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ORIGINAL RESEARCH

Some notes on the critical Hardy inequalities

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Abstract In this paper, we study the critical Hardy inequalities with Bessel pairs with exact remainder terms and their applications. We then establish the relations between the critical Hardy inequalities and the subcritical Hardy inequalities on higher dimensional spaces. We also set up some Hardy identities that can be used to deduce the critical Hardy equality via a limiting process.

Keywords Critical Hardy inequalities · Bessel pair · Remainder term · Stability

Mathematics Subject Classification 26D10 · 46E35 · 35A23

1 Introduction

The main subject of this article is the well-known Hardy inequality that plays key roles in various fields of mathematics such as spectral theory, partial differential equations, geometric estimates, etc (see, e.g., [3, 4, 10, 11, 19, 20]):

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx,\ u \in C_0^\infty\left(\mathbb{R}^N\right). \tag{1}$$

Both the Hardy type inequalities and their higher order versions have received tremendous attention in the literature. There is a vast literature, especially, in the context of the Hardy type inequalities. Hence it is impossible to provide a complete list of references on the subject in our short note. Therefore, we just refer the interested reader, for instance, to [12, 29, 37] for historical backgrounds and to the monographs [5, 24, 28, 30, 38, 39, 42] for the applications as well as the improvements and variants of the Hardy type inequalities and Rellich type inequalities.

It is well-known that the constant $\left(\frac{N-2}{2}\right)^2$ is sharp but is never achieved. Hence, many efforts have been devoted to improve the aforementioned Hardy inequalities. The interested reader is referred to [1, 2, 6, 7, 13, 14, 16–18, 22, 23, 36, 44], to name just a few. We just state here a recent result in [15] that can give a simple interpretation as well as the remainder and the nonexistence of optimizers of the Hardy inequality:

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Theorem A *Let $0<R\le\infty$, V and W be positive C^1-functions on $(0,R)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)}dr=\infty$ and $\int_0^R r^{N-1}V(r)dr<\infty$. Then if $(r^{N-1}V, r^{N-1}W)$ is a $(1-$dimensional$)$ Bessel pair on $(0,R)$, that is, if the ordinary differential equation*

$$y''(r)+\left(\frac{N-1}{r}+\frac{V_r(r)}{V(r)}\right)y'(r)+\frac{W(r)}{V(r)}y(r)=0$$

has a positive solution $\varphi_{V,W;R}$ on the interval $(0,R)$, then we have

$$\int_{B_R^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx=\int_{B_R^N} W(|x|)|u|^2dx+\int_{B_R^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2\varphi_{V,W;R}^2dx$$

$$\int_{B_R^N} V(|x|)|\nabla u|^2dx=\int_{B_R^N} W(|x|)|u|^2dx+\int_{B_R^N} V(|x|)\left|\nabla\left(\frac{u}{\varphi_{V,W;R}}\right)\right|^2\varphi_{V,W;R}^2dx$$

for all $u\in C_0^\infty\left(B_R^N\right)$.

Here B_R^N is the ball with radius R centered at the origin in $\mathbb{R}^N$.

Applying Theorem A to $V=1, W=\left(\frac{N-2}{2}\right)^2\frac{1}{r^2}$ and $\varphi_{V,W;\infty}=r^{-\frac{N-2}{2}}$, we recover the equalities that could be found in [8, 35]:

$$\int_{\mathbb{R}^N}|\nabla u|^2dx=\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}dx+\int_{\mathbb{R}^N}\left|\nabla\left(|x|^{\frac{N-2}{2}}u\right)\right|^2\left|\frac{1}{|x|^{\frac{N-2}{2}}}\right|^2dx$$

$$\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx=\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}dx+\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{N-2}{2}}u\right)\right|^2\left|\frac{1}{|x|^{\frac{N-2}{2}}}\right|^2dx.$$

As a consequence, we can say that $c|x|^{-\frac{N-2}{2}}$, $c\in\mathbb{C}$, are the "virtual" optimizers for (1) while $|x|^{-\frac{N-2}{2}}\phi\left(\frac{x}{|x|}\right)$, $\phi:\mathbb{S}^N\to\mathbb{C}$, are the "virtual" extremizers for the following improved Hardy inequality

$$\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx\ge\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}dx. \tag{2}$$

Actually, in [9, 40], the authors set up stability versions of the Hardy inequalities by adding the *"distance to the optimizers"* non-standard remainder terms to the RHS of (1) and (2).

In the critical case $N=2$, however, the above inequalities (1) and (2) fail for any positive constant. Nevertheless, on bounded domains, we have the following version of the critical Hardy inequality:

$$\int_{B_R^2}|\nabla u|^2dx\ge\frac14\int_{B_R^2}\frac{|u|^2}{|x|^2\left(1+\ln\frac{R}{|x|}\right)^2}dx. \tag{3}$$

The constant $\frac14$ is optimal but is not attained. Moreover, this inequality is equivalent to the critical Sobolev-Lorentz inequality [21] and an estimate of capacity [25].

