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ORIGINAL RESEARCH

Loxodromes on non-degenerate helicoidal surfaces in Minkowski space–time

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Abstract In this article, we first study a class of helicoidal surfaces in Minkowski space–time $\mathbb{E}_1^4$. Next, we find the parametrizations of loxodromes on the non-degenerate helicoidal surfaces in $\mathbb{E}_1^4$. In addition, we give some results and examples by using Mathematica.

Keywords Loxodrome · Helicoidal surface · Minkowski space–time

Mathematics Subject Classification 53B25

1 Introduction

Loxodromes are special curves which meet each meridians on the Earth's surface at the same angle. Until today, a lot of authors studied on loxodromes on rotational surfaces and helicoidal surfaces in both Euclidean 3-space $\mathbb{E}^3$ and Minkowski 3-space $\mathbb{E}_1^3$ (see [1, 4–7, 10, 17, 18, 20, 22, 24]).

Rotational surfaces in Euclidean 4-space $\mathbb{E}^4$ was first introduced by Moore [19] in 1919. Afterwards, the rotational surfaces in $\mathbb{E}^4$ were studied by many authors (see [2, 3, 13, 25, 26]). Minkowski space–time $\mathbb{E}_1^4$ was proposed as a model for space–time in the theory of special relativity by Hermann Minkowski in 1907 (see for details [23]). Space-like and time-like rotational surfaces in $\mathbb{E}_1^4$ were studied by Dursun and Bektaş [12, 14]. Also, the definition and parametrization of a helicoidal surface in $\mathbb{E}^4$ were given by Hieu and Thang [16]. The parametrizations of loxodromes on helicoidal surfaces in $\mathbb{E}^4$ were investigated by Babaarslan [8]. Also, loxodromes on space-like and time-like rotational surfaces in $\mathbb{E}_1^4$ were studied by Babaarslan and Selvi [9] and Babaarslan and Gümüş [11], respectively.

In this present paper, we study a class of helicoidal surfaces which are generalizations of the rotational surfaces in $\mathbb{E}_1^4$. We find the parametrizations of loxodromes on the non-degenerate helicoidal surfaces in $\mathbb{E}_1^4$. Moreover, we give some results and some examples by using Mathematica.

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2 Preliminaries

In this section, we give the necessary definitions and formulas concerning curves and surfaces in Minkowski space–time $\mathbb{E}_1^4$. Also, we find the parametrizations of three types of helicoidal surfaces in $\mathbb{E}_1^4$.

Let $\mathbb{E}_1^4$ be Minkowski space–time with the Lorentzian metric

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_4 b_4,$$

where $\mathbf{a} = (a_1, a_2, a_3, a_4)$ and $\mathbf{b} = (b_1, b_2, b_3, b_4) \in \mathbb{E}_1^4$.

A vector $\mathbf{a} \in \mathbb{E}_1^4$ is said to be space-like, time-like or light-like if $\langle \mathbf{a}, \mathbf{a} \rangle > 0$, $\langle \mathbf{a}, \mathbf{a} \rangle < 0$ or $\langle \mathbf{a}, \mathbf{a} \rangle = 0$, respectively. Also, the pseudo norm of $\mathbf{a} \in \mathbb{E}_1^4$ is given by $\|\mathbf{a}\| = \sqrt{|\langle \mathbf{a}, \mathbf{a} \rangle|}$ and it is said to be a unit vector if $\|\mathbf{a}\| = 1$.

Let $\beta : I \to \mathbb{E}_1^4$ be a curve in $\mathbb{E}_1^4$, where $I \subset \mathbb{R}$ is an open interval. Then, the causal character of β is space-like, time-like or light-like if $\langle \beta', \beta' \rangle > 0$, $\langle \beta', \beta' \rangle < 0$ or $\langle \beta', \beta' \rangle = 0$, respectively (see [21]).

Let $S : U \to \mathbb{E}_1^4$ be a smooth immersed surface in $\mathbb{E}_1^4$, where $U \subset \mathbb{R}^2$ is an open set. S is space-like, time-like or light-like if its metric induced from Lorentzian metric is positive definite, non-degenerate but not positive definite or degenerate, respectively (see [15]).

We assume that $\beta : I \to H$ is a smooth curve in a hyperplane $H \subset \mathbb{E}_1^4$ and P is a plane in H. If β is rotated about P, then we have a rotational surface in $\mathbb{E}_1^4$. Also, we assume that when β rotates about P, it simultaneously translates along a line l parallel to P so that the speed of the translation is proportional to the speed of rotation. Then, we have a helicoidal surface in $\mathbb{E}_1^4$ (see [16] for the Euclidean case).

From this definition, we obtain three types of helicoidal surfaces in $\mathbb{E}_1^4$ as follows:

Helicoidal Surfaces of Type I.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard ortonormal basis of $\mathbb{E}_1^4$. We assume that H_1 is a hyperplane spanned by $\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4\}$, P_1 is a plane spanned by $\{\mathbf{e}_3, \mathbf{e}_4\}$ and l_1 is parallel to the axis spanned by $\{\mathbf{e}_4\}$. Then, the space-like rotation which leaves the time-like plane P_1 invariant is given by the following (space-like) rotational matrix

$$\begin{bmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad 0 \le v < 2\pi$$

[12, 14].

We consider the profile curve $\beta_1(u) = (x(u), 0, z(u), w(u))$ in H_1, where $u \in I \subset \mathbb{R}$ and $x(u) > 0$. Then, the parametrization of (the helicoidal surface of type I) M_1 is given by

$$X_1(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(u) \\ 0 \\ z(u) \\ w(u) \end{bmatrix} + cv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so

$$X_1(u, v) = (x(u) \cos v, x(u) \sin v, z(u), w(u) + cv), \tag{1}$$

where $c \in \mathbb{R}^+$. When w is a constant function, M_1 is called right helicoidal surface of type I. Also, when z is a constant function, M_1 is just a helicoidal surface in $\mathbb{E}_1^3$.

Helicoidal Surfaces of Type II.

We assume that H_2 is a hyperplane spanned by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4\}$, P_2 is a plane spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$ and l_2 is parallel to the axis spanned by $\{\mathbf{e}_1\}$. Then, the hyperbolic rotation which leaves the space-like plane P_2 invariant is given by the following (hyperbolic) rotational matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh v & \sinh v \\ 0 & 0 & \sinh v & \cosh v \end{bmatrix}, \quad v \in \mathbb{R}$$

[12, 14].

We consider the profile curve $\beta_2(u) = (x(u), y(u), 0, w(u))$ in H_2, where $u \in I \subset \mathbb{R}$ and $w(u) > 0$. Then, the parametrization of (the helicoidal surface of type II) M_2 is

$$X_2(u,v) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh v & \sinh v \\ 0 & 0 & \sinh v & \cosh v \end{bmatrix} \begin{bmatrix} x(u) \\ y(u) \\ 0 \\ w(u) \end{bmatrix} + cv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so

$$X_2(u,v) = (x(u) + cv, y(u), w(u)\sinh v, w(u)\cosh v), \tag{2}$$

where $c \in \mathbb{R}^+$. When x is a constant function, M_2 is called right helicoidal surface of type II. Also, when y is a constant function, M_2 is just a helicoidal surface in $\mathbb{E}^3_1$.

