(R)$ is a complete bipartite graph with vertex partition $EZ(R)^* = \{(1, 0), (-1, 0)\} \cup \{(0, 1), (0, -1)\}$. This shows that $E\Gamma(\mathbb{Z} \times \mathbb{Z})$ is finite. Note that the ring $\mathbb{Z} \times \mathbb{Z}$ is infinite. □

Let R be a ring. Then it is well-known that $S = R \backslash Z(R)$ is a multiplicatively closed subset of R. The ring $S^{-1}R$ is called the *total quotient ring* of R and is denoted by $Tot(R)$ or by $T(R)$. Let $f : R \to T(R)$ be the usual homomorphism of rings defined by $f(r) = \frac{r}{1}$. Then it is not hard to show that f is injective. If $r \in Z(R)^*$ is such that $r \in EZ(R)^*$, then there exists $s \in R^*$ such that $Ann_R(r) = Rs$ and $Ann_R(s) = Rr$. Observe that $\frac{r}{1} \in Z(T(R))^*$ and it follows from Atiyah and Macdonald (1969, Proposition 3.14) that $Ann_{T(R)}(\frac{r}{1}) =$

$S^{-1}(Ann_R(r)) = T(R)(\frac{s}{1})$ and $Ann_{T(R)}(\frac{s}{1}) = S^{-1}(Ann_R(s)) = T(R)(\frac{r}{1})$. This shows that if $r \in EZ(R)^*$, then $\frac{r}{1} \in EZ(T(R))^*$ and moreover, if $r, s \in EZ(R)^*$ are such that r and s are adjacent in $E\Gamma(R)$, then $\frac{r}{1}$ and $\frac{s}{1}$ are adjacent in $E\Gamma(T(R))$. Hence, we obtain that for a ring R with $EZ(R)^* \neq \emptyset$, $E\Gamma(R)$ can be identified with a subgraph of $E\Gamma(T(R))$.

Recall from Deo (1994, page 14) that two graphs G and G' are said to be *isomorphic* (to each other) if there is a one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. In other words, suppose that the edge e is incident on vertices v_1 and v_2 in G, then the corresponding edge e' in G' must be incident on the vertices v_1' and v_2' that correspond to v_1 and v_2, respectively. Let R be ring such that $EZ(R)^* \neq \emptyset$. As $EZ(R)^* \subseteq Z(R)^*$, it follows that $Z(R)^* \neq \emptyset$. For a ring R with $Z(R)^* \neq \emptyset$, it was proved in Anderson, Levy, and Shapiro (2003, Theorem 2.2) that the graphs $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic. In Example 2.34, we mention a ring R such that $E\Gamma(R)$ and $E\Gamma(T(R))$ are not isomorphic.

Example 2.34 Let $R = \mathbb{Z} \times \mathbb{Z}$. Then $E\Gamma(R)$ and $E\Gamma(T(R))$ are not isomorphic.

Proof It is not hard to verify that $T(R) \cong \mathbb{Q} \times \mathbb{Q}$ as rings.

It is noted in Remark 2.33 that $E\Gamma(R)$ is a complete bipartite graph with vertex partition $EZ(R)^* = \{(1,0), (-1,0)\} \cup \{(0,1), (0,-1)\}$. We know from the proof of Corollary 2.22 that $E\Gamma(\mathbb{Q} \times \mathbb{Q})$ is a complete bipartite graph with vertex partition $EZ(\mathbb{Q} \times \mathbb{Q})^* = \{(\alpha, 0) \mid \alpha \in \mathbb{Q}^*\} \cup \{(0, \beta) \mid \beta \in \mathbb{Q}^*\}$. It is now clear that $E\Gamma(R)$ and $E\Gamma(T(R))$ are not isomorphic. Observe that $E\Gamma(R)$ is a finite graph, whereas $E\Gamma(T(R))$ is an infinite graph. □

3 Some more results on $E\Gamma(R)$, where R is a reduced ring

Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. The aim of this section is to discuss some results regarding $girth(E\Gamma(R))$, $\omega(E\Gamma(R))$, and $\chi(E\Gamma(R))$. Also, we deduce some properties of $E\Gamma(R)$, where R is a von Neumann regular ring which is not a field.

Let $G = (V, E)$ be a graph such that G contains a cycle. Recall from Balakrishnan and Ranganathan (2000, page 159) that the *girth of* G denoted by $girth(G)$ is defined to be equal to the length of the shortest cycle in G. If a graph G does not contain any cycle, then we define $girth(G) = \infty$.

Let $G = (V, E)$ be a graph. Recall from Balakrishnan and Ranganathan (2000, Definition 1.2.2) that a *clique of* G is a complete subgraph of G. Suppose that there exists $m \in \mathbb{N}$ such that any clique of G is a clique on at most m vertices. Then recall from Balakrishnan and Ranganathan (2000, page 185) that the *clique number of* G denoted by $\omega(G)$ is defined as the largest positive integer $n \geq 1$ such that G contains a clique on n vertices. We set $\omega(G) = \infty$ if G contains a clique on n vertices for all $n \geq 1$.