It is worth noting that (1) is invariant under the scaling $u_\lambda(x)=\lambda^{\frac{N-2}{2}}u(\lambda x)$ while there is no natural scale invariance feature for (3). Hence, in [26], Ioku and Ishiwata proved the following version of the critical Hardy inequality

$$\int_{B_R^2}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx\ge\frac14\int_{B_R^2}\frac{|u|^2}{|x|^2\left(\ln\frac{R}{|x|}\right)^2}dx. \tag{4}$$

In particular, they showed that (4) is invariant under the scaling $u_\lambda(x)=\lambda^{-\frac12}u\left(|x|^{\lambda-1}x\right)$. It was also proved in [43] an interesting phenomenon that the critical Hardy inequality (4) is equivalent to the subcritical (2) in higher dimensional spaces. Also, the following global version of (4) was studied in [27, 41]: for any $R > 0$:

$$\int_{\mathbb{R}^2} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{\left| u(x) - u\left(R \frac{x}{|x|} \right) \right|^2}{|x|^2 \left| \ln \frac{R}{|x|} \right|^2} dx, \ \ u \in C_0^\infty\left(\mathbb{R}^2 \setminus \{0\} \right). \tag{5}$$

Motivated by the Hardy identities and inequalities for Bessel pairs in [22, 31–34], the relations between the critical Hardy inequalities and the subcritical Hardy inequalities on higher dimensional spaces in [43], we propose in this note several versions of the critical Hardy identities and inequalities and also study their applications. More precisely, in section 2, we set up some critical Hardy identities for Bessel pairs and use them to derive stability results for the subcritical Hardy inequalities. In section 3, we study the relations of critical Hardy inequalities for Bessel pairs and the subcritical Hardy inequalities for Bessel pairs on higher dimensional spaces. In section 4, we apply Theorem A to set up some variants of the Hardy equalities that can be combined with a limiting procedure to derive the critical Hardy inequality.

2 Critical Hardy inequalities for Bessel pairs

Theorem 1 *Assume that V and W are positive C^1-functions on $(0, \infty) \setminus \{R\}$ such that the ordinary differential equation*

$$\left(V(r) r^{N-1} \varphi'(r) \right)' + W(r) r^{N-1} \varphi(r) = 0$$

has a positive solution φ on $(0, \infty) \setminus \{R\}$. Then for all $u \in C_0^\infty\left(\mathbb{R}^N \setminus \{0\} \right)$ satisfying that for all $\sigma \in \mathbb{S}^{N-1}$:

$$\lim_{r \to R} V(r) r^{N-1} \frac{\varphi'(r)}{\varphi(r)} |u(r\sigma) - u(R\sigma)|^2 = 0,$$

we have

$$\begin{aligned} &\int_{\mathbb{R}^N} V(|x|) \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 dx - \int_{\mathbb{R}^N} W(|x|) \left| u(x) - u\left(R \frac{x}{|x|} \right) \right|^2 dx \\ &= \int_{\mathbb{R}^N} V(|x|) \varphi^2(|x|) \left| \frac{x}{|x|} \cdot \nabla \left(\frac{u(x) - u\left(R \frac{x}{|x|} \right)}{\varphi(|x|)} \right) \right|^2 dx \end{aligned} \tag{6}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N} V(|x|) \left| \nabla \left(u(x) - u\left(R \frac{x}{|x|} \right) \right) \right|^2 dx - \int_{\mathbb{R}^N} W(|x|) \left| u(x) - u\left(R \frac{x}{|x|} \right) \right|^2 dx \\ &= \int_{\mathbb{R}^N} V(|x|) \varphi^2(|x|) \left| \nabla \left(\frac{u(x) - u\left(R \frac{x}{|x|} \right)}{\varphi(|x|)} \right) \right|^2 dx. \end{aligned} \tag{7}$$

Proof Let $u(x) - u\left(R\frac{x}{|x|}\right) = v(x)\varphi(|x|)$. Then by integrating by parts, we have that for all $\sigma \in \mathbb{S}^{N-1}$:

$$\begin{aligned}
&\int_0^R W(r)|u(r\sigma) - u(R\sigma)|^2 r^{N-1} dr \\
&= \int_0^R W(r)|v(r\sigma)|^2 |\varphi(r)|^2 r^{N-1} dr \\
&= -\int_0^R |v(r\sigma)|^2 \varphi(r)\left(V(r) r^{N-1}\varphi'(r)\right)' dr \\
&= \int_0^R V(r) r^{N-1}\varphi'(r)\left[|v(r\sigma)|^2\varphi(r)\right]' dr \\
&= \int_0^R V(r) r^{N-1}[v(r\sigma)\varphi'(r) + \varphi(r)v_r(r\sigma)]^2 - \int_0^R V(r) r^{N-1}[\varphi(r)v_r(r\sigma)]^2 \\
&= \int_0^R V(r) r^{N-1}\left(\frac{d}{dr}[u(r\sigma) - u(R\sigma)]\right)^2 - \int_0^R V(r) r^{N-1}[\varphi(r)v_r(r\sigma)]^2 \\
&= \int_0^R V(r) r^{N-1}\left(\frac{d}{dr}u(r\sigma)\right)^2 - \int_0^R V(r) r^{N-1}[\varphi(r)v_r(r\sigma)]^2.
\end{aligned}$$

We note here that we used the assumptions of u to deal with boundary terms in the above integrating by parts process. Thus,

$$\begin{aligned}
&\int_{B_R^N} W(|x|)\left|u(x) - u\left(R\frac{x}{|x|}\right)\right|^2 dx - \int_{B_R^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2 dx \\
&= \int_{B_R^N} V(|x|)\varphi^2\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u(x) - u\left(R\frac{x}{|x|}\right)}{\varphi}\right)\right|^2 dx.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_R^\infty W(r)|u(r\sigma)-u(R\sigma)|^2 r^{N-1}dr\\
&=\int_R^\infty W(r)|v(r\sigma)|^2|\varphi(r)|^2 r^{N-1}dr\\
&=-\int_R^\infty |v(r\sigma)|^2\varphi(r)\big(V(r)r^{N-1}\varphi'(r)\big)'dr\\
&=\int_R^\infty V(r)r^{N-1}\varphi'(r)\Big[|v(r\sigma)|^2\varphi(r)\Big]'dr\\
&=\int_R^\infty V(r)r^{N-1}[v(r\sigma)\varphi'(r)+\varphi(r)v_r(r\sigma)]^2-\int_R^\infty V(r)r^{N-1}[\varphi(r)v_r(r\sigma)]^2\\
&=\int_R^\infty V(r)r^{N-1}\left(\frac{d}{dr}[u(r\sigma)-u(R\sigma)]\right)^2-\int_R^\infty V(r)r^{N-1}[\varphi(r)v_r(r\sigma)]^2\\
&=\int_R^\infty V(r)r^{N-1}\left(\frac{d}{dr}u(r\sigma)\right)^2-\int_R^\infty V(r)r^{N-1}[\varphi(r)v_r(r\sigma)]^2
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B_R^N} W(|x|)\left|u(x)-u\left(R\frac{x}{|x|}\right)\right|^2dx-\int_{\mathbb{R}^N\setminus B_R^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2dx\\
&=\int_{\mathbb{R}^N\setminus B_R^N} V(|x|)\varphi^2\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u(x)-u\left(R\frac{x}{|x|}\right)}{\varphi}\right)\right|^2dx.
\end{aligned}$$