Helicoidal Surfaces of Type III.

Let us take the pseudo-orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \xi_3, \xi_4\}$ of $\mathbb{E}^4_1$ such that $\xi_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_4 - \mathbf{e}_3)$ and $\xi_4 = \frac{1}{\sqrt{2}}(\mathbf{e}_3 + \mathbf{e}_4)$. We assume that H_3 is a hyperplane spanned by $\{\mathbf{e}_1, \xi_3, \xi_4\}$, P_3 is a plane spanned by $\{\mathbf{e}_1, \xi_3\}$ and l_3 is parallel to the axis spanned by $\{\xi_3\}$. Then, the orthogonal transformation T_3 of $\mathbb{E}^4_1$ which leaves the light-like plane P_3 invariant is defined by $T_3(\mathbf{e}_1) = \mathbf{e}_1, T_3(\mathbf{e}_2) = \mathbf{e}_2 + \sqrt{2}v\xi_3,\ T_3(\xi_3) = \xi_3$ and $T_3(\xi_4) = \sqrt{2}v\mathbf{e}_2 + v^2\xi_3 + \xi_4$ ([12, 14]).

We consider the profile curve $\beta_3(u) = x(u)\mathbf{e}_1 + z(u)\xi_3 + w(u)\xi_4$ in H_3, where $u \in I \subset \mathbb{R}$ and $w(u) > 0$. Then, the parametrization of (the helicoidal surface of type III) M_3 can be given by

$$X_3(u,v) = x(u)\mathbf{e}_1 + \sqrt{2}vw(u)\mathbf{e}_2 + (z(u) + v^2w(u) + cv)\xi_3 + w(u)\xi_4, \tag{3}$$

where $c \in \mathbb{R}^+$. When w is a constant function, M_3 is called right helicoidal surface of type III.

Note that, when $c = 0$, the helicoidal surfaces above reduce to rotational surfaces in $\mathbb{E}^4_1$ (see [12, 14]).

The line element of a helicoidal surface M is given by

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2, \tag{4}$$

where $g_{11} = \langle X_u, X_u\rangle$, $g_{11} = g_{12} = \langle X_u, X_v\rangle$ and $g_{22} = \langle X_v, X_v\rangle$ are the coefficients of line element for the coordinates u, v in M. Besides, M is space-like iff $g_{11}g_{22} - g_{12}^2 > 0$ and it is time-like iff $g_{11}g_{22} - g_{12}^2 < 0$.

The arc-length of space-like or time-like curve on M between u_1 and u_2 can be given by

$$s = \left| \int_{u_1}^{u_2} \sqrt{\left| g_{11} + 2g_{12}\frac{dv}{du} + g_{22}\left(\frac{dv}{du}\right)^2 \right|} du \right| \tag{5}$$

(see [21]).

Also, a curve on a non-degenerate (space-like or time-like) helicoidal surface M in $\mathbb{E}^4_1$ is said to be a loxodrome if the curve meets each non-light-like meridians at the same Lorentzian angle.

3 Loxodromes on the space-like helicoidal surfaces

In this section, we find the parametrizations of loxodromes on space-like helicoidal surfaces of type I, II and III in $\mathbb{E}^4_1$, respectively.

Let's start by giving the following definition.

Definition 1 Let $\mathbf{a}$ and $\mathbf{b}$ be space-like vectors which span a space-like plane in $\mathbb{E}^4_1$. Then, the Lorentzian space-like angle $\psi \in [0, \pi]$ between $\mathbf{a}$ and $\mathbf{b}$ is given by $\langle \mathbf{a}, \mathbf{b}\rangle = \|\mathbf{a}\|\|\mathbf{b}\| \cos\psi$ [23].

3.1 Loxodromes on the space-like helicoidal surfaces of type I

We now consider the helicoidal surface M_1 parametrized by (1). We assume that $x'^2(u) + z'^2(u) - w'^2(u) = 1$, namely, β_1 is a unit speed curve.

The partial derivatives of $X_1(u, v)$ are given by

$$X_{1u}(u) = (x'(u)\cos v, x'(u)\sin v, z'(u), w'(u)),$$

and

$$X_{1v}(v) = (-x(u)\sin v, x(u)\cos v, 0, c).$$

By a direct computation, we have

$$\begin{aligned} g_{11} &= 1, \\ g_{12} &= -cw'(u), \\ g_{22} &= x^2(u) - c^2. \end{aligned}$$

M_1 is space-like iff $x^2(u) - c^2(x'^2(u) + z'^2(u)) > 0$ for all $u \in I \subset \mathbb{R}$.

Applying formula (4), we get

$$ds^2 = du^2 - 2cw'(u)dudv + \left(x^2(u) - c^2\right)dv^2.$$

Assume that the loxodrome is given by $\alpha(t) = X_1(u(t), v(t))$. At the point $p_1 = X_1(u, v)$, where the loxodrome meets the meridians at a constant angle ψ, one obtains

$$\cos\psi = \frac{du - cw'(u)dv}{\sqrt{du^2 - 2cw'(u)dudv + (x^2(u) - c^2)dv^2}}.$$

Hence, the differential equation of the loxodrome is given by

$$\left(\cos^2\psi\left(x^2(u) - c^2\right) - c^2w'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\sin^2\psi w'(u)\frac{dv}{du} = \sin^2\psi. \tag{6}$$

Consequently, the general solution of the equation above is

$$v = \int_{u_0}^{u} \frac{-2c\sin^2\psi w'(u) \pm \sqrt{\sin^2 2\psi(x^2(u) - c^2(w'^2(u) + 1))}}{2\cos^2\psi(x^2(u) - c^2) - 2c^2w'^2(u)}du. \tag{7}$$

Thus, we have the following theorem.

Theorem 1 *The parametrization of loxodrome on the space-like helicoidal surface of type I in* $\mathbb{E}^4_1$ *is*

$$\alpha(u) = (x(u)\cos v(u), x(u)\sin v(u), z(u), w(u) + cv(u)),$$

where $v(u)$ is given by Eq. (7).

3.2 Loxodromes on the space-like helicoidal surfaces of type II

We consider the helicoidal surface M_2 parametrized by (2):

$$X_2(u, v) = (x(u) + cv,\ y(u),\ w(u)\sinh v,\ w(u)\cosh v).$$

Then, we have

$$X_{2u}(u) = (x'(u),\ y'(u),\ w'(u)\sinh v,\ w'(u)\cosh v).$$

Note that, the causal character of $X_2(u)$ is same as $\beta_2(u)$ since

$$\langle X_{2u}(u), X_{2u}(u)\rangle = x'^2(u) + y'^2(u) - w'^2(u) = 1.$$

Also, we get

$$X_{2v}(v) = (c, 0, w(u)\cosh v, w(u)\sinh v).$$

Moreover, we have

$$\begin{aligned} g_{11} &= 1, \\ g_{12} &= cx'(u), \\ g_{22} &= c^2 + w^2(u). \end{aligned}$$

M_2 is space-like iff $w^2(u) + c^2(y'^2(u) - w'^2(u)) > 0$ for all $u \in I \subset \mathbb{R}$.