Let $G = (V, E)$ be a graph. Recall from Balakrishnan and Ranganathan (2000, page 129) that a *vertex coloring of* G is a map $f : V \to S$, where S is a set of distinct colors. A vertex coloring $f : V \to S$ is said to be *proper* if adjacent vertices of G receive distinct colors of S; that is, if a and b are adjacent vertices of G, then $f(a) \neq f(b)$. It is useful to recall from Balakrishnan and Ranganathan (2000, Definition 7.1.2) that the *chromatic number of* G denoted by $\chi(G)$ is the minimum number of colors needed for a proper vertex coloring of G. It is well-known that for any graph G, $\omega(G) \leq \chi(G)$.

Lemma 3.1 *Let R be a ring such that $EZ(R)^* \neq \emptyset$. Then $girth(E\Gamma(R)) \in \{3, 4, \infty\}$.*

Proof If $E\Gamma(R)$ does not contain any cycle, then $girth(E\Gamma(R)) = \infty$. Suppose that $E\Gamma(R)$ contains a cycle. Let $n \geq 3$ be the length of the shortest cycle in $E\Gamma(R)$. Let $x_1 - x_2 - x_3 - \cdots - x_n - x_1$ be a cycle of length n in $E\Gamma(R)$. If $n = 3$, then we obtain that $girth(E\Gamma(R)) = 3$. Suppose that $n \geq 4$. We know from the proof of Proposition 2.3 that x_1 and x_4 are adjacent in $E\Gamma(R)$ and so, we get $x_1 - x_2 - x_3 - x_4 - x_1$, a cycle of length 4 in $E\Gamma(R)$. Hence, $girth(E\Gamma(R)) = 4$. This proves that $girth(E\Gamma(R)) \in \{3, 4, \infty\}$.

One can also apply the following argument for proving this lemma. We know from Theorem 2.15 that if g is any component of $E\Gamma(R)$, then g is either complete or a complete bipartite graph. Therefore, $girth(E\Gamma(R)) \in \{3, 4, \infty\}$. □

Proposition 3.2 *Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. Then $girth(E\Gamma(R)) \in \{4, \infty\}$.*

Proof We know from Lemma 3.1 that for any ring T with $EZ(T)^* \neq \emptyset$, $girth(E\Gamma(T)) \in \{3, 4, \infty\}$. Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. If $a - b$ is any edge of $E\Gamma(R)$, then we know from Lemma 2.11 that for any $c \in R^*$, either $ca \neq 0$ or $cb \neq 0$. Thus if $c \notin \{a, b\}$, then either c and a or c and b are not adjacent in $\Gamma(R)$ and hence either c and a or c and b are not adjacent in $E\Gamma(R)$. This proves that $E\Gamma(R)$ cannot contain any cycle of length 3. Therefore, we obtain that $girth(E\Gamma(R)) \in \{4, \infty\}$.

Indeed, we know from Theorem 2.15 that each component of $E\Gamma(R)$ is a complete bipartite graph. Hence, $girth(g) \in \{4, \infty\}$ for any component g of $E\Gamma(R)$ and so, $girth(E\Gamma(R)) \in \{4, \infty\}$. □

Corollary 3.3 *Let R be a von Neumann regular ring which is not a field. Then* $girth(E\Gamma(R)) \in \{4, \infty\}$.

Proof It is already noted in the proof of Lemma 2.1 that there exists $e \in R\backslash\{0, 1\}$ such that $e = e^2$. As $Ann(e) = R(1 - e)$ and $Ann(1 - e) = Re$, we obtain that $e, 1 - e \in EZ(R)^*$ and so, $EZ(R)^* \neq \emptyset$. Since R is reduced, it follows from Proposition 3.2 that $girth(E\Gamma(R)) \in \{4, \infty\}$. □

Let R be a reduced ring which is not an integral domain. If $\omega(\Gamma(R)) < \infty$, then it follows from Beck (1988, Theorem 3.8) that $Min(R)$ is finite and $\omega(\Gamma(R)) = \chi(\Gamma(R)) = |Min(R)|$.

Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. Using Theorem 2.15, we prove in Corollary 3.4 that $\omega(E\Gamma(R)) = \chi(E\Gamma(R)) = 2$. If R is a von Neumann regular ring which is not a field, then we verify in the in particular part of Corollary 3.4 that $\omega(E\Gamma(R)) = \chi(E\Gamma(R)) = 2$.

Corollary 3.4 *Let R be a reduced ring such that* $EZ(R)^* \neq \emptyset$. *Then* $\omega(E\Gamma(R)) = \chi(E\Gamma(R)) = 2$.

In particular, if R is a von Neumann regular ring which is not a field, then $\omega(E\Gamma(R)) = \chi 9E\Gamma(R)) = 2$..

Proof We know from Theorem 2.15 that each component of $E\Gamma(R)$ is a complete bipartite graph. Therefore, we obtain that $\omega(g) = \chi(g) = 2$ for each component g of $E\Gamma(R)$ and so, $\omega(E\Gamma(R)) = \chi(E\Gamma(R)) = 2$.

We next verify the in particular part of this corollary. It is already noted in the proof of Corollary 3.3 that R is reduced and $EZ(R)^* \neq \emptyset$. Therefore, we obtain that $\omega(E\Gamma(R)) = \chi(E\Gamma(R)) = 2$. □

Recall from Atiyah and Macdonald (1969, Exercise 11, page 11) that a ring R is said to be *Boolean* if $a = a^2$ for each $a \in R$. It is clear that any Boolean ring is von Neumann regular. Let R be a Boolean ring. If R is not a field, then we prove in Proposition 3.5 that $girth(E\Gamma(R)) = \infty$.