Hence we get the desired equality (6)

To prove (7), we let $v(x)=u(x)-u\left(R\frac{x}{|x|}\right)$. Assume that the decomposition of v into the spherical harmonics is $v=\sum_{k=0}^{\infty}v_k=\sum_{k=0}^{\infty}f_k(r)\phi_k(\sigma)$. where $\phi_k(\sigma)$ are the orthonormal eigenfunctions of the Laplace-Beltrami operator with corresponding eigenvalues $c_k=k(N+k-2)$, $k\geq 0$. We note that the corresponding components f_k are in $C_0^\infty(\mathbb{R}^N)$ and satisfy $f_k(r)=O(r^k)$, $f_k'(r)=O(r^{k-1})$ as $r\downarrow 0$. In particular

$$\phi_0(\sigma)=1,\ c_0=0 \text{ and } f_0(r)=\frac{1}{|\partial B_r|}\int_{\partial B_r}uds.$$

Also, for any $k\in\mathbb{N}$:

$$\Delta v_k=\left(\Delta f_k(r)-c_k\frac{f_k(r)}{r^2}\right)\phi_k(\sigma).$$

Then

$$\int_{\mathbb{R}^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla v\right|^2 dx = \int_0^\infty\int_{\mathbb{S}^{N-1}} V(r)\left|\sum_{k=0}^{\infty}\frac{\partial v_k}{\partial r}\right|^2 r^{N-1}drd\sigma$$
$$= \int_0^\infty\int_{\mathbb{S}^{N-1}} V(r)\left|\sum_{k=0}^{\infty} f_k'(r)\phi_k(\sigma)\right|^2 r^{N-1}drd\sigma$$
$$= \sum_{k=0}^{\infty}\left|\mathbb{S}^{N-1}\right|\int_0^\infty V(r)\left|f_k'(r)\right|^2 r^{N-1}dr$$
$$= \sum_{k=0}^{\infty}\int_{\mathbb{R}^N} V(|x|)|\nabla f_k(|x|)|^2.$$

Next, we note that

$$\int_{\mathbb{R}^N} vV(|x|)\Delta v dx$$
$$= \int_0^\infty\int_{\mathbb{S}^{N-1}} V(r)\sum_{k=0}^{\infty} f_k(r)\phi_k(\sigma)\sum_{k=0}^{\infty}\left(f_k''(r)+\frac{N-1}{r}f_k'(r)-c_k\frac{f_k(r)}{r^2}\right)\phi_k(\sigma)r^{N-1}drd\sigma$$
$$= \sum_{k=0}^{\infty}\left|\mathbb{S}^{N-1}\right|\int_0^\infty V(r)f_k(r)\left(f_k''(r)+\frac{N-1}{r}f_k'(r)-c_k\frac{f_k(r)}{r^2}\right)r^{N-1}drd\sigma$$

and

$$\int_{\mathbb{R}^N} v\nabla V(|x|)\cdot\nabla v dx$$
$$= \int_0^\infty\int_{\mathbb{S}^{N-1}}\left(\sum_{k=0}^{\infty} f_k(r)\phi_k(\sigma)\right)\frac{V'(r)}{r}\left[(r,\sigma)\cdot\sum_{k=0}^{\infty}\left(f_k'(r)\phi_k(\sigma),\frac{f_k(r)}{r}\nabla_{\mathbb{S}^{N-1}}\phi_k(\sigma)\right)\right]r^{N-1}drd\sigma$$
$$= \sum_{k=0}^{\infty}\left|\mathbb{S}^{N-1}\right|\int_0^\infty V'(r)f_k(r)f_k'(r)r^{N-1}drd\sigma.$$

Hence

$$\int_{\mathbb{R}^N} V(|x|)|\nabla v|^2 dx = -\int_{\mathbb{R}^N} v\nabla\cdot(V(|x|)\nabla v)dx$$
$$= -\int_{\mathbb{R}^N} vV(|x|)\Delta v dx - \int_{\mathbb{R}^N} v\nabla V(|x|)\cdot\nabla v dx$$
$$= -\sum_{k=0}^{\infty}\left|\mathbb{S}^{N-1}\right|\int_0^\infty\left[V(r)f_k(r)\left(f_k''(r)+\frac{N-1}{r}f_k'(r)-c_k\frac{f_k(r)}{r^2}\right)+V'(r)f_k(r)f_k'(r)\right]r^{N-1}dr$$
$$= \sum_{k=0}^{\infty}\int_{\mathbb{R}^N} V(|x|)|\nabla f_k(|x|)|^2 + c_k V(|x|)\frac{|f_k(|x|)|^2}{|x|^2}dx.$$

