Indian J Pure Appl Math (2021) 52:1145–1153
https://doi.org/10.1007/s13226-021-00026-7

ORIGINAL RESEARCH

Skewed converse and Laplacian spectral radius of weighted directed graphs

Debajit Kalita

Received: 17 December 2019 / Accepted: 13 October 2020 / Published online: 14 June 2021

Abstract The notion of skewed converse of a weighted directed graph is introduced in this article. Several characterizations of singularity of the skewed converse of the weighted directed graphs are provided here. This article provides some upper bounds and lower bounds on the Laplacian spectral radius of weighted directed graphs. Characterizations of the weighted directed graphs that achieve these bounds are also supplied here.

Keywords Laplacian matrix · Mixed graph · Skewed converse · Spectral radius · Weighted directed graph

Mathematics Subject Classification 05C50 · 05C05 · 15A18

1 Introduction

All our graphs are simple. The notion of weighted directed graph was first introduced by Bapat et al. in [2]. A *weighted directed graph* is a directed graph with simple underlying undirected graph and edges having complex weights of unit modulus, see [2]. Thus a weighted directed graph can be obtained as follows: Take a simple graph, assign a direction to each of its edges and finally assign a complex number of unit modulus to each directed edges. The resulting graph is a weighted directed graph. Let G be a weighted directed graph on the vertices $1, 2\ldots, n$. We denote the set of vertices and the set of edges of G by $V(G)$ and $E(G)$, respectively. From the definition, it follows that in a weighted directed graph both the edges (i, j) and (j, i) cannot be present simultaneously. We use w_{ij} to denote the weight of an edge (i, j). The *adjacency matrix* (see [2]), $A(G) = [a_{ij}]$ of G is an $n \times n$ matrix with ij-th entry

$$a_{ij} = \begin{cases} w_{ij} & \text{if } (i,j) \in E(G) \\ \overline{w}_{ji} & \text{if } (j,i) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

Sometimes we use '$i \sim j$' to mean that vertices i and j are adjacent. By order and size of G we mean the number of vertices and the number of edges in G, respectively. The *degree* d_i of a vertex i in a weighted directed graph G is the sum of the absolute values of the weights of the edges incident with i. Since weights

Communicated by Ravindra B Bapat, Prof.

D. Kalita: This author acknowledges SERB, DST, Government of India, for providing financial support under the scheme MATRICS, with Grant Number MTR/2018/000352.

D. Kalita (✉)
Department of Mathematical Sciences, Tezpur University, Napaam, Sonitpur, Assam 784028, India
E-mail: kdebajit@tezu.ernet.in

of the edges are of unit modulus, it follows that d_i is the number of edges incident with i. By $D(G)$ we denote the diagonal matrix with d_i, the degree of the i-th vertex, as the i-th diagonal entry. The *Laplacian matrix* (see [2]), denoted by $L(G)$, of G is defined as $L(G) = D(G) - A(G)$. Observe that $L(G)$ coincides with the usual Laplacian matrix and the signless Laplacian (see [3]) of an unweighted undirected graph if weights of all the edges in G are 1 and if weights of all the edges in G are -1, respectively. Moreover, if weights of the edges in G are ± 1, then viewing the edges of weight 1 as oriented and the edges of weight -1 as unoriented we see that $L(G)$ coincides with the Laplacian matrix of a mixed graph (see [1]). Hence the study of the Laplacian eigenvalues of the weighted directed graphs is more generalized than that of the unweighted undirected graphs, mixed graphs and the signless Laplacian matrix.

Recall that the Laplacian matrix of an unweighted undirected graph is always singular. However, the signless Laplacian of an unweighted undirected graph is not always singular. The signless Laplacian of a graph is singular if and only if the graph is bipartite (see [3, Proposition 2.1]). In [2], the authors proved that the Laplacian matrix of a weighted directed graph is not always singular. They provided several characterization of singularity of the Laplacian matrix of weighted directed graphs in [2]. A connected weighted directed graph is called *singular* (resp. *nonsingular*) if its corresponding Laplacian matrix is singular (resp. nonsingular).