Proposition 3.5 *Let R be a Boolean ring which is not a field. Then* $girth(E\Gamma(R)) = \infty$.

Moreover, if g is any component of $E\Gamma(R)$, *then g is a complete graph with two vertices.*

Proof We are indebted to the referee for suggesting the following simple proof of this proposition.

As any element of R is idempotent, it follows that $EZ(R)^* = Z(R)^* = R\backslash\{0, 1\}$. Let $e \in R\backslash\{0, 1\}$. Then $e = e^2$. Let $f = 1 - e$. From $Ann(e) = Rf$ and $Ann(f) = Re$, we get that $e - f$ is an edge of $E\Gamma(R)$. We claim that $f \in EZ(R)^*$ is the only vertex which is adjacent to e in $E\Gamma(R)$. Let $e' \in R\backslash\{0, 1\}$ be such that e' is adjacent to e in $E\Gamma(R)$. Then from $Rf = Ann(e) = Re'$, we obtain that $f = e'$. This shows that the degree of each vertex in $E\Gamma(R)$ equals 1. Therefore, $E\Gamma(R)$ does not contain any cycle and so, $girth(E\Gamma(R)) = \infty$.

We next verify the moreover part of this proposition. Let g be a component of $E\Gamma(R)$. Let $e \in V(g)$. Then $f = 1 - e \in V(g)$ and $e - f$ is an edge of g. As g is connected and the degree of each vertex in $E\Gamma(R)$ equals 1, we obtain that g is a complete graph with two vertices. □

Let $R_n = \mathbb{Z}_2$ for each $n \in \mathbb{N}$. Let $R = \prod_{n=1}^{\infty} R_n$. Then R is a Boolean ring and hence, we obtain from Proposition 3.5 that $girth(E\Gamma(R)) = \infty$. We provide Example 3.6 to illustrate that $girth(E\Gamma(R)) = \infty$ for a von Neumann regular ring R need not imply that R is Boolean.

Example 3.6 Let $R = \mathbb{Z}_3 \times \mathbb{Z}_2$. Then R is von Neumann regular, $girth(E\Gamma(R)) = \infty$ but R is not Boolean.

Proof Since R is a direct product of fields, it is clear that R is von Neumann regular. We know from Corollary 2.22 that $E\Gamma(R)$ is a complete bipartite graph with vertex partition $V(E\Gamma(R)) = \{(\alpha, 0) \mid \alpha \in \{1, 2\}\} \cup \{(0, 1)\}$. Therefore, $E\Gamma(R)$ is a star graph and so, $girth(E\Gamma(R)) = \infty$. Note that $(2, 1)^2 = (1, 1) \neq (2, 1)$. Hence, R is not Boolean. □

Let R be a von Neumann regular ring which is not a field. In Theorem 3.7, we provide some necessary and sufficient conditions for $E\Gamma(R)$ to satisfy the property that $girth(E\Gamma(R)) = 4$.

Theorem 3.7 *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent*:

(1) *girth*$(E\Gamma(R)) = 4$.
(2) *There exist* $u \in U(R)$ *and an idempotent element* e *with* $e \in R\backslash\{0,1\}$ *such that* $u - 1 \notin \{Re, R(1-e)\}$.
(3) *There exists an idempotent element* $e \in R\backslash\{0,1\}$ *such that* $|U(Re)| \geq 2$ *and* $|U(R(1-e))| \geq 2$.

Proof (1) $\Rightarrow$ (2) Let $x_1 - x_2 - x_3 - x_4 - x_1$ be a cycle of length 4 in $E\Gamma(R)$. We know from Lemma 2.6 that for each $i \in \{1,2,3,4\}$, there exist $u_i \in U(R)$ and an idempotent element e_i of R such that $x_i = u_i e_i$. Since $x_i \in Z(R)^*$, it is clear that $e_i \notin \{0,1\}$ for each $i \in \{1,2,3,4\}$. Now, $Re_2 = Rx_2 = Ann(x_1) = Ann(e_1) = R(1-e_1)$ and so, $e_2 = 1 - e_1$. Similarly, it follows from $x_2 - x_3$ is an edge of $E\Gamma(R)$ that $e_3 = e_1$ and from the fact that $x_3 - x_4$ is an edge of $E\Gamma(R)$, we get that $e_4 = 1 - e_1$. From $u_1e_1 = x_1 \neq x_3 = u_3e_1$, we obtain that either $u_1 - 1 \notin R(1-e_1)$ or $u_3 - 1 \notin R(1-e_1)$. From $x_2 = u_2(1-e_1) \neq x_4 = u_4(1-e_1)$, it follows that either $u_2 - 1 \notin Re_1$ or $u_4 - 1 \notin Re_1$. Without loss of generality, we can assume that $u_1 - 1 \notin R(1-e_1)$ and $u_2 - 1 \notin Re_1$. Note that $(u_1e_1 + u_2(1-e_1))(u_2^{-1}(1-e_1) + u_1^{-1}e_1) = e_1 + 1 - e_1 = 1$. Thus $u = u_1e_1 + u_2(1-e_1) \in U(R)$. From $u - 1 = u_1e_1 + u_2(1-e_1) - e_1 - (1-e_1) = e_1(u_1 - 1) + (1-e_1)(u_2 - 1)$ and from the assumption that $u_1 - 1 \notin R(1-e_1)$ and $u_2 - 1 \notin Re_1$, it follows that $u - 1 \notin \{Re_1, R(1-e_1)\}$. With $e = e_1$, it is clear that $e \notin \{0,1\}$, $e = e^2$ and $u \in U(R)$ is such that $u - 1 \notin \{Re, R(1-e)\}$.

