Indian J Pure Appl Math (2021) 52:1180–1185
https://doi.org/10.1007/s13226-021-00089-6

ORIGINAL RESEARCH

On a sumset problem of dilates

Sandeep Singh Chahal · Ram Krishna Pandey

Received: 10 August 2020 / Accepted: 9 November 2020 / Published online: 14 June 2021

Abstract Let A be a nonempty finite set of integers. For a real number m, the set $m \cdot A = \{ma : a \in A\}$ denotes the set of m-dilates of A. In 2008, Bukh initiated an interesting problem of finding a lower bound for the sumset of dilated sets, i.e., a lower bound for $|m_1 \cdot A + m_2 \cdot A + \cdots + m_n \cdot A|$, where $m_1, m_2, \ldots, m_n$ are integers. In this paper, we consider the case $|3 \cdot A + k \cdot A|$, where k is a prime number ≥ 5. Under some assumptions on A, first we give a general lower bound on the cardinality of $3 \cdot A + k \cdot A$ then, for large sets A, under the same assumptions, we improve this general lower bound. The results also hold true for the general sumset $q \cdot A + k \cdot A$, where q is any odd prime $< k$, under the similar assumptions on A.

Keywords Additive combinatorics · Sumsets of dilates

Mathematics Subject Classification 11B13 · 11P70

1 Introduction

Let $h \geq 2$ and let $A_1, A_2, \ldots, A_h$ be sets of integers. The *sumset* of these sets, is a set of integers, defined by

$$A_1 + A_2 + \cdots + A_h := \{a_1 + a_2 + \cdots + a_h : a_i \in A_i \text{ for } i = 1, 2, \ldots, h\}.$$

If A is a set of integers such that $A_i = A$ for $i = 1, 2, \ldots, h$, then we denote the sumset $A_1 + A_2 + \cdots + A_h$ by hA and it is called the *h-fold sumset of A*.

Let k be a nonzero real number. The *k-dilation*, $k \cdot A$ of A is defined by $k \cdot A = \{ka : a \in A\}$. Bukh [2] studied the sumset of t dilated sets of the finite set A of integers and proved in his main theorem that

$$|\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_t \cdot A| \geq (|\lambda_1| + |\lambda_2| + \cdots + |\lambda_t|)|A| - o(|A|),$$

Communicated by Sanoli Gun, Phd.

Research was partially supported by SERB-NPDF (PDF/2016/001483). The corresponding author whose research was supported by SERB (MTR/2019/000299).

S. S. Chahal
Department of Mathematics, Akal University, Talwandi Sabo 151302, India
E-mail: sandeepinsan86@gmail.com

R. K. Pandey (✉)
Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, India
E-mail: ram.pandey@ma.iitr.ac.in

where $(\lambda_1, \lambda_2, \ldots, \lambda_t) \in \mathbb{Z}^t$ with $\gcd(\lambda_1, \lambda_2, \ldots, \lambda_t) = 1$. Here and elsewhere, $|A|$ denotes the cardinality of the set A and $|\lambda|$ denotes the absolute value of the real number λ.

Confining ourselves to the sum of only two dilates, it is enough to consider only the normalized sums [10] $m \cdot A + k \cdot A$ satisfying $k \geq m \geq 1$ and $\gcd(m, k) = 1$. We divide the study of $m \cdot A + k \cdot A$ into cases depending on the values of m.

Case 1. $(m = 1)$. If $k = 1$, then it is a 2-fold sum of A. Nathanson [9] gave the sharp lower bound for $|hA|$, where $h \geq 2$ and studied the extremal case satisfying the bound, i.e., the sets for which the lower bound is achieved. For $k = 2$, Nathanson [11] and Cilleruelo et al. [4] proved that $|A + 2 \cdot A| \geq 3|A| - 2$ and if the equality holds, then A is an arithmetic progression or a singleton. For $k = 3$, Cilleruelo et al. [4] proved that $|A + 3 \cdot A| \geq 4|A| - 4$. If the equality holds, then $A = 3 \cdot \{0, 1, \ldots, n\} + \{0, 1\}$ or $A = \{0, 1, 3\}$ or $A = \{0, 1, 4\}$ or A is an affine transform of one of these sets. The lower bound was also proved using different methods in [1, 6]. For $k = 4$, Du et al. [5] proved that $|A + 4 \cdot A| \geq 5|A| - 6$ for $|A| \geq 5$.