Let G be a weighted directed graph of order n. The Laplacian matrix of G is Hermitian and positive semidefinite (see [2]). Thus the Laplacian eigenvalues of G are real and nonnegative. We denote by $\lambda_i(G)$ the i-th smallest Laplacian eigenvalue of G. Hence $0 \leq \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. The largest eigenvalue $\lambda_n(G)$ of $L(G)$ is called the Laplacian *spectral radius* of G. Henceforth, we denote the Laplacian spectral radius of G by $\rho(G)$ instead of $\lambda_n(G)$. By Rayleigh's Theorem (see [5]),

$$\rho(G) = \max_{x \in \mathbb{C}^n, x \neq 0} \frac{x^* L(G) x}{x^* x}. \tag{1.1}$$

Thus for a vector $x \in \mathbb{C}^n$, we have $\rho(G) \geq \dfrac{x^* L(G) x}{x^* x}$ and the equality holds if and only if x is an eigenvector of $L(G)$.

Let G be a weighted directed graph. Suppose that λ is an eigenvalue of $L(G)$ with an eigenvector x. Then $L(G)x = \lambda x$. Thus for a vertex i we have the following equation.

$$[d_i - \lambda] x_i = \sum_{i \sim j} a_{ij} x_j, \text{ where } a_{ij} \text{ is the } ij\text{-th entry of } A(G). \tag{1.2}$$

We call this equation an *eigenvector equation* of x at the vertex i.

The following lemma is essentially contained in [2, Theorem 8].

Lemma 1.1 [2, *Theorem* 8] *Let G be a connected weighted directed graph on vertices $1, \ldots, n$. Then $L(G)$ is singular if and only if there is a diagonal matrix D with diagonal entries of unit modulus such that $D^* L(G) D$ is the usual Laplacian matrix of the underlying unweighted undirected graph.*

In view of Lemma 1.1, a singular weighted directed graph can be viwed as an unweighted undirected graph and vice versa, as far as study of Laplacian eigenvalues are concerned. Upper and lower bounds for the spectral radius of unweighted undirected graphs have been investigated to a great extent in the literature, see for example in [4, 6–9] and references therein.

The article is organized as follows. In Sect. 2, we introduce the notion of skewed converse of weighted directed graphs. In this section, we provide several characterizations of singularity of the skewed converse of weighted directed graphs. In Sect. 3, we supply some bounds on the spectral radius for weighted directed graphs. We also characterize the weighted directed graphs which achieve these bounds. The results clearly extends some known results for unweighted undirected graphs, to the class of all weighted directed graphs.

2 Skewed converse of weighted directed graph

In this section we introduce the notion of skewed converse of weighted directed graphs. We provide chracterizations of singularity of the skewed converse of the weighted directed graphs in this section.

Let G be a weighted directed graph. The skewed converse of G is the weighted directed graph $\overleftarrow{G}$ obtained from G as follows:

(a) Reverse the orientation of every edge of G, and

(b) If an edge (u, v) of G has the weight w, then assign the weight $-\mathsf{w}$ to the edge (v, u) of $\overleftarrow{G}$.

Thus if G contains an edge (u, v) of weight w, then the skewed converse of G contains an edge (v, u) of weight $-\mathsf{w}$. We use w_C and $\mathsf{w}_{\overleftarrow{C}}$ to denote the weight of a cycle C in G and $\overleftarrow{G}$, respectively.

Definition 2.1 (see [2]) A i_1-i_k-*walk* W in a weighted directed graph G is a finite sequence $i_1, \ldots, i_k$ of vertices such that, for $1 \le p \le k-1$, either $i_p \sim i_{p+1}$. We call $\mathsf{w}_W = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k}$, the *weight* of the walk W, where a_{ij} are the entries of $A(G)$.

Lemma 2.2 *Let G be a weighted directed graph and let C be a cycle of length k in G. Then $\mathsf{w}_{\overleftarrow{C}} = (-1)^k \overline{\mathsf{w}}_C$.*

Proof Let $A(G) = [a_{ij}]$ and $A(\overleftarrow{G}) = [h_{ij}]$. Observe that $\overleftarrow{G}$ contains an edge (v, u) of weight $-\mathsf{w}$ whenever G contains an edge (u, v) of weight w. Thus $h_{ij} = -\overline{a}_{ij}$ for every pair of distinct vertices i, j. Let $C = [v_1, v_2, \ldots, v_k, v_1]$. Thus

$$\begin{aligned}\mathsf{w}_{\overleftarrow{C}} &= h_{v_1 v_2} h_{v_2 v_3} \cdots h_{v_k v_1} \\ &= (-\overline{a}_{v_1 v_2})(-\overline{a}_{v_2 v_3}) \cdots (-\overline{a}_{v_k v_1}) \\ &= (-1)^k \overline{a}_{v_1 v_2} \overline{a}_{v_2 v_3} \cdots \overline{a}_{v_k v_1} \\ &= (-1)^k \overline{\mathsf{w}}_C.\end{aligned}$$

□

The following lemmas provide characterizations of singularity of weighted directed graphs which are cruical for further development.