(2) $\Rightarrow$ (3) We are assuming that there exist $u \in U(R)$ and an element $e \notin \{0,1\}$ with $e = e^2$ such that $u - 1 \notin \{Re, R(1-e)\}$. Observe that $e \neq ue$ and $e, ue \in U(Re)$; $1 - e \neq u(1-e)$ and $1 - e, u(1-e) \in U(R(1-e))$. This proves that $|U(Re)| \geq 2$ and $|U(R(1-e))| \geq 2$.

(3) $\Rightarrow$ (1) Let $x \in R$ be such that $xe \in U(Re)$ with $xe \neq e$. Hence, there exists $y \in R$ such that $(xe)(ye) = e$. Thus $u_1 = xe + 1 - e$ is such that $u_1(ye + 1 - e) = xye + 1 - e = e + 1 - e = 1$. Hence, $u_1 \in U(R)$ and $xe = u_1e \neq e$. Similarly, from $|U(R(1-e))| \geq 2$, it follows that there exists $u_2 \in U(R)$ such that $u_2(1-e) \neq 1 - e$. Observe that with $f = 1 - e$, $e - f - u_1e - u_2f - e$ is a cycle of length 4 in $E\Gamma(R)$. It now follows from Corollary 3.3 that $girth(E\Gamma(R)) = 4$. □

Corollary 3.8 *Let* $n \geq 2$ *and let* $F_1, F_2, \ldots, F_n$ *be fields. Let* $R = F_1 \times F_2 \times \cdots \times F_n$. *Then* $girth(E\Gamma(R)) \in \{4, \infty\}$ *and* $girth(E\Gamma(R)) = 4$ *if and only if there exist at least two distinct* $i, j \in \{1,2,\ldots,n\}$ *such that* $|F_i| \geq 3$ *and* $|F_j| \geq 3$.

Proof It is already noted in a paragraph which appears just preceding the statement of Lemma 2.8 that $R = F_1 \times F_2 \times \cdots \times F_n$ is von Neumann regular. Therefore, we obtain from Corollary 3.3 that $girth(E\Gamma(R)) \in \{4, \infty\}$.

Suppose that $girth(E\Gamma(R)) = 4$. Then it follows from (1) $\Rightarrow$ (3) of Theorem 3.7 that there exists an idempotent element e of R with $e \notin \{(0,0,\ldots,0),(1,1,\ldots,1)\}$ such that $|U(Re)| \geq 2$ and $|U(R((1,1,\ldots,1) - e))| \geq 2$. As $\{0,1\}$ is the set of all idempotent elements of a field, it follows that $e = (x_1, x_2, \ldots, x_n)$ with $x_k \in \{0,1\}$ for each $k \in \{1,2,\ldots,n\}$ and there exists $t \in \mathbb{N}$ with $t < n$ such that the number of $k \in \{1,2,\ldots,n\}$ with $x_k = 0$, equals $n - t$. Without loss of generality, we can assume that $x_1 = \cdots = x_{n-t} = 0$. Then $Re = (0) \times \cdots \times (0) \times F_{n-t+1} \times \cdots \times F_n$ and $R((1,1,\ldots,1) - e) = F_1 \times \cdots \times F_{n-t} \times (0) \times \cdots \times (0)$. From $|U(Re)| \geq 2$ and $|U(R((1,1,\ldots,1) - e))| \geq 2$, it follows that $|F_i| \geq 3$ for some $i \in \{n-t+1,\ldots,n\}$ and $|F_j| \geq 3$ for some $j \in \{1,\ldots,n-t\}$. This shows that there exist distinct $i, j \in \{1,2,\ldots,n\}$ such that $|F_i| \geq 3$ and $|F_j| \geq 3$.

Conversely, assume that there exist distinct $i, j \in \{1,2,\ldots,n\}$ such that $|F_i| \geq 3$ and $|F_j| \geq 3$. Without loss of generality, we can assume that $i = 1$ and $j = 2$. It is possible to choose $\alpha_1 \in F_1^*\backslash\{1\}$ and $\alpha_2 \in F_2^*\backslash\{1\}$. Let $e_1 = (1,0,\ldots,0)$. Observe that $Re_1 = F_1 \times (0) \times \cdots \times (0)$ and $R((1,1,\ldots,1) - e_1) = (0) \times F_2 \times \cdots \times F_n$. It is clear that $|U(Re_1)| \geq 2$ and $|U(R((1,1,\ldots,1) - e_1))| \geq 2$. Hence, we obtain from (3) $\Rightarrow$ (1) of Theorem 3.7 that $girth(E\Gamma(R)) = 4$. □

Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. We know from Proposition 3.2 that $girth(E\Gamma(R)) \in \{4, \infty\}$. With the help of Lemma 2.14, in Corollary 3.9, we characterize R such that $girth(E\Gamma(R)) = 4$.