We now can deduce that

$$\int_{\mathbb{R}^N} V(|x|)|\nabla v|^2 dx = \int_{\mathbb{R}^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla v\right|^2 dx + \sum_{k=0}^{\infty}\int_{\mathbb{R}^N} c_k V(|x|)\frac{|f_k(|x|)|^2}{|x|^2}dx.$$

Similarly, by noting that $\frac{v(x)}{\varphi(|x|)} = \sum_{k=0}^{\infty}\frac{f_k(r)}{\varphi(r)}\phi_k(\sigma)$ we also have

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(|x|)\varphi^2(|x|)\left|\nabla\frac{v}{\varphi}\right|^2 dx\\
&= \int_{\mathbb{R}^N} V(|x|)\varphi^2(|x|)\left|\frac{x}{|x|}\cdot\nabla\frac{v}{\varphi}\right|^2 dx + \sum_{k=0}^{\infty}\int_{\mathbb{R}^N} c_k V(|x|)\varphi^2(|x|)\frac{\left|\frac{f_k(|x|)}{\varphi(|x|)}\right|^2}{|x|^2}dx\\
&= \int_{\mathbb{R}^N} V(|x|)\varphi^2(|x|)\left|\frac{x}{|x|}\cdot\nabla\frac{v}{\varphi}\right|^2 dx + \sum_{k=0}^{\infty}\int_{\mathbb{R}^N} c_k V(|x|)\frac{|f_k(|x|)|^2}{|x|^2}dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(|x|)|\nabla v|^2 dx\\
&= \int_{\mathbb{R}^N} V(|x|)\left|\frac{x}{|x|}\cdot\nabla v\right|^2 dx + \int_{\mathbb{R}^N} V(|x|)\varphi^2(|x|)\left|\nabla\frac{v}{\varphi}\right|^2 dx - \int_{\mathbb{R}^N} V(|x|)\varphi^2(|x|)\left|\frac{x}{|x|}\cdot\nabla\frac{v}{\varphi}\right|^2 dx\\
&= \int_{\mathbb{R}^N} W(|x|)|v|^2 dx + \int_{\mathbb{R}^N} V(|x|)\varphi^2(|x|)\left|\nabla\left(\frac{v(x)}{\varphi(|x|)}\right)\right|^2 dx.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\int_{\mathbb{R}^N} V(|x|)\left|\nabla\left(u(x) - u\left(R\frac{x}{|x|}\right)\right)\right|^2 dx - \int_{\mathbb{R}^N} W(|x|)\left|u(x) - u\left(R\frac{x}{|x|}\right)\right|^2 dx\\
&= \int_{\mathbb{R}^N} V(|x|)\varphi^2(|x|)\left|\nabla\left(\frac{u(x) - u\left(R\frac{x}{|x|}\right)}{\varphi(|x|)}\right)\right|^2 dx.
\end{aligned}$$

□

As a consequence of our main result, we obtain the following identity that establishes the stability of subcritical Hardy inequalities:

Theorem 2 *For $u \in C_0^\infty(\mathbb{R}^N)$, $N \geq 3$, one has*

$$\int_{\mathbb{R}^N} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}dx$$
$$= \frac{1}{4}\int_{\mathbb{R}^N}\frac{\left||x|^{\frac{N-2}{2}}u(x) - R^{\frac{N-2}{2}}u\left(R\frac{x}{|x|}\right)\right|^2}{|x|^N\left|\ln\frac{|x|}{R}\right|^2}dx$$
$$+ \int_{\mathbb{R}^N}\frac{1}{|x|^{N-2}}\left|\ln\frac{|x|}{R}\right|\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u(x) - R^{\frac{N-2}{2}}u\left(R\frac{x}{|x|}\right)}{\sqrt{\left|\ln\frac{|x|}{R}\right|}}\right)\right|^2 dx.$$

As a by-product,

$$\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}dx \geq \frac{1}{4}\sup_{R>0}\int_{\mathbb{R}^N}\frac{\left|u(x) - R^{\frac{N-2}{2}}u\left(R\frac{x}{|x|}\right)|x|^{-\frac{N-2}{2}}\right|^2}{|x|^2\left|\ln\frac{|x|}{R}\right|^2}dx.$$

Proof We note that with $V(r) = \frac{1}{r^{N-2}}$ and $W = \frac{1}{4r^N\left|\ln\frac{r}{R}\right|^2}$, then $\varphi = \sqrt{\left|\ln\frac{r}{R}\right|}$. In this case, we have that for any $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ and any $\sigma \in \mathbb{S}^{N-1}$:

$$\lim_{r\to R}\left|V(r)r^{N-1}\frac{\varphi'(r)}{\varphi(r)}|u(r\sigma) - u(R\sigma)|^2\right| = \lim_{r\to R}\frac{1}{r^{N-2}}r^{N-1}\left|\frac{\left(\sqrt{\left|\ln\frac{r}{R}\right|}\right)'}{\sqrt{\left|\ln\frac{r}{R}\right|}}\right||u(r\sigma) - u(R\sigma)|^2$$
$$\lesssim \lim_{r\to R}\frac{1}{\left|\ln\frac{r}{R}\right|}\|\nabla u\|_\infty^2(R-r)^2 = 0$$