The cases $k \geq 3$ are much more involved. In fact, Cilleruelo et al. [4] also conjectured that for sufficiently large set A, $|A + k \cdot A| \geq (k + 1)|A| - \lceil k(k + 2)/4 \rceil$. Cilleruelo et al. [3] gave a positive answer to the above conjecture for k a prime. Later, Du et al. [5] also gave a positive answer to this conjecture in the cases when k is a power of a prime and k is a product of two distinct primes. To the best of our knowledge that is all known in the case $m = 1$.

Case 2. $(m = 2)$. Hamidoune and Rué [7] proved that if k is an odd prime and $|A| > 8k^k$, then

$$|2 \cdot A + k \cdot A| \geq (k + 2)|A| - k^2 - k + 2.$$

Extending the result of Hamidoune and Rué [7], Ljujic [8] obtained the same lower bound for the case k a power of an odd prime and the case k a product of two distinct odd primes. Again this is where we would like to stop in this case.

Case 3. $(m = 3)$. This case is handled in this paper in a special situation and then it is generalized for $m = q$, where q is an odd prime $< k$. Under some assumptions on A, first we give a general lower bound on the cardinality of $3 \cdot A + k \cdot A$ then, for large sets A, under the same assumptions, we improve this general bound. The results also hold true for the general sumset $q \cdot A + k \cdot A$, where q is any odd prime $< k$, under the similar assumptions on A. More precisely, we prove Lemma 3.2 and Theorem 3.6 and generalize them in the Remarks 3.5 and 3.7, respectively.

In Section 2, we give notation and some already known results that are used to prove our main results in Section 3.

2 Notation and preliminary results

Throughout this paper we use the following notation.

Let A be a finite set of integers and let k be a prime number > 3. Since the cardinality of sums of dilates is invariant under the affine transformation, we can assume that $0 \in A$ and $\gcd(A) = 1$. Let $\hat{A}$ be the natural projection of A on $\mathbb{Z}/k\mathbb{Z}$. If $|\hat{A}| = j$, we denote by $A_1, A_2, \ldots, A_j$ the distinct congruence classes of A modulo k also called *k-components* of A. So, we have $A = A_1 \cup A_2 \cup \cdots \cup A_j$. Let $c_k(A)$ denote the number of distinct congruence classes of A modulo k, i.e., $c_k(A) = |\hat{A}|$. We also assume that $|A_1| \geq |A_2| \geq \cdots \geq |A_j|$. Let $A_i = k \cdot X_i + u_i$, where $0 \leq u_i < k$ for all i. Define $E = \{1 \leq i \leq j : 0 < c_k(X_i) < k\}$, $F = \{1 \leq i \leq j : c_k(X_i) = k\}$. Let also $A_E = \cup_{i \in E} A_i$ and $A_F = \cup_{i \in F} A_i$. Observe that $j \geq 2$ whenever $|A| \geq 2$. Furthermore, define $\Delta_{rs} = (3 \cdot A_r + k \cdot A) \setminus (3 \cdot A_r + k \cdot A_s)$ for $1 \leq r, s \leq j$. Then

$$|3 \cdot A_r + k \cdot A| = |3 \cdot A_r + k \cdot A_s| + |\Delta_{rs}| = |3 \cdot X_r + k \cdot X_s| + |\Delta_{rs}|.$$

Proposition 2.1 has been taken from [7, Proposition 3.2] and it is used at several places in the proofs of the results of Section 3.

Proposition 2.1 *Let A and B be finite set of integers and let m and n be coprime integers. Then* $|n \cdot A + m \cdot B| \geq c_n(B)|A| + c_m(A)|B| - c_m(A)c_n(B)$.

3 Main Results

Lemma 3.1 (including its proof) is a suitable transformation of already known result [3, Lemma 2.4] which fits into our current problem.

Lemma 3.1

(i) Let k be a prime number. If $i \in E$, then $|\triangle_{ii}| \geq |A_s|$ for any $s \neq i$.
(ii) $\sum_{i \in E} |\Delta_{ii}| \geq (|E| - 1)|A_1| + |A_2|$.