The line element of M_2 is

$$ds^2 = du^2 + 2cx'(u)dudv + \left(c^2 + w^2(u)\right)dv^2.$$

The angle ψ between the loxodrome and the meridian is given by

$$\cos\psi = \frac{du + cx'(u)dv}{\sqrt{du^2 + 2cx'(u)dudv + (c^2 + w^2(u))dv^2}}.$$

It follows from this formula, we have

$$\left(\cos^2\psi\left(c^2 + w^2(u)\right) - c^2x'^2(u)\right)\left(\frac{dv}{du}\right)^2 - 2c\sin^2\psi x'(u)\frac{dv}{du} = \sin^2\psi. \tag{8}$$

Consequently, we get

$$v = \int_{u_0}^{u} \frac{2c\sin^2\psi x'(u) \pm \sqrt{\sin^2 2\psi(w^2(u) - c^2(x'^2(u) - 1))}}{2\cos^2\psi(c^2 + w^2(u)) - 2c^2x'^2(u)} du. \tag{9}$$

As a result, we get the following theorem.

Theorem 2 *The parametrization of loxodrome on the space-like helicoidal surface of type II in* $\mathbb{E}_1^4$ *is*

$$\alpha(u) = (x(u) + cv(u),\ y(u),\ w(u)\sinh v(u),\ w(u)\cosh v(u)),$$

where $v(u)$ *is given by Eq.* (9).

3.3 Loxodromes on the space-like helicoidal surfaces of type III

We consider the helicoidal surface M_3 parametrized by (3). Therefore, the meridian is

$$X_3(u) = x(u)\mathbf{e}_1 + \sqrt{2}vw(u)\mathbf{e}_2 + (z(u) + v^2w(u) + cv)\xi_3 + w(u)\xi_4.$$

Since the meridian is space-like, we get

$$\langle X_{3u}(u), X_{3u}(u)\rangle = x'^2(u) - 2z'(u)w'(u) = 1.$$

Also, we have

$$\begin{aligned} g_{11} &= 1, \\ g_{12} &= -cw'(u), \\ g_{22} &= 2w^2(u). \end{aligned}$$

M_3 is space-like iff $2w^2(u) - c^2w'^2(u) > 0$ for all $u \in I \subset \mathbb{R}$.

The line element of M_3 is

$$ds^2 = du^2 - 2cw'(u)dudv + 2w^2(u)dv^2.$$

At the point $p_3 = X_3(u, v)$, we have

$$\cos\psi = \frac{du - cw'(u)dv}{\sqrt{du^2 - 2cw'(u)dudv + 2w^2(u)dv^2}}.$$

From this equation, we arrive

$$\left(2\cos^2\psi w^2(u) - c^2 w'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\sin^2\psi w'(u)\frac{dv}{du} = \sin^2\psi. \tag{10}$$

Its general solution is

$$v = \int_{u_0}^{u} \frac{-2c\sin^2\psi w'(u) \pm \sqrt{\sin^2 2\psi(2w^2(u) - c^2 w'^2(u))}}{4\cos^2\psi w^2(u) - 2c^2 w'^2(u)} du. \tag{11}$$

So, we get the following theorem.

Theorem 3 *The parametrization of loxodrome on the space-like helicoidal surface of type III in $\mathbb{E}^4_1$ is*

$$\alpha(u) = x(u)\mathbf{e}_1 + \sqrt{2}v(u)w(u)\mathbf{e}_2 + (z(u) + v^2(u)w(u) + cv(u))\xi_3 + w(u)\xi_4,$$

where $v(u)$ is given by Eq. (11).

As a result, we have

Corollary 1 *The arc-length of the loxodromes on space-like right helicoidal surfaces in $\mathbb{E}^4_1$ is given by*

$$s = \left|\frac{u_2 - u_1}{\cos\psi}\right|.$$

Also, we have the following remark.

Remark 1 If we take $c = 0$ in the equations (6), (8) and (10), respectively, we arrive the differential equations of loxodromes on the space-like rotational surfaces in $\mathbb{E}^4_1$ (see [9]).

4 Space-like loxodromes on the time-like helicoidal surfaces which have space-like meridians

In this section, we find the parametrizations of space-like loxodromes on the time-like helicoidal surfaces of type I, II and III which have space-like meridians in $\mathbb{E}^4_1$, respectively.

We give the following definition for later use.

Definition 2 Let $\mathbf{a}$ and $\mathbf{b}$ be space-like vectors which span a time-like plane in $\mathbb{E}^4_1$. Then, the Lorentzian time-like angle $\eta \in \mathbb{R}^+$ between $\mathbf{a}$ and $\mathbf{b}$ is given by $|\langle \mathbf{a}, \mathbf{b}\rangle| = \|\mathbf{a}\|\|\mathbf{b}\|\cosh\eta$ [23].

4.1 Space-like loxodromes on the time-like helicoidal surfaces of type I which have space-like meridians

We consider the helicoidal surface M_1 parametrized by (1). The coefficients of line element of M_1 are same as that are given in the subsection (3.1).

M_1 is time-like iff $x^2(u) - c^2(x'^2(u) + z'^2(u)) < 0$ for all $u \in I \subset \mathbb{R}$.

The angle η between the loxodrome and the meridian is expressed by

$$\pm\cosh\eta = \frac{du - cw'(u)dv}{\sqrt{du^2 - 2cw'(u)dudv + (x^2(u) - c^2)dv^2}}.$$

Using this equation, we have the following equation

$$\left(-\cosh^2\eta\left(x^2(u) - c^2\right) + c^2 w'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\sinh^2\eta w'(u)\frac{dv}{du} = \sinh^2\eta \tag{12}$$

whose general solution is

$$v = \int_{u_0}^{u} \frac{-2c\sinh^2\eta w'(u) \pm \sqrt{\sinh^2 2\eta(-x^2(u) + c^2(w'^2(u) + 1))}}{-2\cosh^2\eta(x^2(u) - c^2) + 2c^2 w'^2(u)} du. \tag{13}$$

Thus, the following theorem is obtained.

Theorem 4 *The parametrization of space-like loxodrome on the time-like helicoidal surface of type I having space-like meridians in $\mathbb{E}_1^4$ is*

$$\gamma(u) = (x(u)\cos v(u), x(u)\sin v(u), z(u), w(u) + cv(u)),$$

where $v(u)$ is given by Eq. (13).

4.2 Space-like loxodromes on the time-like helicoidal surfaces of type II which have space-like meridians

We consider the helicoidal surface M_2 parametrized by (2). The coefficients of line element of M_2 are same as that are given in the subsection (3.2).

M_2 is time-like iff $w^2(u) + c^2(y'^2(u) - w'^2(u)) < 0$ for all $u \in I \subset \mathbb{R}$.