Hence for any $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$:

$$\int_{\mathbb{R}^N}\frac{1}{|x|^{N-2}}\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2 dx = \frac{1}{4}\int_{\mathbb{R}^N}\frac{\left|u(x) - u\left(R\frac{x}{|x|}\right)\right|^2}{|x|^N\left|\ln\frac{|x|}{R}\right|^2}dx$$
$$+ \int_{\mathbb{R}^N}\frac{1}{|x|^{N-2}}\left|\ln\frac{|x|}{R}\right|\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u(x) - u\left(R\frac{x}{|x|}\right)}{\sqrt{\left|\ln\frac{|x|}{R}\right|}}\right)\right|^2 dx$$

and

$$\int_{\mathbb{R}^N}\frac{1}{|x|^{N-2}}\left|\nabla\left(u(x) - u\left(R\frac{x}{|x|}\right)\right)\right|^2 dx = \frac{1}{4}\int_{\mathbb{R}^N}\frac{\left|u(x) - u\left(R\frac{x}{|x|}\right)\right|^2}{|x|^N\left|\ln\frac{|x|}{R}\right|^2}dx$$
$$+ \int_{\mathbb{R}^N}\frac{1}{|x|^{N-2}}\left|\ln\frac{|x|}{R}\right|\left|\nabla\left(\frac{u(x) - u\left(R\frac{x}{|x|}\right)}{\sqrt{\left|\ln\frac{|x|}{R}\right|}}\right)\right|^2 dx$$

As an application, using Theorem A with $V=1$, $W=\left(\frac{N-2}{2}\right)^2\frac{1}{r^2}$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx-\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}dx\\
&=\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{N-2}{2}}u\right)\right|^2\left|\frac{1}{|x|^{\frac{N-2}{2}}}\right|^2dx\\
&=\frac{1}{4}\int_{\mathbb{R}^N}\frac{\left||x|^{\frac{N-2}{2}}u(x)-R^{\frac{N-2}{2}}u\left(R\frac{x}{|x|}\right)\right|^2}{|x|^N\left|\ln\frac{|x|}{R}\right|^2}dx\\
&+\int_{\mathbb{R}^N}\frac{1}{|x|^{N-2}}\left|\ln\frac{|x|}{R}\right|\left|\frac{x}{|x|}\cdot\nabla\left(\frac{|x|^{\frac{N-2}{2}}u(x)-R^{\frac{N-2}{2}}u\left(R\frac{x}{|x|}\right)}{\sqrt{\left|\ln\frac{|x|}{R}\right|}}\right)\right|^2dx
\end{aligned}$$

which implies

$$\int_{\mathbb{R}^N}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx-\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^2}dx\geq\frac{1}{4}\sup_{R>0}\int_{\mathbb{R}^N}\frac{\left|u(x)-R^{\frac{N-2}{2}}u\left(R\frac{x}{|x|}\right)|x|^{-\frac{N-2}{2}}\right|^2}{|x|^2\left|\ln\frac{|x|}{R}\right|^2}dx.$$

□

3 Connections of the critical and the subcritical Hardy type inequalities

Let $m>2$. We recall that (V,W) is a $m-$dimensional Bessel pair on $(0,\infty)$ if the ordinary differential equation

$$y''(r)+\left(\frac{m-1}{r}+\frac{V_r(r)}{V(r)}\right)y'(r)+\frac{W(r)}{V(r)}y(r)=0$$

has a positive solution $\varphi_{V,W;\infty}$ on the interval $(0,\infty)$. The main aim of this section is to prove that

Theorem 3 *If* $\left(V(R\exp(-r^{-\alpha}))r^{-m+\alpha+2},\alpha^2(R\exp(-r^{-\alpha}))^2W(R\exp(-r^{-\alpha}))r^{-m-\alpha}\right)$ *is* $m-$*dimensional Bessel pair on* $(0,\infty)$ *for some* $\alpha>0$ *and* $R>0$, *then for* $v\in C_0^\infty\left(B_R^2\setminus\{0\}\right)$:

$$\int_{B_R^2}V(|y|)\left|\frac{y}{|y|}\cdot\nabla v(y)\right|^2\geq\int_{B_R^2}W(|y|)|v(y)|^2.$$

Moreover, there exists a $m-$dimensional Bessel pair $\left(\widetilde{V},\widetilde{W}\right)$ on $(0,\infty)$ such that for all $v\in C_0^\infty\left(B_R^2\setminus\{0\}\right)$, there exists $u\in C_0^\infty(\mathbb{R}^m\setminus\{0\})$ such that

$$\begin{aligned}
&\frac{\omega_m}{\omega_2}\alpha\left[\int_{B_R^2}V(|y|)\left|\frac{y}{|y|}\cdot\nabla v(y)\right|^2dy-\int_{B_R^2}W(|y|)|v(y)|^2dy\right]\\
&=\int_{\mathbb{R}^m}\widetilde{V}(|x|)\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2dx-\int_{\mathbb{R}^m}\widetilde{W}(|x|)|u(x)|^2dx.
\end{aligned}$$

Proof For $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^2$ we use the polar coordinates:

$$y = (s, \psi) \text{ where } s = |y| \text{ and } 0 \leq \psi < 2\pi$$
$$x = (r, \theta_1, ..., \theta_{m-1}) \text{ where } r = |x| \text{ and } 0 \leq \theta_1, ..., \theta_{m-2} < \pi,\ 0 \leq \theta_{m-1} < 2\pi.$$

Now, for $v \in C_0^\infty(B_R^2 \setminus \{0\})$, we set

$$u(r, \theta_1, ..., \theta_{m-1}) = u(r, \theta_{m-1}) = v(s, \psi)$$
$$r = \left(\ln \frac{R}{s}\right)^{-\frac{1}{\alpha}} \text{ i.e. } s = R\exp(-r^{-\alpha}),\ \alpha > 0$$
$$\theta_{m-1} = \psi.$$

We have that

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial s}\frac{\partial s}{\partial r} = \frac{\partial v}{\partial s}\alpha s r^{-\alpha-1}$$

We note that u is independent of $\theta_1, ..., \theta_{m-2}$.