Lemma 2.3 [2, *Lemma* 1] *Let G be a connected weighted directed graph on vertices $1\ldots, n$. Then $L(G)$ is singular if and only if the weight of any 1-i-walk is the same. Furthermore, when $L(G)$ is singular, 0 is a simple eigenvalue with an eigenvector $\mathbf{x}$, where $\mathbf{x}_1 = 1$ and $\mathbf{x}_i =$ weight of the i- 1 walk in G for $i = 2, \ldots, n$.*

Definition 2.4 [2] A cycle C in a weighted directed graph is said to be non-singular if its weight $\mathsf{w}_C \neq 1$.

Lemma 2.5 [2, *Corollary* 9] *Let G be a connected weighted directed graph. Then G is non-singular if and only if it contains a nonsingular cycle. In particular, a weighted directed tree is always singular.*

The following result is one of our main result of this section which provides a characterization of singularity of the skewed converse of singular weighted directed graphs.

Theorem 2.6 *Let G be a weighted directed graph. Suppose that G is singular. Then $\overleftarrow{G}$ is singular if and only if the underlying graph of G is bipartite.*

Proof First suppose that $\overleftarrow{G}$ is singular. Let C be a cycle of length k in the underlying graph of G. By Lemma2.2, $\mathsf{w}_{\overleftarrow{C}} = (-1)^k \overline{\mathsf{w}}_C$. Thus $\mathsf{w}_C \mathsf{w}_{\overleftarrow{C}} = (-1)^k$. Since both G and $\overleftarrow{G}$ are singular, it follows that $\mathsf{w}_{\overleftarrow{C}} = \mathsf{w}_C = 1$, by Lemma 2.5. Thus $1 = (-1)^k$, which implies that k is even. Hence length of every cycle in the underlying graph of G is even, that is, the underlying graph of G is bipartite.

Conversely, suppose that the underlying graph of G is bipartite. Thus length of every cycle in the underlying graph of G is even. Let C be a cycle of length k, where k is even. Since G is singular, $\mathsf{w}_C = 1$, by Lemma 2.5. Using Lemma 2.2 we have $\mathsf{w}_{\overleftarrow{C}} = (-1)^k \overline{\mathsf{w}}_C$. Thus $\mathsf{w}_{\overleftarrow{C}} = 1$. Hence weight of any cycle in $\overleftarrow{G}$ is 1. By Lemma 2.5, $\overleftarrow{G}$ is singular. □

The following result is one of our main result, which provides a characterization of singularity of the skewed converse of non-singular weighted directed graphs.

Theorem 2.7 *Let G be a weighted directed graph. Suppose that G is nonsingular. Then $\overleftarrow{G}$ is singular if and only if the weight of all the cycles in G is real, length of each nonsingular cycle in G is odd and the length of each singular cycle in G is even.*

Proof First suppose that $\overleftarrow{G}$ is singular. Let C be a cycle of length k in G. By Lemma 2.2, $\mathsf{w}_C \mathsf{w}_{\overleftarrow{C}} = (-1)^k$. By Lemma 2.5, $\mathsf{w}_{\overleftarrow{C}} = 1$. So $\mathsf{w}_C = (-1)^k$. Thus $\mathsf{w}_C = 1$ if and only if k is even, that is, the cycle C is singular if and only if k is even. Hence the weight of any cycle in G is real, more precisely it is ± 1, the length of each nonsingular cycle in G is odd and the length of each singular cycle in G is even.

Conversely, suppose that the weight of all the cycles in G is real, length of each nonsingular cycle in G is odd and the length of each singular cycle in G is even. Let C be any cycle of length k in G. Using Lemma 2.2, we see that $\mathsf{w}_C \mathsf{w}_{\overleftarrow{C}} = (-1)^k$. Since w_C is real, we see that $\mathsf{w}_C = \pm 1$. If $\mathsf{w}_C = 1$, then C is singular in G, by Lemma2.5. So length of C, that is, k is even. Thus in this case $\mathsf{w}_{\overleftarrow{C}} = 1$. On the otherhand, if $\mathsf{w}_C = -1$, then C is nonsingular in G, by Lemma 2.5. So length of C, that is, k is odd. Thus in this case also $\mathsf{w}_{\overleftarrow{C}} = 1$. Hence weight of any cycle in $\overleftarrow{G}$ is 1. By Lemma 2.5, $\overleftarrow{G}$ singular. □

3 Bounds on Laplacian spectral radius

In this section, we supply some upper bounds and a lower bound on the Laplacian spectral radius of weighted directed graphs. We also characterize the weighted directed graphs which achieve these bounds.