Proof (i) Let $i \in E$. Let a and b be distinct elements in $\mathbb{Z}/k\mathbb{Z}$. If $\hat{X_i} + a = \hat{X_i} + b$, then $\hat{X_i} = \hat{X_i} + b - a = \hat{X_i} + 2(b-a) = \cdots = \hat{X_i} + (k-1)(b-a)$. So $\hat{X_i} = \mathbb{Z}/k\mathbb{Z}$ as k is a prime. This contradicts the choice of i. So $\hat{X_i} + a \neq \hat{X_i} + b$, which implies $3 \cdot \hat{X_i} + a \neq 3 \cdot \hat{X_i} + b$. Thus, $|(3 \cdot \hat{X_i} + a) \setminus (3 \cdot \hat{X_i} + b)| \geq 1$. Now we have

$$\begin{aligned}
|\Delta_{ii}| &= |(3 \cdot A_i + k \cdot A) \setminus (3 \cdot A_i + k \cdot A_i)| \\
&\geq |(3 \cdot (k \cdot X_i + u_i) + k \cdot A_s) \setminus (3 \cdot (k \cdot X_i + u_i) + k \cdot A_i)| \\
&= |(3 \cdot X_i + A_s) \setminus (3 \cdot X_i + A_i)| \\
&= |(3 \cdot X_i + k \cdot X_s + u_s) \setminus (3 \cdot X_i + k \cdot X_i + u_i)| \\
&\geq |X_s||(3 \cdot \hat{X_i} + \hat{u_s}) \setminus (3 \cdot \hat{X_i} + \hat{u_i})| \geq |X_s| = |A_s|.
\end{aligned}$$

(ii) From (i), we have $|\Delta_{ii}| \geq |A_1|$ for all $i \in E$ provided $i \neq 1$, when $1 \in E$. In that case we have $|\Delta_{11}| \geq A_2|$. Hence the lemma. □

Lemma 3.2 *Let $k > 3$ be a prime number and A be a nonempty finite set of integers with its k-components $A_1, A_2, \ldots, A_j$, where $A_i = k \cdot X_i + u_i$ for $1 \leq i \leq j$. If $c_3(X_i) = 3$ for any i, then $c_3(A_i) = 3 = c_3(A)$.*

Proof Let $X_i = X_i^{(1)} \cup X_i^{(2)} \cup X_i^{(3)}$, where for $1 \leq l \leq 3$, $X_i^{(l)} = 3 \cdot Z_i^l + v_l$, with $\{v_1, v_2, v_3\} = \{0, 1, 2\}$. Now

$$\begin{aligned}
A_i &= k \cdot X_i + u_i \\
&= (k \cdot X_i^{(1)} + u_i) \cup (k \cdot X_i^{(2)} + u_i) \cup (k \cdot X_i^{(3)} + u_i) \\
&= (3k \cdot Z_i^1 + kv_1 + u_i) \cup (3k \cdot Z_i^2 + kv_2 + u_i) \cup (3k \cdot Z_i^3 + kv_3 + u_i).
\end{aligned}$$

Since $\{v_1, v_2, v_3\}$ is a complete residue system modulo 3, hence $\{kv_1 + u_i, kv_2 + u_i, kv_3 + u_i\}$ is also a complete residue system modulo 3. So, $c_3(A_i) = 3$. But $A_i \subset A$, so we also have $c_3(A) = 3$. This completes the proof. □

Remark 3.3 It can be observed that, Lemma 3.2 holds not only for 3-components but it holds in general for q-components, where q is an odd prime different from k.

Lemma 3.4 *Let A be a nonempty finite set of integers satisfying $c_3(X_i) = 3$ for each $1 \leq i \leq |\hat{A}|$ and k be a prime number > 3. Then*

$$|3 \cdot A + k \cdot A| \geq (k+3)|A| - 9(k-1)!.$$

Proof Let t be the largest integer such that for each finite set X of integers

$$|3 \cdot X + k \cdot X| \geq (t+3)|A| - 9(t-1)!.$$

Let $t < k$. Then there is a set A satisfying $|3 \cdot A + k \cdot A| < (t+4)|A| - 9t!$. Without loss of generality, we may assume that $j = |\hat{A}| \geq 2$. Proposition 2.1 with Lemma 3.2 gives that $|3 \cdot A + k \cdot A| \geq (j+3)|A| - 3j$. So, $t \geq j$. We have,

$$|3 \cdot A + k \cdot A| = \sum_{i \in F} |3 \cdot A_i + k \cdot A| + \sum_{i \in E} |3 \cdot A_i + k \cdot A|. \tag{1}$$