Let

$$V(s) = \widetilde{V}\left(\left(\ln \frac{R}{s}\right)^{-\frac{1}{\alpha}}\right)\left(\ln \frac{R}{s}\right)^{-\frac{m-\alpha-2}{\alpha}}$$
$$W(s) = \frac{1}{\alpha^2}\frac{\widetilde{W}\left(\left(\ln \frac{R}{s}\right)^{-\frac{1}{\alpha}}\right)}{s^2\left(\ln \frac{R}{s}\right)^{\frac{m}{\alpha}+1}}.$$

Then by polar coordinate on $\mathbb{R}^m : x = (r, \theta_1, ..., \theta_{m-1})$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^m} \widetilde{V}(|x|)\left|\frac{x}{|x|} \cdot \nabla u(x)\right|^2 dx \\
&= \int_0^\infty \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{m-2}=0}^{\pi} \int_{\theta_{m-1}=0}^{2\pi} \widetilde{V}(r)|\partial_r u(r, \theta_{m-N+1}, ..., \theta_{m-1})|^2 \left(\prod_1^{m-1} (\sin\theta_j)^{m-1-j} d\theta_j\right) r^{m-1} dr \\
&= \left[\prod_1^{m-2} \int_{\theta_j=0}^{\pi} (\sin\theta_j)^{m-1-j} d\theta_j\right] \int_0^\infty \int_{\theta_{m-1}=0}^{2\pi} \widetilde{V}(r)|\partial_r u(r, \theta_{m-1})|^2 d\theta_{m-1} r^{m-1} dr \\
&= \alpha\left[\prod_1^{m-2} \int_{\theta_j=0}^{\pi} (\sin\theta_j)^{m-1-j} d\theta_j\right] \int_0^R \int_{\psi=0}^{2\pi} \widetilde{V}\left(\left(\ln \frac{R}{s}\right)^{-\frac{1}{\alpha}}\right)\left(\ln \frac{R}{s}\right)^{-\frac{m-\alpha-2}{\alpha}} |\partial_s v(s, \psi)|^2 d\psi s ds \\
&= \frac{\omega_m}{\omega_2}\alpha \int_{B_R^2} \widetilde{V}\left(\left(\ln \frac{R}{|y|}\right)^{-\frac{1}{\alpha}}\right)\left(\ln \frac{R}{|y|}\right)^{-\frac{m-\alpha-2}{\alpha}} \left|\frac{y}{|y|} \cdot \nabla v(y)\right|^2 dy \\
&= \frac{\omega_m}{\omega_2}\alpha \int_{B_R^2} V(|y|)\left|\frac{y}{|y|} \cdot \nabla v(y)\right|^2 dy.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{R}^m} \widetilde{W}(|x|)|u(x)|^2 dx \\
&= \int_0^\infty \int_{\theta_1=0}^{\pi} \dots \int_{\theta_{m-2}=0}^{\pi} \int_{\theta_{m-1}=0}^{2\pi} \widetilde{W}(r)|u(r,\theta_1,\dots,\theta_{m-1})|^2 \left(\prod_1^{m-1} (\sin\theta_j)^{m-1-j} d\theta_j\right) r^{m-1} dr \\
&= \left[\prod_1^{m-2} \int_{\theta_j=0}^{\pi} (\sin\theta_j)^{m-1-j} d\theta_j\right] \int_0^\infty \int_{\theta_{m-1}=0}^{2\pi} \widetilde{W}(r)|u(r,\theta_{m-1})|^2 d\theta_{m-1} r^{m-1} dr \\
&= \frac{1}{\alpha}\frac{\omega_m}{\omega_2} \int_0^R \int_{\psi=0}^{2\pi} \widetilde{W}\left(\left(\ln\frac{R}{s}\right)^{-\frac{1}{\alpha}}\right)|v(s,\psi)|^2 d\psi \left(\ln\frac{R}{s}\right)^{-\frac{m}{\alpha}-1} \frac{1}{s} ds \\
&= \frac{1}{\alpha}\frac{\omega_m}{\omega_2} \int_{B_R^2} \widetilde{W}\left(\left(\ln\frac{R}{|y|}\right)^{-\frac{1}{\alpha}}\right) \frac{1}{|y|^2} \frac{1}{\left(\ln\frac{R}{|y|}\right)^{\frac{m}{\alpha}+1}} |v(y)|^2 dy \\
&= \frac{\omega_m}{\omega_2}\alpha \int_{B_R^2} W(|y|)|v(y)|^2 dy
\end{aligned}$$

Since $\left(V(R\exp(-r^{-\alpha}))r^{-m+\alpha+2}, \alpha^2(R\exp(-r^{-\alpha}))^2 W(R\exp(-r^{-\alpha}))r^{-m-\alpha}\right)$ is $m-$dimensional Bessel pair on $(0,\infty)$, $\left(\widetilde{V},\widetilde{W}\right)$ is a $m-$dimensional Bessel pair on $(0,\infty)$. Hence we have

$$\begin{aligned}
&\frac{\omega_m}{\omega_2}\alpha\left[\int_{B_R^2} V(|y|)\left|\frac{y}{|y|}\cdot\nabla v(y)\right|^2 dy - \int_{B_R^2} W(|y|)|v(y)|^2 dy\right] \\
&= \int_{\mathbb{R}^m} \widetilde{V}(|x|)\left|\frac{x}{|x|}\cdot\nabla u(x)\right|^2 dx - \int_{\mathbb{R}^m} \widetilde{W}(|x|)|u(x)|^2 dx \geq 0.
\end{aligned}$$