Lemma 3.1 [5, *Corollary* 4.3.3] *Let A, B be two $n \times n$ Hermitian matrices. Assume that B is positive semidefinite and that the eigenvalues of A and $A+B$ are in increasing order. Then $\lambda_k(A) \le \lambda_k(A+B)$ for $k = 1, 2, \ldots, n$.*

Lemma 3.2 *Let G be a weighted directed graph of order n. Suppose that $\widehat{G} = G - e$, where $e = (i,j)$. Then $\rho(\widehat{G}) \le \rho(G)$.*

Proof Observe that $L(G) = L(\widehat{G}) + (e_i - \overline{\mathsf{w}} e_j)(e_i - \overline{\mathsf{w}} e_j)^*$, where e_i and e_j are the i-th and j-th standard unit basis vector of $\mathbb{R}^n$. Note that $(e_i - \overline{\mathsf{w}} e_j)(e_i - \overline{\mathsf{w}} e_j)^*$ is positive semidefinite. Thus $\lambda_k(\widehat{G}) \le \lambda_k(G)$ for $k = 1, 2, \ldots, n$, by Lemma 3.1 Hence the lemma holds. □

As a consequence of Lemma 3.2, we obtain the following lemma.

Lemma 3.3 *Let G be a weighted directed graph of order n. Let H be a subgraph of G. Then $\rho(H) \le \rho(G)$.*

Proof The proof follows from repeated applications of Lemma 3.2. □

Lemma 3.4 [5, *Geršgorin*] *Let $B = [b_{ij}]$ be $n \times n$ matrix and let $R_i = \sum_{\substack{j=1 \\ i \neq j}}^{n} |b_{ij}|$ for $i = 1, \ldots, n$. Then all the eigenvalues of B lies in $\bigcup_{i=1}^{n} \{|z - a_{ii}| \le R_i\}$.*

Let G be a weighted directed graph. Let $A(G) = (a_{ij})$ and $A(\overleftarrow{G}) = (b_{ij})$. Observe that $b_{ij} = -\overline{a}_{ij}$. By equation 1.2, it follows that, if λ is an eigenvalue of $L(\overleftarrow{G})$ with an eigenvector x, then the *eigenvector equation* at a vertex i of $L(\overleftarrow{G})$ is

$$[d_i - \lambda] x_i = \sum_{i \sim j} (-\overline{a}_{ij}) x_j \text{ or } \lambda x_i = d_i x_i + \sum_{i \sim j} \overline{a}_{ij} x_j \tag{3.1}$$

The following theorem provides an upper bound for laplacian sepctral radius of weighted directed graphs. Moreover, it characterize the weighted directed graphs which achieve the upper bound.

Theorem 3.5 *Let G be a weighted directed graph of order n and let Δ be the maximum degree of G. Then $\rho(G) \le 2\Delta$. Furthermore, $\rho(G) = 2\Delta$ if and only if G is regular and the skewed converse of G is singular.*

Proof Observe that the degree d_i of a vertex i in G is the sum of the absolute values of the off-diagonal entries of $L(G)$. Thus by Lemma 3.4, we see that there exists a vertex u of G such that $|\rho(G) - d_u| \le d_u$. This implies that $\rho(G) \le 2\Delta$. Hence we have $\Delta + 1 \le \rho(G) \le 2\Delta$.

Suppose that G is regular and $\overleftarrow{G}$ is singular. Then G is Δ-regular. Let $V(G) = \{1, 2, \ldots, n\}$. Let x be the vector defined as $x_1 = 1$ and $x_i =$ weight of the $i-1$ walk in $\overleftarrow{G}$ for $i = 2, \ldots, n$. By Lemma 2.3, x is a null vector of $\overleftarrow{G}$.

$$(\Delta - 2\Delta)x_u = -\Delta x_u = \sum_{u\sim j} -x_u = \sum_{u\sim j}(-a_{uj}\overline{a}_{uj})x_u$$
$$= \sum_{u\sim j}(-a_{uj}a_{ju})x_u$$
$$= \sum_{u\sim j} -a_{uj}x_j.$$

Thus $L(G)x = 2\Delta x$, that is, 2Δ is an eigenvalue of $L(G)$. Hence $\rho(G) = 2\Delta$.