Corollary 3.9 *Let* R *be a reduced ring such that* $EZ(R)^* \neq \emptyset$. *Then* $girth(E\Gamma(R)) = 4$ *if and only if there exists an edge* $a - b$ *of* $E\Gamma(R)$ *such that* $|U(\frac{R}{Ra})| > 1$ *and* $|U(\frac{R}{Rb})| > 1$.

Proof Now, R is a reduced ring with $EZ(R)^* \neq \emptyset$. We know from Lemma 2.14 that each component of $E\Gamma(R)$ is a complete bipartite graph. Observe that $girth(E\Gamma(R)) = 4$ if and only if there exists at least one component g of $E\Gamma(R)$ such that g is not a star graph. We know from the proof of Lemma 2.14 that if g is any component of $E\Gamma(R)$, then there exists an edge $a - b$ of g such that $V(g) = V_1 \cup V_2$, where $V_1 =$

$\{x \in V(g) \mid Rx = Ra\}$ and $V_2 = \{y \in V(g) \mid Ry = Rb\}$. Note that a component g of $E\Gamma(R)$ is not a star graph if and only if $|V_i| > 1$ for each $i \in \{1, 2\}$. We know from the proof of moreover part of Lemma 2.14 that $|V_1| > 1$ (respectively, $|V_2| > 1$) if and only if $|U(\frac{R}{Rb})| > 1$ (respectively, $|U(\frac{R}{Ra})| > 1$). From the above given arguments, it is clear that $girth(E\Gamma(R)) = 4$ if and only if there exists an edge $a - b$ of $E\Gamma(R)$ such that $|U(\frac{R}{Ra})| > 1$ and $|U(\frac{R}{Rb})| > 1$. □

Let $G = (V, E)$ be a graph. Recall from Balakrishnan and Ranganathan (2000) that G is said to be *weakly perfect* if $\chi(G) = \omega(G)$. Recall from Balakrishnan and Ranganathan (2000) that G is said to be *perfect* if any induced subgraph of G is weakly perfect. That is, if H is any induced subgraph of G, then $\chi(H) = \omega(H)$. Let $n \in \mathbb{N}$. Let P be a path of G. If P is a path with n vertices, then it is denoted by P_n. It was proved in Endean, Henry, and Manlove (2007, Theorem 1.1) that a graph G is perfect if G does not contain P_4 as an induced subgraph. Let R be a ring such that $EZ(R)^* \neq \emptyset$. We know from Theorem 2.15 that if g is any component of $E\Gamma(R)$, then g is either complete or a complete bipartite graph and so, $\chi(H) = \omega(H)$ for any induced subgraph H of $E\Gamma(R)$. Therefore, $E\Gamma(R)$ is perfect. As a corollary to the proof of Proposition 2.3 and Endean, Henry, and Manlove (2007, Theorem 1.1), we provide in Corollary 3.10, another argument for the fact that $E\Gamma(R)$ is perfect, where R is any ring with $EZ(R)^* \neq \emptyset$.

Corollary 3.10 *Let R be a ring such that $EZ(R)^* \neq \emptyset$. Then $E\Gamma(R)$ is perfect.*

Proof We prove $E\Gamma(R)$ is perfect by showing that $E\Gamma(R)$ does not admit P_4 as an induced subgraph. Let H be any induced subgraph of $E\Gamma(R)$. Suppose that $H = P_4$. Let H be the path given by $x_1 - x_2 - x_3 - x_4$. It follows from the proof of Proposition 2.3 that x_1 and x_4 are adjacent in $E\Gamma(R)$ and so, they are adjacent in H. This is a contradiction. This shows that $E\Gamma(R)$ does not admit P_4 as an induced subgraph and so, we obtain from Endean, Henry, and Manlove (2007, Theorem 1.1) that $E\Gamma(R)$ is perfect. □

4 Some more remarks on $E\Gamma(R)$, where R is a reduced ring

Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. The aim of this section is to discuss some more properties of $E\Gamma(R)$.

Let $G = (V, E)$ be a simple graph. Recall from Anderson, Levy, and Shapiro [2], Levy and Shapiro [19] that two distinct vertices u, v of G are said to be *orthogonal*, written $u \perp v$, if u and v are adjacent in G and there is no vertex w of G which is adjacent to both u and v in G; that is, the edge $u - v$ is not an edge of any triangle in G. A vertex v of G is said to be a *complement of* u if $u \perp v$. Moreover, recall from Anderson, Levy, and Shapiro [2] that G is said to be *complemented* if each vertex of G admits a complement in G. Furthermore, G is said to be *uniquely complemented*, if G is complemented and whenever the vertices u, v, w of G are such that $u \perp v$ and $u \perp w$, then a vertex x of G is adjacent to v in G if and only if x is adjacent to w in G. In Section 3 of Anderson, Levy, and Shapiro [2], D.F. Anderson et al. determined rings T for which the zero-divisor graphs $\Gamma(T)$ are complemented or uniquely complemented.

We deduce Corollaries 4.1 and 4.2 from Theorem 2.15.