At the point $p_2 = X_2(u, v)$, we have

$$\pm\cosh\eta = \frac{du + cx'(u)dv}{\sqrt{du^2 + 2cx'(u)dudv + (c^2 + w^2(u))dv^2}}.$$

From this formula, we get

$$\left(-\cosh^2\eta\left(c^2 + w^2(u)\right) + c^2 x'^2(u)\right)\left(\frac{dv}{du}\right)^2 - 2c\sinh^2\eta x'(u)\frac{dv}{du} = \sinh^2\eta. \tag{14}$$

Thus, we find

$$v = \int_{u_0}^{u} \frac{2c\sinh^2\eta x'(u) \pm \sqrt{\sinh^2 2\eta(-w^2(u) + c^2(x'^2(u) - 1))}}{-2\cosh^2\eta(c^2 + w^2(u)) + 2c^2 x'^2(u)} du. \tag{15}$$

As a result, we have

Theorem 5 *The parametrization of space-like loxodrome on the time-like helicoidal surface of type II having space-like meridians in $\mathbb{E}_1^4$ is*

$$\gamma(u) = (x(u) + cv(u),\ y(u),\ w(u)\sinh v(u),\ w(u)\cosh v(u)),$$

where $v(u)$ is given by Eq. (15).

4.3 Space-like loxodromes on the time-like helicoidal surfaces of type III which have space-like meridians

We consider the helicoidal surface M_3 parametrized by (3). The coefficients of line element of M_3 are same as that are given in the subsection (3.3).

M_3 is time-like iff $2w^2(u) - c^2 w'^2(u) < 0$ for all $u \in I \subset \mathbb{R}$.

Similarly, we have

$$\pm\cosh\eta = \frac{du - cw'(u)dv}{\sqrt{du^2 - 2cw'(u)dudv + 2w^2(u)dv^2}}.$$

By using this equation, we arrive

$$\left(-2\cosh^2\eta w^2(u) + c^2 w'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\sinh^2\eta w'(u)\frac{dv}{du} = \sinh^2\eta. \tag{16}$$

So, we have

$$v = \int_{u_0}^{u} \frac{-2c\sinh^2\eta w'(u) \pm \sqrt{\sinh^2 2\eta(-2w^2(u) + c^2w'^2(u))}}{-4\cosh^2\eta w^2(u) + 2c^2w'^2(u)} du. \tag{17}$$

As a result, we get

Theorem 6 *The parametrization of space-like loxodrome on the time-like helicoidal surface of type III having space-like meridians in $\mathbb{E}_1^4$ is*

$$\gamma(u) = x(u)\mathbf{e}_1 + \sqrt{2}v(u)w(u)\mathbf{e}_2 + (z(u) + v^2(u)w(u) + cv(u))\xi_3 + w(u)\xi_4,$$

where $v(u)$ is given by Eq. (17).

5 Space-like loxodromes on the time-like helicoidal surfaces which have time-like meridians

In this section, we find the parametrizations of space-like loxodromes on the time-like helicoidal surfaces of type I, II and III having time-like meridians in $\mathbb{E}_1^4$, respectively.

Firstly, we give the following definition.

Definition 3 Let $\mathbf{a}$ and $\mathbf{b}$ be space-like and time-like vector in $\mathbb{E}_1^4$, respectively. Then, the Lorentzian time-like angle $\varphi \in \mathbb{R}^+ \cup \{0\}$ between $\mathbf{a}$ and $\mathbf{b}$ is given by $|\langle \mathbf{a}, \mathbf{b}\rangle| = \|\mathbf{a}\|\|\mathbf{b}\| \sinh\varphi$ [23].

5.1 Space-like loxodromes on the time-like helicoidal surfaces of type I which have time-like meridians

We consider the helicoidal surface of M_1 parametrized by (1). So, the meridian is

$$X_1(u) = (x(u)\cos v, x(u)\sin v, z(u), w(u) + cv),$$

and it is time-like iff $x'^2(u) + z'^2(u) - w'^2(u) = -1$ for all $u \in I \subset \mathbb{R}$.

Also, we have

$$\begin{aligned} g_{11} &= -1, \\ g_{12} &= -cw'(u), \\ g_{22} &= x^2(u) - c^2. \end{aligned}$$

Note that, M_1 is time-like since $-x^2(u) - c^2(x'^2(u) + z'^2(u)) < 0$ for all $u \in I \subset \mathbb{R}$.

The line element of M_1 is

$$ds^2 = -du^2 - 2cw'(u)dudv + \left(x^2(u) - c^2\right)dv^2.$$

At the point $p_1 = X_1(u, v)$, we have

$$\pm\sinh\varphi = \frac{-du - cw'(u)dv}{\sqrt{-du^2 - 2cw'(u)dudv + (x^2(u) - c^2)dv^2}}.$$

From here, we find the equation

$$\left(\sinh^2\varphi\left(x^2(u) - c^2\right) - c^2w'^2(u)\right)\left(\frac{dv}{du}\right)^2 - 2c\cosh^2\varphi w'(u)\frac{dv}{du} = \cosh^2\varphi \tag{18}$$

whose solution is

$$v = \int_{u_0}^{u} \frac{2c\cosh^2\varphi w'(u) \pm \sqrt{\sinh^2 2\varphi(x^2(u) + c^2(w'^2(u) - 1))}}{2\sinh^2\varphi(x^2(u) - c^2) - 2c^2w'^2(u)} du. \tag{19}$$

Thus, we have the following theorem.

Theorem 7 *The parametrization of space-like loxodrome on the time-like helicoidal surface of type I which have time-like meridians in $\mathbb{E}_1^4$ is*

$$\sigma(u) = (x(u)\cos v(u), x(u)\sin v(u), z(u), w(u) + cv(u)),$$

where $v(u)$ is given by Eq. (19).

5.2 Space-like loxodromes on the time-like helicoidal surfaces of type II which have time-like meridians

We consider the helicoidal surface M_2 parametrized by (2). So, the meridian is

$$X_2(u) = (x(u) + cv,\ y(u),\ w(u)\sinh v,\ w(u)\cosh v).$$

Since it is time-like, we get

$$\langle X_{2u}(u), X_{2u}(u)\rangle = x'^2(u) + y'^2(u) - w'^2(u) = -1.$$

Also, we obtain

$$\begin{aligned} g_{11} &= -1,\\ g_{12} &= cx'(u),\\ g_{22} &= c^2 + w^2(u). \end{aligned}$$

M_2 is time-like since $-w^2(u) - c^2(1 + x'^2(u)) < 0$ for all $u \in I \subset \mathbb{R}$.
The line element of M_2 is

$$ds^2 = -du^2 + 2cx'(u)dudv + \left(c^2 + w^2(u)\right)dv^2.$$

Also, we have

$$\pm\sinh\varphi = \frac{-du + cx'(u)dv}{\sqrt{-du^2 + 2cx'(u)dudv + (c^2 + w^2(u))dv^2}}.$$

Using this equation, we obtain that

$$\left(\sinh^2\varphi\left(c^2 + w^2(u)\right) - c^2x'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\cosh^2\varphi x'(u)\frac{dv}{du} = \cosh^2\varphi. \tag{20}$$

Therefore, we find that

$$v = \int_{u_0}^{u}\frac{2c\cosh^2\varphi x'(u) \pm \sqrt{\sinh^2 2\varphi(w^2(u) + c^2(x'^2(u) + 1))}}{-2\sinh^2\varphi(c^2 + w^2(u)) + 2c^2x'^2(u)}du. \tag{21}$$

As a result, we have

Theorem 8 *The parametrization of space-like loxodrome on the time-like helicoidal surface of type II having time-like meridians in $\mathbb{E}^4_1$ is*

$$\sigma(u) = (x(u) + cv(u),\ y(u),\ w(u)\sinh v(u),\ w(u)\cosh v(u)),$$

where $v(u)$ is given by Eq. (21).