Conversely, assume that $\rho(G) = 2\Delta$. Let x be an eigenvector corresponding to the eigenvalue 2Δ of $L(G)$. Let $|x_w| = \max\{|x_u| : u \in V(G)\}$. Then $d_w|x_w| \leq |2\Delta - d_w||x_w|$. By eigenvector equation 1.2 at w, we have $(d_w - 2\Delta)x_w = \sum_{w\sim j} a_{wj}x_j$. Thus

$$d_w|x_w| \leq |2\Delta - d_w||x_w| = \left|\sum_{w\sim j} a_{wj}x_j\right| \leq \sum_{w\sim j}|a_{wj}x_j| = \sum_{w\sim j}|x_j| \leq d_w|x_w|.$$

This implies that $d_w|x_w| = \sum_{w\sim j}|x_j|$. Observe that $d_w|x_w| = \sum_{w\sim j}|x_w|$. So $\sum_{w\sim j}|x_w| = \sum_{w\sim j}|x_j|$, which implies that $\sum_{w\sim j}|x_w| - |x_j| = 0$. It follows that $|x_w| = |x_j|$ for each vertex j adjacent to w. Since G is connected, we see that $|x_i| = |x_j|$ for all $i, j \in V(G)$. Without loss of generality, assume that $|x_i| = 1$, for all $i \in V(G)$.

By the eigenvector equation (see eq.1.2) of x at i for the eigenvalue 2Δ of $L(G)$, we see that

$$2\Delta x_i = d_i x_i + \sum_{i\sim j}(-a_{ij})x_j. \tag{3.2}$$

Thus

$$2\Delta = |2\Delta x_i| = \left|d_i x_i + \sum_{i\sim j}(-a_{ij})x_j\right| \leq |d_i x_i| + \sum_{i\sim j}\left|(-a_{ij})x_j\right| = 2d_i \leq 2\Delta. \tag{3.3}$$

Hence $d_i = \Delta$ for all i, that is, G is regular.

Now from equation 3.2, we have $\Delta x_i = \sum_{i\sim j}(-a_{ij})x_j$ for all $i \in V(G)$. Thus $\Delta\overline{x}_i = \sum_{i\sim j}(-\overline{a}_{ij})\overline{x}_j$ for all $i \in V(G)$, that is, $\Delta\overline{x}_i + \sum_{i\sim j}(\overline{a}_{ij})\overline{x}_j = 0.\overline{x}_i$ for all $i \in V(\overleftarrow{G})$. Thus $\overline{x}$ is an eigenvector of $L(\overleftarrow{G})$ corresponding eigenvalue 0. Hence $\overleftarrow{G}$ is singular. □

In view of Lemma 1.1, we see that singular weighted directed graphs are nothing but the unweighted undirected graphs. Hence we have the following lemma due to Merris [7], for singular weighted directed graphs.

Lemma 3.6 [7] *Let G be a singular connected weighted graph on n vertices with at least one edge. Then $\Delta + 1 \leq \rho(G)$, where Δ is the maximum degree of G. The equality holds if and only if $\Delta = n - 1$.*

The following lemma says that the equality in the bound given in Lemma 3.6 need no hold for a weighted directed graph with $\Delta = n - 1$.

Lemma 3.7 *Consider the star graph $K_{1,n-1}$ with all its edge weight 1. Let H be the weighted directed graph obtained from $K_{1,n-1}$ by joining two of its pendant vertices with a directed edge of weight $\mathsf{w} \neq 1$. Then $\rho(H) > n$.*

Proof Assume that the vertices of $K_{1,n-1}$ are $1, 2\ldots, n$ such that degree of 1 is $n - 1$. Without loss of generality let $H = K_{1,n-1} + (2, 3)$, where weight of the edge (2, 3) is $\mathsf{w} \neq 1$. Then

$$L(H) = L(K_{1,n-1}) + \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -w & 0 & \cdots & 0 \\ 0 & -\overline{w} & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that $x = \begin{bmatrix} 1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots \frac{-1}{n-1} \end{bmatrix}^t$ is an eigenvector of $L(K_{1,n-1})$ corresponding to the eigenvalue n. Thus $\frac{x^*L(K_{1,n-1})x}{x^*x} = n$. We see that

$$\frac{x^*L(H)x}{x^*x} = n + \frac{2(1 - \operatorname{Re} \mathsf{w})}{n(n-1)}.$$