Corollary 4.1 *Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. Then $E\Gamma(R)$ is uniquely complemented.*

Proof We know from Theorem 2.15 that g is a complete bipartite graph for each component g of $E\Gamma(R)$. Since any complete bipartite graph is uniquely complemented, it follows that $E\Gamma(R)$ is uniquely complemented. □

Let $G = (V, E)$ be a simple graph. Recall from Rahimi (2013, Definition 2.9) that a vertex a of G is said to be a *Smarandache vertex* or an *S-vertex* if there exist distinct vertices x, y, and b of G such that $a - x, a - b$, and $b - y$ are edges in G but there is no edge between x and y in G. In Rahimi [21], A.M. Rahimi investigated S-vertices of the zero-divisor graph of a commutative ring and the zero-divisor graph of a ring with respect to an ideal.

Corollary 4.2 *Let R be a reduced ring such that $EZ(R)^* \neq \emptyset$. Then no vertex of $E\Gamma(R)$ is an S-vertex.*

Proof Let g be any component of $E\Gamma(R)$. We know from Theorem 2.15 that g is a complete bipartite graph. Since no vertex of a complete bipartite graph is an S-vertex, we obtain that no vertex of $E\Gamma(R)$ is an S-vertex. □

Let $G=(V,E)$ be a graph. Recall from Deo (1994, page 172) that a non-empty subset A of V is a *dominating set in* G if A dominates every vertex v in G in the following sense: Either $v\in A$ or v is adjacent in G to one or more members of A. Recall from Deo (1994, page 172) that a *minimal dominating set in* G is a dominating set in G from which no vertex can be removed without destroying its dominance property. Suppose that a graph $G=(V,E)$ admits a finite dominating set. The *domination number of* G, denoted by $\gamma(G)$ is the number of vertices in the smallest minimal dominating set of G. We define $\gamma(G)=\infty$ if G admits no finite dominating set. A good amount of work was done on the dominating sets of zero-divisor graphs of rings, to mention a few, refer Jaffari Rad, Heidar Jaffari, and Mojdeh [15], Mojdeh and Rahimi [20], Samei [22]. With the help of Proposition 2.23 and Theorem 2.26, we prove Corollary 4.3.

Corollary 4.3 *Let R be a reduced ring which is not an integral domain. Let $n\in\mathbb{N}$ be such that $|Min(R)|=n$. Suppose that each $\mathfrak{p}\in Min(R)$ is principal. Then $2^{n-1}-1\leq\gamma(E\Gamma(R))\leq 2^n-2$.*

Proof Since R is a reduced ring which is not an integral domain, it follows that $n\geq 2$. Let $Min(R)=\{\mathfrak{p}_i \mid i\in\{1,2,\ldots,n\}\}$. We know from (3) $\Rightarrow$ (1) of Proposition 2.23 (in the case $n=2$) and Theorem 2.26 (in the case $n\geq 3$) that $E\Gamma(R)$ has exactly $2^{n-1}-1$ components and each component is a complete bipartite graph. Let $\{g_j \mid j\in\{1,\ldots,2^{n-1}-1\}\}$ denote the set of all components of $E\Gamma(R)$. Observe that if G is any complete bipartite graph, then $\gamma(G)\leq 2$. Observe that $\gamma(E\Gamma(R))=\sum_{j=1}^{2^{n-1}-1}\gamma(g_j)$. As $1\leq\gamma(g_j)\leq 2$ for each $j\in\{1,\ldots,2^{n-1}-1\}$, we obtain that $2^{n-1}-1\leq\gamma(E\Gamma(R))\leq(2^{n-1}-1)2=2^n-2$. □

Corollary 4.4 *Let R be a reduced ring which is not an integral domain. Let $n\in\mathbb{N}$ be such that $|Min(R)|=n$. Suppose that each $\mathfrak{p}\in Min(R)$ is principal. Then the following statements hold*:

(1) $\gamma(E\Gamma(R))=2^{n-1}-1$ *if and only if each component of $E\Gamma(R)$ is a star graph.*
(2) $\gamma(E\Gamma(R))=2^n-2$ *if and only if each component of $E\Gamma(R)$ is not a star graph.*
(3) *Let $k\geq 1$ be the number of components g of $E\Gamma(R)$ such that g is a star graph. Then $\gamma(E\Gamma(R))=2^n-2-k$.*

Proof It is already noted in the proof of Corollary 4.3 that $n\geq 2$ and $\{g_j \mid j\in\{1,\ldots,2^{n-1}-1\}\}$ is the set of all components of $E\Gamma(R)$ and g_j is a complete bipartite graph for each $j\in\{1,\ldots,2^{n-1}-1\}$. Observe that $\gamma(E\Gamma(R))=\sum_{j=1}^{2^{n-1}-1}\gamma(g_j)=\sum_{j=1}^{2^{n-1}-1}t_j$, where $t_j\in\{1,2\}$ for each $j\in\{1,\ldots,2^{n-1}-1\}$. Note that $t_j=1$ if and only if g_j is a star graph.

(1) It is now clear that $\gamma(E\Gamma(R))=2^{n-1}-1$ if and only if g_j is a star graph for each $j\in\{1,\ldots,2^{n-1}-1\}$.

(2) Observe that $\gamma(E\Gamma(R))=2^n-2$ if and only if $t_j=2$ for each $j\in\{1,\ldots,2^{n-1}-1\}$. Therefore, $\gamma(E\Gamma(R))=2^n-2$ if and only if g_j is not a star graph for each $j\in\{1,\ldots,2^{n-1}-1\}$.