Now, we estimate both the sums appearing in (1). We have

$$\begin{aligned}\sum_{i\in F}|3\cdot A_i+k\cdot A| &\geq \sum_{i\in F}|3\cdot A_i+k\cdot A_1|\\ &=\sum_{i\in F}|3\cdot X_i+k\cdot X_1|\\ (\text{from Proposition 2.1}) &\geq \sum_{i\in F}(c_3(X_1)|X_i|+c_k(X_i)|X_1|-c_3(X_1)c_k(X_i))\\ &=\sum_{i\in F}(3|A_i|+k|A_1|-3k)\\ (\text{since} k\geq t+1 \text{and} |A_1|\geq 3) &\geq 3|A_F|+(t+1)|F|(|A_1|-3)\\ (\text{since} t|F||A_1|\geq t|A_F|) &\geq (t+3)|A_F|+|F||A_1|-3(t+1)|F|.\end{aligned}\tag{2}$$

Next, we have

$$\begin{aligned}\sum_{i\in E}|3\cdot A_i+k\cdot A| &=\sum_{i\in E}(|3\cdot A_i+k\cdot A_i|+|\Delta_{ii}|)\\ &\geq \sum_{i\in E}((t+3)|A_i|-9(t-1)!)+\sum_{i\in E}|\Delta_{ii}|\\ (\text{from Lemma 3.1(ii)}) &\geq (t+3)|A_E|-9(t-1)!|E|+(|E|-1)|A_1|+|A_2|.\end{aligned}\tag{3}$$

Substituting (2) and (3) in (1), we get

$$\begin{aligned}|3\cdot A+k\cdot A| &\geq (t+3)|A|+(|E|+|F|-1)|A_1|+|A_2|-3(t+1)|F|-9(t-1)!|E|\\ &\geq (t+4)|A|-3(t+1)|F|-9(t-1)!|E|.\end{aligned}$$

Now using $|E|=j-|F|\leq t-|F|$, we have

$$\begin{aligned}3(t+1)|F|+9(t-1)!|E| &\leq 3(t+1)|F|+9(t-1)!(t-|F|)\\ &\leq 9t!+|F|(3(t+1)-9(t-1)!)\\ &\leq 9t!.\end{aligned}$$

Hence $|3\cdot A+k\cdot A|\geq (t+4)|A|-9t!$, which contradicts the choice of A. This completes the proof. □

Remark 3.5 It can be observed that, Lemma 3.4 holds not only for 3-components but it holds in general for q-components, where q is an odd prime $<k$. More precisely, we can see that "if A is a nonempty finite set of integers satisfying $c_q(X_i)=q$ for each $1\leq i\leq|\hat{A}|$ and k be a prime number $>q$, then $|q\cdot A+k\cdot A|\geq (k+q)|A|-3q(k-1)!$."

Theorem 3.6 *Let $k\geq 5$ be a prime number and A be a nonempty finite set of integers with $c_3(X_i)=3$ for each $1\leq i\leq|\hat{A}|$, where $\hat{A}$ is the projection of A on $\mathbb{Z}/k\mathbb{Z}$. Then for $|A|\geq 9|\hat{A}|^2(k-1)!$,*

$$|3\cdot A+k\cdot A|\geq (k+3)|A|-3k|\hat{A}|.$$

Proof Let $|\hat{A}|=j$. If $j=k$, then using Proposition 2.1, we have $|3\cdot A+k\cdot A|\geq (k+3)|A|-3k$. So, let $j<k$.

Case 1. $E=\{1\}$.

Using the definition of Δ_{11}, Lemma 3.1 (i), Lemma 3.4, and Proposition 2.1, we have

$$\begin{aligned}
|3\cdot A+k\cdot A| =&|3\cdot A_1+k\cdot A|+|3\cdot A_2+k\cdot A|+\sum_{3\le i\le j}|3\cdot A_i+k\cdot A|\\
\ge&|3\cdot A_1+k\cdot A_1|+|\Delta_{11}|+|3\cdot A_2+k\cdot A_1|\\
&+\sum_{3\le i\le j}|3\cdot A_i+k\cdot A_i|\\
\ge&(3+k)|A_1|-9(k-1)!+|A_2|+|3\cdot X_2+k\cdot X_1|\\
&+\sum_{3\le i\le j}((3+k)|A_i|-9(k-1)!)\\
\ge&(3+k)|A_1|-9(k-1)!+|A_2|+3|A_2|+k|A_1|-3k\\
&+\sum_{3\le i\le j}((3+k)|A_i|-9(k-1)!)\\
\ge&(3+k)|A|+|A_1|-9(j-1)(k-1)!-3k\\
\ge&(3+k)|A|+\frac{|A|}{j}-9j(k-1)!\\
\ge&(3+k)|A|,
\end{aligned}$$

since $|A|\ge 9j^2(k-1)!$.