Since $\mathsf{w} \neq 1$ and $|\mathsf{w}| = 1$, we see that $1 - \operatorname{Re}(\mathsf{w}) > 0$. Thus $\frac{x^*L(H)x}{x^*x} > n$. Hence

$$\rho(H) = \max_{x \in \mathbb{C}^n, x \neq 0} \frac{x^*L(H)x}{x^*x} > n.$$

□

Remark 3.8 Let G be a non-singular unicyclic weighted directed graph. Let C be the non-singular cycle in G. Suppos that $T = G - e$, where $e = (u, v)$ is an edge of C. Clearly, T is a weighted directed tree. By Lemma 2.5, T is singular. By Lemma 1.1, there exists a diagonal matrix D with diagonal entries of unit modulus such that $D^*L(T)D$ is the usual Laplacian matrix of the underlying unweighted undirected graph of T, that is $D^*L(T)D$ is the Laplacian matrix of weighted directed graph with all edges of weight 1 whose underlying graph is same as that of T. Thus $D^*L(G)D$ is the Laplacian matrix of the non-singular unicyclic weighted directed graph, denoted by ${}^DG$ obtained as follows:

(a) Take the weighted directed tree ${}^DT$ with all edges of weight 1 whose underlying graph is same as that of T,
(b) Add the edge $e = (u, v)$ in ${}^DT$ and assign it the weight $\mathsf{w}_C \neq 1$.

Hence the Laplacian eigenvalues of G and ${}^DG$ are same.

The following theorem provides a lower bound for Laplacian sepctral radius of weighted directed graphs. Moreover, it characterizes the weighted directed graphs which achieve the lower bound.

Theorem 3.9 *Let G be a weighted directed graph of order n and let Δ be the maximum degree of G. Then $\Delta + 1 \leq \rho(G)$. Furthermore, $\rho(G) = \Delta + 1$ if and only if $\Delta = n - 1$ and G is singular.*

Proof Let $A(G) = [a_{ij}]$. Observe that the underlying graph of G contains a subgraph $H = K_{1,\Delta}$. Without loss of generality assume that vertices of H are $1, 2, \ldots, \Delta, \Delta + 1$ such that degree of the vertex 1 is Δ. Let $\hat{\mathbf{x}} \in \mathbb{C}^{\Delta+1}$ be the vector defined as $x_1 = \Delta$ and for $i = 2, \ldots \Delta + 1$,

$$x_i = \begin{cases} -\mathsf{w}_{i1} & \text{if } (i, 1) \in E(H) \\ -\overline{\mathsf{w}}_{1i} & \text{if } (1, i) \in E(H) \end{cases}$$

Observe that $x_i = -\overline{a}_{1i}$ for $i = 2, \ldots, \Delta + 1$. Thus

$$(\Delta - (\Delta + 1))x_1 = -\Delta = \sum_{i=2}^{\Delta+1} a_{1i}x_i.$$

and for $i = 2, \ldots, \Delta + 1$

$$(1 - (\Delta + 1))x_i = -\Delta x_i = \Delta \overline{a}_{1i} = \Delta a_{i1} = a_{i1}x_1.$$

It follows that $\Delta + 1$ is a Laplacian eigenvalue of $K_{1,\Delta}$. Thus $\Delta + 1 \leq \rho(K_{1,\Delta})$. By Lemma 3.3, $\rho(K_{1,\Delta}) \leq \rho(G)$. Hence $\Delta + 1 \leq \rho(G)$.

Suppose that $\rho(G) = \Delta + 1$. To the contrary, assume that $\Delta < n - 1$. Then G contains a tree T obtained from $K_{1,\Delta}$ by inserting a new vertex and making it adjacent to a pendant vertex of $K_{1,\Delta}$. Note that order of T

is $\Delta + 2$ and maximum degree of T is Δ. Thus $\Delta + 1 < \rho(T)$, by Lemma 3.6. By Lemma 3.3, $\Delta + 1 < \rho(T) \leq \rho(G)$, a contradiction. Hence $\Delta = n - 1$.

Next, to the contrary assume that G is non-singular. Then G contains a non-singular cycle, by Lemma2.5. Since $\Delta = n - 1$ and G has a non-singular cycle, we see that G contains a spanning subgraph H obtained from the weighted directed star garph $K_{1,n-1}$ by joining two of its pendant vertices with a directed edge. By Remark 3.8, ${}^{D}H$ is obtained from $K_{1,n-1}$ with all edges of weight 1 by joining two of its pendant vertices with a directed edge of weight $\mathsf{w} \neq 1$. Note that $\rho(H) = \rho({}^{D}H)$. By Lemma 3.7, we have $\Delta + 1 = n < \rho({}^{D}H)$. By Lemma 3.3, $\rho(H) \leq \rho(G)$. Thus $\Delta + 1 < \rho(G)$, a contradiction. Hence G is singular.