□

Example 1 Let $V=1$ and $W=\frac{1}{4}\frac{1}{\left(\ln\frac{R}{r}\right)^2 r^2}$, then with $\alpha=m-2$

$$V(R\exp(-r^{-\alpha}))r^{-m+\alpha+2}=1$$

and

$$\begin{aligned}
&\alpha^2(R\exp(-r^{-\alpha}))^2 W(R\exp(-r^{-\alpha}))r^{-m-\alpha} \\
&= \frac{1}{4}\alpha^2(R\exp(-r^{-\alpha}))^2 \frac{1}{\left(\ln\frac{R}{R\exp(-r^{-\alpha})}\right)^2 (R\exp(-r^{-\alpha}))^2} r^{-m-\alpha} \\
&= \frac{\alpha^2}{4} r^{-2}.
\end{aligned}$$

Hence we recover the following identity that could be found in [43]:

$$\begin{aligned}
&\int_{\mathbb{R}^m} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \left(\frac{m-2}{2}\right)^2 \int_{\mathbb{R}^m} \frac{|u|^2}{|x|^2} dx \\
&= \frac{\omega_m}{\omega_2}(m-2)\left(\int_{B_R^2} \left|\frac{y}{|y|}\cdot\nabla v(y)\right|^2 dy - \frac{1}{4}\int_{B_R^2} \frac{|v(y)|^2}{|y|^2\left|\ln\frac{R}{|y|}\right|^2} dy\right)
\end{aligned}$$

where in the polar coordinate

$$u(r,\theta_1,\ldots,\theta_{m-1}) = u(r,\theta_{m-1}) = v(s,\psi)$$
$$r = \left(\ln\frac{R}{s}\right)^{-\frac{1}{\alpha}} \text{ i.e. } s = R\exp(-r^{-\alpha})$$
$$\theta_{m-1} = \psi.$$

Noting that by Theorem A, we have

$$\int_{\mathbb{R}^m} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \left(\frac{m-2}{2}\right)^2 \int_{\mathbb{R}^m} \frac{|u|^2}{|x|^2} dx = \int_{\mathbb{R}^m} \left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{m-2}{2}}u\right)\right|^2 \left|\frac{1}{|x|^{\frac{m-2}{2}}}\right|^2 dx.$$

Using polar coordinates, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^m} \left|\frac{x}{|x|}\cdot\nabla\left(|x|^{\frac{m-2}{2}}u\right)\right|^2 \left|\frac{1}{|x|^{\frac{m-2}{2}}}\right|^2 dx\\
&= \int_0^\infty \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{m-2}=0}^{\pi} \int_{\theta_{m-1}=0}^{2\pi} \frac{1}{r^{m-2}} \left|\partial_r\left[r^{\frac{m-2}{2}}u(r,\theta_{m-N+1},\ldots,\theta_{m-1})\right]\right|^2 \left(\prod_1^{m-1}(\sin\theta_j)^{m-1-j}d\theta_j\right) r^{m-1}dr\\
&= \left[\prod_1^{m-2}\int_{\theta_j=0}^{\pi}(\sin\theta_j)^{m-1-j}d\theta_j\right] \int_0^\infty \int_{\theta_{m-1}=0}^{2\pi} \left[r^{\frac{m-2}{2}}\partial_r u(r,\theta_{m-1}) + \frac{m-2}{2}r^{\frac{m-4}{2}}u(r,\theta_{m-1})\right]^2 d\theta_{m-1}rdr\\
&= \frac{\omega_m}{\omega_2}\int_0^R \int_{\psi=0}^{2\pi} \left[\left(\ln\frac{R}{s}\right)^{\frac{\alpha+2}{2\alpha}} \partial_s v(s,\psi)\alpha + \frac{\alpha}{2s}\left(\ln\frac{R}{s}\right)^{-\frac{\alpha-2}{2\alpha}} v(s,\psi)\right]^2 d\psi \left(\ln\frac{R}{s}\right)^{-\frac{\alpha+2}{\alpha}} \frac{1}{\alpha} sds\\
&= \frac{\omega_m}{\omega_2}\alpha\int_0^R \int_{\psi=0}^{2\pi} \left[\left(\ln\frac{R}{s}\right)\partial_s v(s,\psi) + \frac{1}{2s}v(s,\psi)\right]^2 d\psi \frac{1}{\left(\ln\frac{R}{s}\right)^2} sds\\
&= \frac{\omega_m}{\omega_2}\alpha\int_0^R \int_{\psi=0}^{2\pi} \left[\partial_s\left(\frac{v(s,\psi)}{\sqrt{\ln\frac{R}{s}}}\right)\right]^2 d\psi \ln\frac{R}{s} sds\\
&= \frac{\omega_m}{\omega_2}\alpha\int_{B_R^2} \left|\frac{y}{|y|}\cdot\nabla\left(\frac{v(y)}{\sqrt{\ln\frac{R}{|y|}}}\right)\right|^2 \ln\frac{R}{|y|}dy.
\end{aligned}$$

Hence, we get the following critical Hardy identity

$$\int_{B_R^2} \left|\frac{y}{|y|}\cdot\nabla v(y)\right|^2 dy - \frac{1}{4}\int_{B_R^2} \frac{|v(y)|^2}{|y|^2\left|\ln\frac{R}{|y|}\right|^2} dy = \int_{B_R^2} \left|\frac{y}{|y|}\cdot\nabla\left(\frac{v(y)}{\sqrt{\ln\frac{R}{|y|}}}\right)\right|^2 \ln\frac{R}{|y|}dy.$$

Example 2 Let $0\le\lambda<m-2$, $V=1$ and $W=\frac{1}{4}\frac{1}{\left(\ln\frac{R}{r}\right)^2 r^2}$, then with $\alpha = m-2-\lambda$

$$V(R\exp(-r^{-\alpha}))r^{-m+\alpha+2} = r^{-\lambda}$$

and

$$
\begin{aligned}
&\alpha^2(R\exp(-r^{-\alpha}))^2 W(R\exp(-r^{-\alpha}))r^{-m-\alpha}\\
&=\frac{1}{4}\alpha^2(R\exp(-r^{-\alpha}))^2\frac{1}{\left(\ln\frac{R}{R\exp(-r^{-\alpha})}\right)^2(R\exp(-r^{-\alpha}))^2}r^{-m-\alpha}\\
&=\frac{1}{4}\alpha^2 r^{-\lambda-2}
\end{aligned}
$$