Case 2. There exists $2\le s\le j$ with $s\in E$.

Using the definition of Δ_{ss}, Lemma 3.1 (i), and Lemma 3.4, we have

$$\begin{aligned}
|3\cdot A+k\cdot A| =&|3\cdot A_s+k\cdot A|+\sum_{i\neq s}|3\cdot A_i+k\cdot A|\\
\ge&|3\cdot A_s+k\cdot A_s|+|\Delta_{ss}|+\sum_{i\neq s}|3\cdot A_i+k\cdot A_i|\\
\ge&(3+k)|A_s|-9(k-1)!+|A_1|+\sum_{i\neq s}((3+k)|A_i|-9(k-1)!)\\
\ge&(3+k)|A|+\frac{|A|}{j}-9j(k-1)!\\
\ge&(3+k)|A|,
\end{aligned}$$

since $|A|\ge 9j^2(k-1)!$.

Case 3. $E=\emptyset$.

We have

$$\begin{aligned}
|3\cdot A+k\cdot A|\ge&\sum_{i\in F}|3\cdot A_i+k\cdot A_i|\\
=&\sum_{i\in F}|3\cdot X_i+k\cdot X_i|\\
\ge&\sum_{i\in F}(c_3(X_i)|X_i|+c_k(X_i)|X_i|-c_3(X_i)c_k(X_i))\\
=&\sum_{i\in F}((3+k)|A_i|-3k)\\
=&(3+k)|A|-3kj.
\end{aligned}$$

This completes the proof of the theorem. □

Remark 3.7 It can be observed that, Theorem 3.6 holds not only for 3-components but it holds in general for q-components, where q is an odd prime $<k$. More precisely, we can see that "if q and k are odd primes with $q<k$ and A is a nonempty finite set of integers with $c_q(X_i)=q$ for each $1\le i\le|\hat{A}|$, where $\hat{A}$ is the projection of A in $\mathbb{Z}/k\mathbb{Z}$, then for $|A|\ge 3q|\hat{A}|^2(k-1)!$,

$$|q \cdot A + k \cdot A| \geq (k+q)|A| - qk|\hat{A}|''.$$

Acknowledgements The authors wish to thank to the anonymous referee for his/her valuable remarks which considerably improved the paper.

References

1. Bhanja J., Chaudhary S., and Pandey R. K., On some direct and inverse results concerning sums of dilates, *Acta Arith.* **188**(2) (2019), 101–109.
2. Bukh B., Sums of dilates, *Combin. Probab. Comput.* **17** (2008), 627–639.
3. Cilleruelo J., Hamidoune Y. O., and Serra O., On sums of dilates, *Combin. Probab. Comput.* **18**(6) (2009), 871–880.
4. Cilleruelo J., Silva M., and Vinuesa C., A sumset problem, *J. Comb. Number Theory* **2**(1) (2010), 79–89.
5. Du S. S., Cao H. Q., and Sun Z. W., On a sumset problem for integers, *Electron. J. Comb.* **21**(1) (2014), #P1.13.
6. Freiman G. A., Herzog M., Longobardi P., Maj M., and Stanchescu Y. V., Direct and inverse problems in additive number theory and in non-abelian group theory, *European J. Combin.* **40** (2014), 42–54.
7. Hamidoune Y. O. and Rué J., A lower bound for the size of a Minkowski sum of dilates, *Combin. Probab. Comput.* **20**(2) (2011), 249–256.
8. Ljujić Ž., A lower bound for the size of a sum of dilates, *J. Comb. Number Theory* **5** (2013), 31–51.
9. Nathanson M. B., *Additive Number Theory: Inverse Problems and the Geometry of Sumsets,* Grad. texts in math. **165**, Springer-Verlag, New York, 1996.
10. Nathanson M. B., O'Bryant K., Orosz B., Ruzsa I., and Silva M., Binary linear forms over finite sets of integers, *Acta Arith.* **129**(4) (2007), 341–361.
11. Nathanson M. B., Inverse problems for linear forms over finite sets of integers, *J. Ramanujan Math. Soc.* **23**(2) (2008), 151–165.