Conversely, assume that $\Delta = n - 1$ and G is singular. By Lemma 3.6, $\rho(G) = \Delta + 1$. □

Combining Theorem 3.5 and Theorem 3.9, we have the following result providing sharp bounds on the Laplacian spectral radius of weighted directed graphs.

Theorem 3.10 *Let G be a weighted directed graph of order n and let Δ be the maximum degree of G. Then*

$$\Delta + 1 \leq \rho(G) \leq 2\Delta.$$

The equality in the left holds if and only if $\Delta = n - 1$ and G is singular. Furthermore, the equality in the right holds if and only if G is regular and the skewed converse of G is singular.

The following result provides an upper bound for the class of weighted directed graphs involving the number of vertices, the number of edges, the minimum and maximum degree. This result extends the bound obtained in [6] to the class of all weighted directed graphs. Moreover, it characterize the weighted directed graphs which achieve the bound.

Theorem 3.11 *Let G be a weighted directed graph of order n and size m. Then*

$$\rho(G) \leq \sqrt{2\Delta^2 + 4m - \delta(n-1) + \Delta(\delta - 1)},$$

where Δ and δ denote the largest and smallest degree of G, respectively. Furthermore, the equality holds if and only if G is regular and the skewed converse of G is singular.

Proof Let x be an eigenvector of $L(G)$ corresponding to $\rho(G)$ with $x^*x = 1$. We have

$$\rho(G)x_u = d_u x_u + \sum_{u \sim v}(-a_{uv})x_v = \sum_{u \sim v}(x_u - a_{uv}x_v).$$

So

$$\rho(G)|x_u| = |\rho(G)x_u| \leq \sum_{u \sim v}(|x_u| + |a_{uv}x_v|) = \sum_{u \sim v}(|x_u| + |x_v|)$$

Thus

$$\rho(G)^2|x_u|^2 \leq \left(\sum_{u \sim v}(|x_u| + |x_v|)\right)^2 \tag{3.4}$$

$$\leq d_u \sum_{u \sim v}(|x_u| + |x_v|)^2 (\text{using Cauchy-Schwarz inequality}) \tag{3.5}$$

$$= d_u^2|x_u|^2 + 2d_u|x_u| \sum_{u \sim v}|x_v| + d_u \sum_{u \sim v}|x_v|^2. \tag{3.6}$$

Observe that

$$2|x_u| \sum_{u \sim v}|x_v| \leq \sum_{u \sim v}(|x_u|^2 + |x_v|^2) = d_u|x_u|^2 + \sum_{u \sim v}|x_v|^2. \tag{3.7}$$

Using equation 3.7 in equation 3.6, we have

$$\rho(G)^2|x_u|^2 \leq d_u^2|x_u|^2 + 2d_u|x_u| \sum_{u \sim v}|x_v| + d_u \sum_{u \sim v}|x_v|^2 \leq 2d_u^2|x_u|^2 + 2d_u \sum_{u \sim v}|x_v|^2. \tag{3.8}$$

Now

$$
\begin{aligned}
\sum_{u\in V} d_u \sum_{v\sim u} |x_v|^2 &= \sum_{u\in V} d_u \left(1 - \sum_{v\nsim u} |x_v|^2\right)\\
&= 2m - \left\{ \sum_{u\in V} d_u |x_u|^2 + \sum_{v\in V} \left(\sum_{u\sim v, v\neq u} d_u \right) |x_v|^2 \right\}\\
&\le 2m - \left\{ \sum_{u\in V} d_u |x_u|^2 + \sum_{v\in V} \delta(n - d_v - 1)|x_v|^2 \right\}\\
&= 2m - \delta(n-1) + (\delta - 1)\sum_{u\in V} d_u |x_u|^2\\
&\le 2m - \delta(n-1) + \Delta(\delta - 1)
\end{aligned}
$$