5.3 Space-like loxodromes on the time-like helicoidal surfaces of type III which have time-like meridians

We consider the helicoidal surface of M_3 parametrized by (3). So, the meridian is

$$X_3(u) = x(u)\mathbf{e}_1 + \sqrt{2}vw(u)\mathbf{e}_2 + (z(u) + v^2w(u) + cv)\xi_3 + w(u)\xi_4,$$

and it is time-like iff $x'^2(u) - 2z'(u)w'(u) = -1$ for all $u \in I \subset \mathbb{R}$.
Furthermore, we have

$$g_{11} = -1,$$
$$g_{12} = -cw'(u),$$
$$g_{22} = 2w^2(u).$$

M_3 is time-like since $-2w^2(u) - c^2w'^2(u) < 0$ for all $u \in I \subset \mathbb{R}$.

Also, the line element of M_3 is

$$ds^2 = -du^2 - 2cw'(u)dudv + 2w^2(u)dv^2.$$

Similarly, we get

$$\pm \sinh \varphi = \frac{-du - cw'(u)dv}{\sqrt{-du^2 - 2cw'(u)dudv + 2w^2(u)dv^2}}.$$

From here, we find the equation

$$\left(2\sinh^2 \varphi w^2(u) - c^2w'^2(u)\right)\left(\frac{dv}{du}\right)^2 - 2c\cosh^2 \varphi w'(u)\frac{dv}{du} = \cosh^2 \varphi \tag{22}$$

whose general solution is

$$v = \int_{u_0}^{u} \frac{2c\cosh^2 \varphi w'(u) \pm \sqrt{\sinh^2 2\varphi(2w^2(u) + c^2w'^2(u))}}{4\sinh^2 \varphi w^2(u) - 2c^2w'^2(u)} du. \tag{23}$$

As a result, we get the following theorem.

Theorem 9 *The parametrization of space-like loxodrome on the time-like helicoidal surface of type III having time-like meridians in* $\mathbb{E}_1^4$ *is*

$$\sigma(u) = x(u)\mathbf{e}_1 + \sqrt{2}v(u)w(u)\mathbf{e}_2 + (z(u) + v^2(u)w(u) + cv(u))\xi_3 + w(u)\xi_4,$$

where $v(u)$ *is given by Eq.* (23).

Also, we have

Corollary 2 *The arc-length of the space-like loxodromes on time-like right helicoidal surfaces which have time-like meridians in* $\mathbb{E}_1^4$ *is given by*

$$s = \left|\frac{u_2 - u_1}{\sinh \varphi}\right|.$$

In addition, we have the following remark.

Remark 2 If we take $c = 0$ in the equations (18), (20) and (22), respectively, we arrive the differential equations of space-like loxodromes on the time-like rotational surfaces in $\mathbb{E}_1^4$ (see [11]).

6 Time-like loxodromes on the time-like helicoidal surfaces which have space-like meridians

In this section, we find the parametrizations of time-like loxodromes on the time-like helicoidal surfaces of type I, II and III which have space-like meridians in $\mathbb{E}_1^4$, respectively.

6.1 Time-like loxodromes on the time-like helicoidal surfaces of type I which have space-like meridians

We consider the helicoidal surface of M_1 parametrized by (1). The coefficients of line element of M_1 are same as that are given in the subsection (3.1).

M_1 is time-like iff $x^2(u) - c^2(x'^2(u) + z'^2(u)) < 0$ for all $u \in I \subset \mathbb{R}$.

At the point $p_1 = X_1(u, v)$, we get

$$\pm \sinh \varphi = \frac{du - cw'(u)dv}{\sqrt{-du^2 + 2cw'(u)dudv - (x^2(u) - c^2)dv^2}}.$$

From the equation above, we have the following

$$\left(\sinh^2 \varphi\left(x^2(u) - c^2\right) + c^2 w'^2(u)\right)\left(\frac{dv}{du}\right)^2 - 2c\cosh^2 \varphi w'(u)\frac{dv}{du} = -\cosh^2 \varphi. \tag{24}$$

Consequently, the general solution is

$$v = \int_{u_0}^{u} \frac{2c\cosh^2 \varphi w'(u) \pm \sqrt{\sinh^2 2\varphi(-x^2(u) + c^2(w'^2(u) + 1))}}{2\sinh^2 \varphi(x^2(u) - c^2) + 2c^2 w'^2(u)} du. \tag{25}$$

Thus, we have the following theorem.

Theorem 10 *The parametrization of time-like loxodrome on the time-like helicoidal surface of type I having space-like meridians in $\mathbb{E}_1^4$ is*

$$\delta(u) = (x(u)\cos v(u), x(u)\sin v(u), z(u), w(u) + cv(u)),$$

where $v(u)$ is given by Eq. (25).

6.2 Time-like loxodromes on the time-like helicoidal surfaces of type II which have space-like meridians

We consider the helicoidal surface of M_2 parametrized by (2). The coefficients of line element of M_2 are same as that are given in the subsection (3.2).

M_2 is time-like iff $w^2(u) + c^2(y'^2(u) - w'^2(u)) < 0$ for all $u \in I \subset \mathbb{R}$.

At the point $p_2 = X_2(u, v)$, we get

$$\pm \sinh \varphi = \frac{du + cx'(u)dv}{\sqrt{-du^2 - 2cx'(u)dudv - (c^2 + w^2(u))dv^2}}.$$

From the formula above, we have

$$\left(\sinh^2 \varphi\left(c^2 + w^2(u)\right) + c^2 x'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\cosh^2 \varphi x'(u)\frac{dv}{du} = -\cosh^2 \varphi. \tag{26}$$

Thus, the general solution is

$$v = \int_{u_0}^{u} \frac{-2c\cosh^2 \varphi x'(u) \pm \sqrt{\sinh^2 2\varphi(-w^2(u) + c^2(x'^2(u) - 1))}}{2\sinh^2 \varphi(c^2 + w^2(u)) + 2c^2 x'^2(u)} du. \tag{27}$$

Thus, we get the following theorem.

Theorem 11 *The parametrization of time-like loxodrome on the time-like helicoidal surface of type II having space-like meridians in $\mathbb{E}_1^4$ is*

$$\delta(u) = (x(u) + cv(u),\ y(u),\ w(u)\sinh v(u),\ w(u)\cosh v(u)),$$

where $v(u)$ is given by Eq. (27).