(3) It is clear that $\gamma(E\Gamma(R))=k+(2^{n-1}-1-k)2=2^n-2-k$. □

Corollary 4.5 *Let R be a reduced ring which satisfies the hypotheses of Corollary* 4.4. *Let $Min(R)=\{\mathfrak{p}_i=Rp_i \mid i\in\{1,2,\ldots,n\}\}$. Let $\mathcal{C}$ denote the collection of all non-empty proper subsets of $\{1,2,\ldots,n\}$. Let the members of $\mathcal{C}$ be arranged so that $\mathcal{C}=\{A_1,\ldots,A_{2^{n-1}-1},B_1=A_1^c,\ldots,B_{2^{n-1}-1}=A^c_{2^{n-1}-1}\}$, where for each subset $j\in\{1,\ldots,2^{n-1}-1\}$, A_j^c denotes the complement of A_j in $\{1,2,\ldots,n\}$. Then the following statements hold*:

(1) $\gamma(E\Gamma(R))=2^{n-1}-1$ *if and only if for each $j\in\{1,\ldots,2^{n-1}-1\}$ either* $|U(\frac{R}{R(\prod_{k\in A_j}p_k)})|=1$ *or* $|U(\frac{R}{R(\prod_{t\in B_j}p_t)})|=1$.

(2) $\gamma(E\Gamma(R))=2^n-2$ *if and only if for each $j\in\{1,\ldots,2^{n-1}-1\}$,* $|U(\frac{R}{R(\prod_{k\in A_j}p_k)})|>1$ *and* $|U(\frac{R}{R(\prod_{t\in B_j}p_t)})|>1$.

Proof We know from the proof of the moreover part of Proposition 2.21 (in the case $n = 2$) and from the proof of Theorem 2.26 (in the case $n \geq 3$) that $\{g_j \mid j \in \{1, \ldots, 2^{n-1} - 1\}\}$ is the set of all components of $E\Gamma(R)$ and for each $j \in \{1, \ldots, 2^{n-1} - 1\}$, g_j is a complete bipartite graph with vertex partition $V(g_j) = V(g_j)^{(1)} \cup V(g_j)^{(2)}$, where $V(g_j)^{(1)} = \{R(\prod_{k \in A_j} p_k)r \mid r \in U_j\}$ and $V(g_j)^{(2)} = \{R(\prod_{t \in B_j} p_t)s \mid s \in V_j\}$ with $U_j = \{r \in R \mid r + R(\prod_{t \in B_j} p_t) \in U(\frac{R}{R(\prod_{t \in B_j} p_t)})\}$ and $V_j = \{s \in R \mid s + R(\prod_{k \in A_j} p_k) \in U(\frac{R}{R(\prod_{k \in A_j} p_k)})\}$. Let $j \in \{1, \ldots, 2^{n-1} - 1\}$. Observe that g_j is a star graph if and only if either $|V(g_j)^{(1)}| = 1$ or $|V(g_i)^{(2)}| = 1$. It can be shown as in the proof of the moreover part of Lemma 2.14 that $|V(g_j)^{(1)}| = 1$ (respectively, $|V(g_j)^{(2)}| = 1$) if and only if $|U(\frac{R}{R(\prod_{t \in B_j} p_t)})| = 1$ (respectively, $|U(\frac{R}{R(\prod_{k \in A_j} p_k)})| = 1$).

(1) We know from Corollary 4.4(1) that $\gamma(E\Gamma(R)) = 2^{n-1} - 1$ if and only if g_j is a star graph for each $j \in \{1, \ldots, 2^{n-1} - 1\}$. Therefore, we obtain that $\gamma(E\Gamma(R)) = 2^{n-1} - 1$ if and only if for each $j \in \{1, \ldots, 2^{n-1} - 1\}$, either $|U(\frac{R}{R(\prod_{k \in A_j} p_k)})| = 1$ or $|U(\frac{R}{R(\prod_{t \in B_j} p_t)})| = 1$.

(2) We know from Corollary 4.4(2) that $\gamma(E\Gamma(R)) = 2^n - 2$ if and only if g_j is not a star graph for each $j \in \{1, \ldots, 2^{n-1} - 1\}$. Therefore, we obtain that $\gamma(E\Gamma(R)) = 2^n - 2$ if and only if for each $j \in \{1, \ldots, 2^{n-1} - 1\}$, $|U(\frac{R}{R(\prod_{k \in A_j} p_k)})| > 1$ and $|U(\frac{R}{R(\prod_{t \in B_j} p_t)})| > 1$. □

Let R be a von Neumann regular ring which is not a field. We prove in Proposition 4.8 that $\gamma(E\Gamma(R)) < \infty$ if and only if $|Max(R)| < \infty$. We use Lemma 4.6 in the proof of Proposition 4.8.