Now equation 3.8 implies that

$$
\begin{aligned}
\rho(G)^2 \sum_{u\in V} |x_u|^2 &\le 2\sum_{u\in V} d_u^2 |x_u|^2 + 2\sum_{u\in V} d_u \sum_{v\sim u} |x_v|^2\\
&\le 2\Delta^2 \sum_{u\in V} |x_u|^2 + 2\sum_{u\in V} d_u \sum_{v\sim u} |x_v|^2\\
&\le 2\Delta^2 \sum_{u\in V} |x_u|^2 + 4m - 2\delta(n-1) + 2\Delta(\delta - 1)\\
&= 2\Delta^2 + 4m - 2\delta(n-1) + 2\Delta(\delta - 1)
\end{aligned}
\tag{3.9}
$$

Since $\sum_{u\in V} |x_u|^2 = 1$, it follows that

$$
\rho(G) \le \sqrt{2\Delta^2 + 4m - 2\delta(n-1) + 2\Delta(\delta - 1)} \tag{3.10}
$$

Assume that the equality holds, that is,

$$
\rho(G) = \sqrt{2\Delta^2 + 4m - 2\delta(n-1) + 2\Delta(\delta - 1)}.
$$

Then inequalities 3.9 must be equalities, which implies that the inequalities 3.8 and 3.4 are equalities. Thus $d_u^2|x_u|^2 + 2d_u|x_u| \sum_{u\sim v} |x_v| + d_u \sum_{u\sim v} |x_v|^2 = 2d_u^2|x_u|^2 + 2d_u \sum_{u\sim v} |x_v|^2$, that is, $d_u^2|x_u|^2 - 2d_u|x_u| \sum_{u\sim v} |x_v| + d_u \sum_{u\sim v} |x_v|^2 = 0$ for any $u \in V(G)$. This implies that $d_u \sum_{u\sim v} |x_u|^2 - 2d_u|x_u| \sum_{u\sim v} |x_v| + d_u \sum_{u\sim v} |x_v|^2 = 0$, that is,

$$
d_u \sum_{u\sim v} (|x_u|^2 - 2|x_u||x_v| + |x_v|^2) = d_u \sum_{u\sim v} (|x_u| - |x_v|)^2 = 0 \text{ for any } u \in V(G).
$$

Thus $|x_u| = |x_v|$ for any vertex v adjacent to u in G. Since G is connected, we see that $|x_u| = |x_v|$ for all $u, v \in V(G)$. So $|x_i| \neq 0$ for all $i \in V(G)$. Let i be a vertex of degree Δ, that is, $d_i = \Delta$. Then the equality in inequality 3.4 implies that $\rho(G)^2|x_i|^2 = 4d_i^2|x_i|^2 = 4\Delta^2|x_i|^2$. Thus $\rho(G) = 2\Delta$. Hence G is regular and $\overleftarrow{G}$ is singular, by Theorem 3.10.

Conversely assume that G is regular and $\overleftarrow{G}$ is singular. Then $\rho(G) = 2\Delta$, by Theorem 3.10. Since G is regular, we see that $\delta = \Delta$. Thus

$$
\rho(G) = 2\Delta = \sqrt{2\Delta^2 + 4m - 2\delta(n-1) + 2\Delta(\delta - 1)}.
$$

Hence the equality holds. □

Acknowledgements The author sincerely thank the referees for carefully reading the manuscript and their suggestions.

References

1. R. B. Bapat, J. W. Grossman, and D. M. Kulkarni. Generalized matrix tree theorem for mixed graphs. *Linear and Multilinear Algebra*, 46:299–312, 1999.
2. R. B. Bapat, D. Kalita and S. Pati. On weighted directed graphs. *Linear Algebra and its Applications*, 436:99–111, 2012.
3. D. Cvetkovic, P. Rowlinson and S. K. Simic, *Signless Laplacian of Finite Graphs*, Linear Algebra and its Applications, 423 (2007), 155-171.
4. R. Grone and R. Merris. The Laplacian spectrum of a graph II. *Siam Journal of Discrete Mathematics*, 7:221–229, 1994.
5. R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.
6. J. S. Li and Y. L. Pan. De Caen's inequality and bounds on the largest Laplacian eigenvalue of a graph, *Linear Algebra and its Applications*, 328:153–160 (2001).
7. R. Merris. Laplacian Matrices of graphs: A survey. Linear Algebra and its Applications, 197,198:143–176, 1994.
8. R. Merris. A note on Laplacian Graph eigenvalues. Linear Algebra and its Applications, 285:33–35, 1998.
9. X. D. Zhang and R. Luo. The Laplacian eigenvalues of mixed graphs *Linear Algebra and its Applications*, 362:109–119 (2003).