6.3 Time-like loxodromes on the time-like helicoidal surfaces of type III which have space-like meridians

We consider the helicoidal surface of M_3 parametrized by (3). The coefficients of line element of M_3 are same as that are given in the subsection (3.3).

One obtains that

$$\pm \sinh \varphi = \frac{du - cw'(u)dv}{\sqrt{-du^2 + 2cw'(u)dudv - 2w^2(u)dv^2}}.$$

From this equation, we arrive the equation

$$\left(-2\sinh^2 \varphi w^2(u) - c^2 w'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\cosh^2 \varphi w'(u)\frac{dv}{du} = \cosh^2 \varphi \tag{28}$$

whose general solution is

$$v = \int_{u_0}^{u} \frac{-2c\cosh^2 \varphi w'(u) \pm \sqrt{\sinh^2 2\varphi(-2w^2(u) + c^2 w'^2(u))}}{-4\sinh^2 \varphi w^2(u) - 2c^2 w'^2(u)} du. \tag{29}$$

Thus, we have

Theorem 12 *The parametrization of time-like loxodrome on the time-like helicoidal surface of type III having space-like meridians in* $\mathbb{E}^4_1$ *is*

$$\delta(u) = x(u)\mathbf{e}_1 + \sqrt{2}v(u)w(u)\mathbf{e}_2 + (z(u) + v^2(u)w(u) + cv(u))\xi_3 + w(u)\xi_4,$$

where $v(u)$ *is given by Eq.* (29).

7 Time-like loxodromes on the time-like helicoidal surfaces which have time-like meridians

In this section, first we introduce the following definition. Next we give the parametrizations of time-like loxodromes on the time-like helicoidal surfaces of type I, II and III which have time-like meridians in $\mathbb{E}^4_1$, respectively.

Definition 4 Let $\mathbf{a}$ and $\mathbf{b}$ be positive (negative) time-like vectors in $\mathbb{E}^4_1$. Then, the Lorentzian time-like angle $\theta \in \mathbb{R}^+$ between $\mathbf{a}$ and $\mathbf{b}$ is given by $\langle \mathbf{a}, \mathbf{b} \rangle = -\|\mathbf{a}\|\|\mathbf{b}\| \cosh\theta$ [23].

7.1 Time-like loxodromes on the time-like helicoidal surfaces of type I which have time-like meridians

We consider the helicoidal surface of M_1 parametrized by (1). The coefficients of line element of M_1 are same as that are given in the subsection (5.1).

The angle θ between the time-like loxodrome and time-like meridian is given by

$$-\cosh\theta = \frac{-du - cw'(u)dv}{\sqrt{du^2 + 2cw'(u)dudv - (x^2(u) - c^2)dv^2}}.$$

From the equation above, we find the following differential equation

$$\left(\cosh^2 \theta\left(x^2(u) - c^2\right) + c^2 w'^2(u)\right)\left(\frac{dv}{du}\right)^2 - 2c\sinh^2 \theta w'(u)\frac{dv}{du} = \sinh^2 \theta \tag{30}$$

whose general solution is

$$v = \int_{u_0}^{u} \frac{2c\sinh^2 \theta w'(u) \pm \sqrt{\sinh^2 2\theta(x^2(u) + c^2(w'^2(u) - 1))}}{2\cosh^2 \theta(x^2(u) - c^2) + 2c^2 w'^2(u)} du. \tag{31}$$

Thus, we have the following theorem.

Theorem 13 *The parametrization of time-like loxodrome on the time-like helicoidal surface of type I having time-like meridians in* $\mathbb{E}^4_1$ *is*

$$\zeta(u) = (x(u)\cos v(u), x(u)\sin v(u), z(u), w(u) + cv(u)),$$

where $v(u)$ *is given by Eq.* (31).

7.2 Time-like loxodromes on the time-like helicoidal surfaces of type II which have time-like meridians

We consider the helicoidal surface of M_2 parametrized by (2). The coefficients of line element of M_2 are same as that are given in the subsection (5.2).

At the point $p_2 = X_2(u, v)$, we have

$$-\cosh\theta = \frac{-du + cx'(u)dv}{\sqrt{du^2 - 2cx'(u)dudv - (c^2 + w^2(u))dv^2}}.$$

Using the equation above, we get the following

$$\left(\cosh^2\theta\left(c^2 + w^2(u)\right) + c^2x'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\sinh^2\theta x'(u)\frac{dv}{du} = \sinh^2\theta. \tag{32}$$

Therefore, we find the general solution as

$$v = \int_{u_0}^{u} \frac{-2c\sinh^2\theta x'(u) \pm \sqrt{\sinh^2 2\theta(w^2(u) + c^2(x'^2(u) + 1))}}{2\cosh^2\theta(c^2 + w^2(u)) + 2c^2x'^2(u)} du. \tag{33}$$

As a result, we have

Theorem 14 *The parametrization of time-like loxodrome on the time-like helicoidal surface of type II having time-like meridians in* $\mathbb{E}_1^4$ *is*

$$\zeta(u) = (x(u) + cv(u),\ y(u),\ w(u)\sinh v(u),\ w(u)\cosh v(u)),$$

where $v(u)$ *is given by Eq.* (33).

7.3 Time-like loxodromes on the time-like helicoidal surfaces of type III which have time-like meridians

We consider the helicoidal surface of M_3 parametrized by (3). The coefficients of line element of M_3 are same as that are given in the subsection (5.3).

Similarly, at the point $p_3 = X_3(u, v)$, we get

$$-\cosh\theta = \frac{-du - cw'(u)dv}{\sqrt{du^2 + 2cw'(u)dudv - 2w^2(u)dv^2}}.$$

From this equation, we have the equation

$$\left(-2\cosh^2\theta w^2(u) - c^2w'^2(u)\right)\left(\frac{dv}{du}\right)^2 + 2c\sinh^2\theta w'(u)\frac{dv}{du} = -\sinh^2\theta \tag{34}$$

whose general solution is

$$v = \int_{u_0}^{u} \frac{-2c\sinh^2\theta w'(u) \pm \sqrt{\sinh^2 2\theta(2w^2(u) + c^2w'^2(u))}}{-4\cosh^2\theta w^2(u) - 2c^2w'^2(u)} du. \tag{35}$$

Thus, we get the following theorem.

Theorem 15 *The parametrization of time-like loxodrome on the time-like helicoidal surface of type III having time-like meridians in* $\mathbb{E}_1^4$ *is*

$$\zeta(u) = x(u)\mathbf{e}_1 + \sqrt{2}v(u)w(u)\mathbf{e}_2 + (z(u) + v^2(u)w(u) + cv(u))\xi_3 + w(u)\xi_4,$$

where $v(u)$ *is given by Eq.* (35).

Also, we have

Corollary 3 *The arc-length of the time-like loxodromes on time-like right helicoidal surfaces having time-like meridians in* $\mathbb{E}_1^4$ *is given by*

$$s = \left|\frac{u_2 - u_1}{\cosh\theta}\right|.$$

Finally, we have the following remark.