Lemma 4.6 *Let R be a ring such that $dimR = 0$ and $|Max(R)| \geq 2$. If $E\Gamma(R)$ has only a finite number of components, then $|Max(R)| < \infty$.*

Proof It is already remarked in the paragraph which appears just preceding the statement of Proposition 2.3 that $EZ(R)^* \neq \emptyset$. We are assuming that $E\Gamma(R)$ has only a finite number of components. Let t be the number of components of $E\Gamma(R)$. We claim that $|Max(R)| \leq t + 1$. Suppose that $|Max(R)| \geq t + 2$. Since $dimR = 0$, we obtain from Lemma 2.2 that there exist zero-dimensional rings $R_1, R_2, R_3, \ldots, R_{t+2}$ such that $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_{t+2}$ as rings. Let us denote the ring $R_1 \times R_2 \times R_3 \times \cdots \times R_{t+2}$ by T. From $R \cong T$ as rings, it follows that the graphs $E\Gamma(R)$ and $E\Gamma(T)$ are isomorphic. Hence, the number of components of $E\Gamma(T)$ equals t. We know from the moreover part of Lemma 2.4 that $E\Gamma(T)$ has at least $t + 2$ components. This is in contradiction to the fact that the number of components of $E\Gamma(T)$ equals t. Therefore, $|Max(R)| \leq t + 1 < \infty$. □

We provide Example 4.7 to illustrate that the hypothesis $dimR = 0$ cannot be omitted in Lemma 4.6.

Example 4.7 Let $S = R \times F$ be as in Example 2.32. Then $\gamma(E\Gamma(S)) \in \{1, 2\}$ but $Max(S)$ is infinite.

Proof In the notation of Example 2.32, $R = \frac{T}{I}$, where $T = K[X_1, X_2, X_3]$ and I is the ideal of T given by $I = TX_1X_2 + TX_1X_3$. It is shown in the proof of Example 2.32 that $E\Gamma(S)$ is a complete bipartite graph. Therefore, $\gamma(E\Gamma(S)) \in \{1, 2\}$. Note that $Max(S) = \{\mathfrak{m} \times F \mid \mathfrak{m} \in Max(R)\} \cup \{R \times (0)\}$. Since $Max(R)$ is infinite, we get that $Max(S)$ is infinite. □

Proposition 4.8 *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent:*

(1) $\gamma(E\Gamma(R)) < \infty$.
(2) $|Max(R)| < \infty$.

Proof Now, R is a von Neumann regular ring which is not a field. Hence, $|Max(R)| \geq 2$. Observe that $dimR = 0$ and R is reduced. Thus $Max(R) = Min(R)$.

$(1) \Rightarrow (2)$ We are assuming that $\gamma(E\Gamma(R)) < \infty$. Let $\gamma(E\Gamma(R)) = t$. Since $dimR = 0$, we obtain from Lemma 4.6 that $|Max(R)| \leq t + 1$ and so, $|Max(R)| < \infty$.

$(2) \Rightarrow (1)$ We are assuming that $|Max(R)| < \infty$. Let $|Max(R)| = n$. Since R is von Neumann regular, we obtain from $(3) \Rightarrow (5)$ of Theorem 2.9 (in the case $n = 2$) and Corollary 2.28 (in the case $n \geq 3$) that $E\Gamma(R)$

has exactly $2^{n-1}-1$ components and each component is a complete bipartite graph. Now, it follows from the proof of Corollary 4.3 that $2^{n-1}-1 \leq \gamma(E\Gamma(R)) \leq 2^n-2$. This shows that $\gamma(E\Gamma(R)) < \infty$. □

Let R be a von Neumann regular ring which is not a field. In Theorem 4.9, we characterize R such that at least one component of $E\Gamma(R)$ is complete.

Theorem 4.9 *Let R be a von Neumann regular ring which is not a field. The following statements are equivalent*:

(1) $E\Gamma(R)$ *has at least one component g such that g is complete.*
(2) $|U(R)| = 1$.
(3) *R is a Boolean ring.*
(4) *Each component of* $E\Gamma(R)$ *is complete.*

Proof Now, R is a von Neumann regular ring which is not a field. It is already noted in the proof of Corollary 3.3 that R is reduced and $EZ(R)^* \neq \emptyset$.

$(1) \Rightarrow (2)$ We are assuming that $E\Gamma(R)$ has at least one component g such that g is complete. Hence, we obtain from the moreover part of Lemma 2.14 that $|U(R)| = 1$.

$(2) \Rightarrow (3)$ Let $a \in R$. Since R is von Neumann regular, we obtain from Lemma 2.6 that there exist $u \in U(R)$ and an idempotent element e of R such that $a = ue$. Since $U(R) = \{1\}$, we get that $a = e$. And so, $a = a^2$. Therefore, R is a Boolean ring.

$(3) \Rightarrow (4)$ For any Boolean ring T with $EZ(T)^* \neq \emptyset$, it is already verified in the proof of the moreover part of Proposition 3.5 that each component of $E\Gamma(T)$ is K_2. Therefore, we get that each component of $E\Gamma(R)$ is complete.

$(4) \Rightarrow (1)$ This is clear. □

Let R be a von Neumann regular ring which is not a field and $n \in \mathbb{N}$ such that $|Max(R)| = n$. It is noted in the paragraph which appears just preceding the statement of Theorem 2.9 that there exist fields $F_1, F_2 \ldots, F_n$ such that $R \cong F_1 \times F_2 \times \cdots \times F_n$ as rings. If at least one component of $E\Gamma(R)$ is complete, then we obtain from $(1) \Rightarrow (2)$ of Theorem 4.9 that $|U(R)| = 1$. Hence, $|F_i| = 2$ for each $i \in \{1, 2, \ldots, n\}$.

Acknowledgements We are very much thankful to the referee for many useful and helpful suggestions.

Declaration

Conflict of interest We declare that there is no conflict of interest.

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