Then we obtain the following equality:

$$
\begin{aligned}
&\int_{\mathbb{R}^m}\frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx-\left(\frac{m-\lambda-2}{2}\right)^2\int_{\mathbb{R}^m}\frac{|u|^2}{|x|^{\lambda+2}}dx\\
&=\frac{\omega_m}{\omega_2}(m-2-\lambda)\left(\int_{B_R^2}\left|\frac{y}{|y|}\cdot\nabla v(y)\right|^2dy-\frac{1}{4}\int_{B_R^2}\frac{|v(y)|^2}{|y|^2\left|\ln\frac{R}{|y|}\right|^2}dy\right)
\end{aligned}
$$

where in the polar coordinate

$$
\begin{aligned}
u(r,\theta_1,\dots,\theta_{m-1})&=u(r,\theta_{m-1})=v(s,\psi)\\
r&=\left(\ln\frac{R}{s}\right)^{-\frac{1}{\alpha}}\text{ i.e. } s=R\exp(-r^{-\alpha})\\
\theta_{m-1}&=\psi.
\end{aligned}
$$

4 Some improved Hardy identities

We begin this section with the following weighted Hardy identity:

Theorem 4 *Let $N>2$ and $0\le\lambda<N-2$. For all $u\in C_0^\infty\left(B_R^N\setminus\{0\}\right)$, we have*

$$
\begin{aligned}
&\int_{B_R^N}\frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx-\frac{1}{4}\int_{B_R^N}\frac{|u|^2}{|x|^{\lambda+2}\left(\frac{1-\left(\frac{R}{|x|}\right)^{2+\lambda-N}}{N-2-\lambda}\right)^2}dx\\
&=\int_{B_R^N}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u}{\sqrt{\left(\frac{|x|}{R}\right)^{2+\lambda-N}-1}}\right)\right|^2\left[\left(\frac{|x|}{R}\right)^{2+\lambda-N}-1\right]dx.
\end{aligned}
\tag{8}
$$

Proof Let $V=\frac{1}{r^{\lambda}}$ and $W=\left(\frac{N-\lambda-2}{2}\right)^2\frac{1}{r^{\lambda+2}\left(1-\left(\frac{R}{r}\right)^{2+\lambda-N}\right)^2}$. Then $(r^{N-1}V,r^{N-1}W)$ is a (1−dimensional) Bessel pair on $(0,R)$ with $\varphi=R^{-\frac{N-\lambda-2}{2}}\sqrt{\left(\frac{r}{R}\right)^{2+\lambda-N}-1}$. Hence, by Theorem A, we obtain

$$
\begin{aligned}
&\int_{B_R^N}\frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2dx-\left(\frac{N-\lambda-2}{2}\right)^2\int_{B_R^N}\frac{|u|^2}{|x|^{\lambda+2}\left(1-\left(\frac{R}{|x|}\right)^{2+\lambda-N}\right)^2}dx\\
&=\int_{B_R^N}\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u}{\sqrt{\left(\frac{|x|}{R}\right)^{2+\lambda-N}-1}}\right)\right|^2\left[\left(\frac{|x|}{R}\right)^{2+\lambda-N}-1\right]dx.
\end{aligned}
$$

Equivalently,

$$\int_{B_R^N} \left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \frac{1}{4}\int_{B_R^N} \frac{|u|^2}{|x|^{\lambda+2}\left(\frac{1-\left(\frac{R}{|x|}\right)^{2+\lambda-N}}{N-2-\lambda}\right)^2} dx$$

$$= \int_{B_R^N} \left|\frac{x}{|x|}\cdot\nabla\left(\frac{u}{\sqrt{\left(\frac{|x|}{R}\right)^{2+\lambda-N}-1}}\right)\right|^2 \left[\left(\frac{|x|}{R}\right)^{2+\lambda-N}-1\right] dx.$$

□

As a consequence, we get

$$\int_{B_R^N} \frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx \geq \frac{1}{4}\int_{B_R^N} \frac{|u|^2}{|x|^{\lambda+2}\left(\frac{1-\left(\frac{R}{|x|}\right)^{2+\lambda-N}}{N-2-\lambda}\right)^2} dx$$

Also, by noting that

$$(N-2-\lambda)\ln\frac{R}{|x|} \geq 1-\left(\frac{R}{|x|}\right)^{2+\lambda-N}$$

we get

$$\int_{B_R^N} \frac{1}{|x|^{\lambda}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx \geq \frac{1}{4}\int_{B_R^N} \frac{|u|^2}{|x|^{\lambda+2}\left(\ln\frac{R}{|x|}\right)^2} dx.$$

Now if we send $\lambda \to N-2$ in (8), we obtain the critical Hardy inequality:

$$\int_{B_R^N} \frac{1}{|x|^{N-2}}\left|\frac{x}{|x|}\cdot\nabla u\right|^2 dx - \frac{1}{4}\int_{B_R^N} \frac{|u|^2}{|x|^{N}\left(\ln\frac{R}{|x|}\right)^2} dx$$

$$= \int_{B_R^N} \frac{1}{|x|^{N-2}}\left|\ln\frac{R}{|x|}\right|\left|\frac{x}{|x|}\cdot\nabla\left(\frac{u(x)}{\sqrt{\left|\ln\frac{R}{|x|}\right|}}\right)\right|^2 dx.$$

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