Remark 3 If we take $c = 0$ in the equations (30), (32) and (34), respectively, we arrive the differential equations of time-like loxodromes on the time-like rotational surfaces in $\mathbb{E}_1^4$ (see [11]).

8 Visualization

In this section, we give some examples by using Mathematica computer programme.

Example 1 We consider the following unit-speed space-like profile curve:

$$\beta_1(u) = \left(\sinh\frac{u}{2}, 0, \frac{\sqrt{3}}{2}u, \cosh\frac{u}{2}\right).$$

When take $c = \frac{1}{2}$, we have the following parametrization of helicoidal surface M_1:

$$X_1(u,v) = \left(\sinh\frac{u}{2}\cos v, \sinh\frac{u}{2}\sin v, \frac{\sqrt{3}}{2}u, \cosh\frac{u}{2} + \frac{v}{2}\right).$$

If we take $u \in (1,3)$, then M_1 is a space-like helicoidal surface.

Taking $u_0 = 2$ and $\psi = \frac{\pi}{2}$, we find $v(u) \cong 8\ln(2.16395\tanh\frac{u}{2})$. Thus, $v \in (-5.07915, 2.54432)$.

Then, the parametrization of space-like loxodrome is

$$\alpha(u) = \left(\sinh\frac{u}{2}\cos v(u), \sinh\frac{u}{2}\sin v(u), \frac{\sqrt{3}}{2}u, \cosh\frac{u}{2} + \frac{v(u)}{2}\right),$$

where $v(u) \cong 8\ln(2.16395\tanh\frac{u}{2})$.

The arc-length of the space-like loxodrome is approximately equal to 6.43309.

The graphs of the projections of space-like helicoidal surface M_1, loxodrome and meridian ($v = 0$) in $\mathbb{E}_1^3$ can be drawn by using Mathematica plotting command as follows (Fig. 1):

$$\textit{Parametric Plot 3D}\ [\{x(u,v) + y(u,v), z(u,v), w(u,v)\}, \{u,a,b\}, \{v,c,d\}].$$

Example 2 We consider the following unit-speed time-like profile curve:

$$\beta_2(u) = \left(1, u, 0, \sqrt{2}u\right).$$

When take $c = \sqrt{2}$, we have the following parametrization of time-like right helicoidal surface M_2:

$$X_2(u,v) = \left(1 + \sqrt{2}v, u, \sqrt{2}u\sinh v, \sqrt{2}u\cosh v\right).$$

Taking $u_0 = 0$ and $\varphi = 1$, we find $v(u) = \frac{\sqrt{2}}{2}\coth 1\,\mathrm{arcsinh}\,u$. If $u \in (-2,2)$, then $v \in (-1.34035, 1.34035)$.

Then, the parametrization of space-like loxodrome is

$$\sigma(u) = \left(1 + \sqrt{2}v(u), u, \sqrt{2}u\sinh v(u), \sqrt{2}u\cosh v(u)\right),$$

where $v(u) = \frac{\sqrt{2}}{2}\coth 1\,\mathrm{arcsinh}\,u$

The arc-length of the space-like loxodrome is approximately equal to 3.40367.

The graphs of the projections of time-like right helicoidal surface M_2, space-like loxodrome and time-like meridian ($v = 0$) in $\mathbb{E}_1^3$ can be drawn by using Mathematica plotting command as follows:

$$\textit{Parametric Plot 3D}\ [\{x(u,v), y(u,v) + z(u,v), w(u,v)\}, \{u,a,b\}, \{v,c,d\}].$$

Example 3 We consider the following unit-speed time-like profile curve:

$$\beta_3(u) = 2\sqrt{e^u}\mathbf{e}_1 + (u - e^{-u})\xi_3 + \frac{1}{2}e^u\xi_4$$

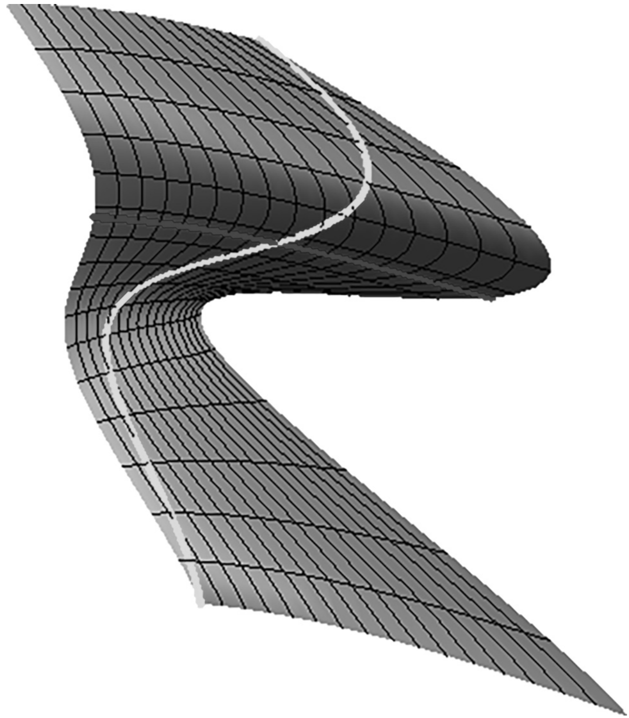

Fig. 1 The projections of space-like helicoidal surface, loxodrome (green), meridian (blue). (Color figure online)

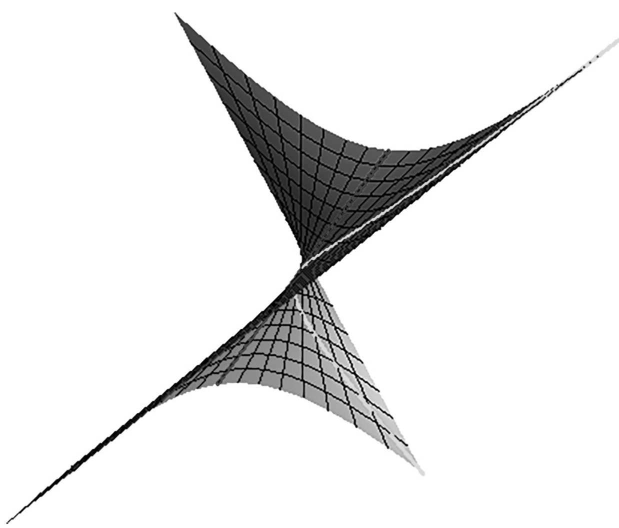

Fig. 2 The projections of time-like right helicoidal surface, space-like loxodrome (green), time-like meridian (blue). (Color figure online)

(see [11]).

When take $c = 1$, we have the following parametrization of time-like helicoidal surface M_3:

$$X_3(u, v) = 2\sqrt{e^u}\mathbf{e}_1 + \frac{\sqrt{2}}{2}ve^u\mathbf{e}_2 + (u - e^{-u} + \frac{1}{2}v^2e^u + v)\xi_3 + \frac{1}{2}e^u\xi_4.$$

Taking $u_0 = 0$ and $\theta = 2$, we find $v(u) \cong 0.715133(e^{-u} - 1)$. If $u \in (-2, 2)$, then $v \in (-0.61835, 4.56903)$.

Then, the parametrization of time-like loxodrome is

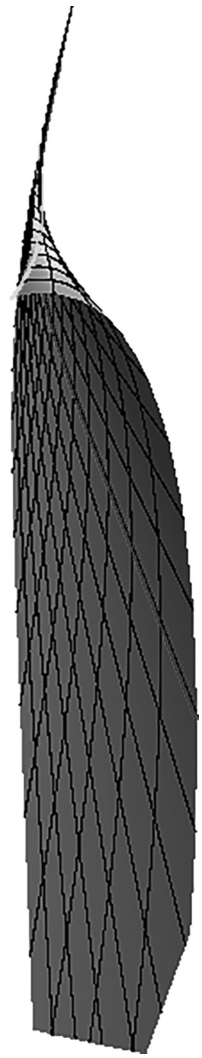

Fig. 3 The projections of time-like helicoidal surface, time-like loxodrome (green), time-like meridian (blue). (Color figure online)

$$\zeta(u) = 2\sqrt{e^u}\mathbf{e}_1 + \frac{\sqrt{2}}{2}v(u)e^u\mathbf{e}_2 + (u - e^{-u} + \frac{1}{2}v^2(u)e^u + v(u))\xi_3 + \frac{1}{2}e^u\xi_4,$$

where $v(u) \cong 0.715133(e^{-u} - 1)$.

The arc-length of the time-like loxodrome is approximately equal to 0.116636.

The graphs of the projections of time-like helicoidal surface M_3, time-like loxodrome and time-like meridian ($v = 2$) in $\mathbb{E}_1^3$ can be drawn by using Mathematica plotting command as follows (Fig. 3):

$$Parametric\,Plot\,3D[\{x(u,v) + y(u,v), z(u,v), w(u,v)\}, \{u,a,b\}, \{v,c,d\}].$$

9 Conclusion

Until today, a lot of authors studied on loxodromes on rotational surfaces and helicoidal surfaces in both $\mathbb{E}^3$ and $\mathbb{E}_1^3$ (see [1, 4–7, 10, 17, 18, 20, 22, 24]). Minkowski space–time $\mathbb{E}_1^4$ was proposed as a model for space–time in the theory of special relativity by Hermann Minkowski in 1907 (see [23]). The parametrizations of loxodromes on the helicoidal surfaces in $\mathbb{E}^4$ were investigated by Babaarslan [8]. Also, loxodromes on space-like and time-like rotational surfaces in $\mathbb{E}_1^4$ were studied by Babaarslan and Selvi [9] and Babaarslan and Gümüş [11], respectively.

In this article, we first obtain three types of helicoidal surfaces in $\mathbb{E}_1^4$. Afterwards, we find the parametrizations of loxodromes on the non-degenerate helicoidal surfaces in $\mathbb{E}_1^4$. Also, we give some results and examples by using Mathematica. Although the general solutions of loxodromes on helicoidal surfaces of type I and type II respectively in $\mathbb{E}_1^4$ look like the case of $\mathbb{E}_1^3$, the importance of the paper is that it generalizes the parametrizations of loxodromes on helicoidal surfaces in $\mathbb{E}_1^3$ to the case of Minkowski space–time $\mathbb{E}_1^4$.

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References

1. J. Alexander, A Rhumb Way to Go, *Math. Mag.,* **77** (2004), 349–356.
2. K. Arslan, B. (Kılıç) Bayram, B. Bulca, Y. H. Kim, C. Murathan and G. Öztürk, Rotational embeddings in $\mathbb{E}^4$ with pointwise 1-type Gauss map, *Turk. J. Math.,* **35** (2011), 493–499.
3. K. Arslan, B. (Kılıç) Bayram, B. Bulca and G. Öztürk, Generalized Rotation Surfaces in $\mathbb{E}^4$, *Results. Math.,* **61** (2012), 315–327.
4. M. Babaarslan and Y. Yayli, Space-like loxodromes on rotational surfaces in Minkowski 3-space, *J. Math. Anal. Appl.,* **409** (2014), 288–298.
5. M. Babaarslan and M. I. Munteanu, Time-like loxodromes on rotational surfaces in Minkowski 3-space, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat.,* **61** (2015), 471–484.
6. M. Babaarslan and Y. Yayli, Differential equation of the loxodrome on a helicoidal surface, *Journal of Navigation,* **68** (2015), 962–970.
7. M. Babaarslan and M. Kayacik, Time-like loxodromes on the helicoidal surfaces in Minkowski 3-space, *Filomat,* **31** (2017), 4405–4414.
8. M. Babaarslan, Loxodromes on helicoidal surfaces and tubes with variable radius in $\mathbb{E}^4$, *Commun. Fac. Sci. Univ. Ank., Ser. A1, Math. Stat.,* **68** (2019), 1950–1958.
9. M. Babaarslan and M. Selvi, *Loxodromes on space-like rotational surfaces in* $\mathbb{E}_1^4$, 17th International Geometry Symposium, June 19-22, 2019, Erzincan Binali Yıldırım University, Erzincan, Turkey.
10. M. Babaarslan and M. Kayacik, Differential equation of the space-like loxodromes on the helicoidal surfaces in Minkowski 3-space, *Differ. Equ. Dyn. Syst.,* **28** (2020), 495–512.
11. M. Babaarslan and M. Gümüş, On parametrizations of loxodromes on time-like rotational surfaces in Minkowski space-time, *Asian-Eur. J. Math.,* **14** (2021), 2150080.
12. B. Bektaş and U. Dursun, Timelike Rotational Surfaces of Elliptic, Hyperbolic and Parabolic Types in Minkowski Space $\mathbb{E}_1^4$ with Pointwise 1-Type Gauss Map, *Filomat,* **29** (2015), 381–392.
13. U. Dursun and N. C. Turgay, General rotational surfaces in Euclidean space $\mathbb{E}^4$ with pointwise 1-type Gauss map, *Math. Commun.,* **17** (2012), 71–81.
14. U. Dursun and B. Bektaş, Spacelike Rotational Surfaces of Elliptic, Hyperbolic and Parabolic Types in Minkowski Space $\mathbb{E}_1^4$ with Pointwise 1-Type Gauss Map, *Math. Phys. Anal. Geom.,* **17** (2014), 247–263.
15. V. Gorkavyy, On minimal lightlike surfaces in Minkowski space-time, *Differ. Geom. Appl.,* **26** (2008), 133–139.
16. D. T. Hieu and N. N. Thang, Bour's Theorem in 4-Dimensional Euclidean Space, *Bull. Korean Math Soc.,* **54** (2017), 2081–2089.
17. S. Kos, D. Vranic and D. Zec, Differential Equation of a Loxodrome on a Sphere, *Journal of Navigation,* **52** (1999), 418–420.
18. S. Kos, R. Filjar and M. Hess, *Differential Equation of the Loxodrome on a Rotational Surface,* Proceedings of the 2009 International Technical Meeting of The Institute of Navigation, 2009, Anaheim, CA.
19. C. L. E. Moore, Surfaces of Rotation in a Space of Four Dimensions, *Ann. Math. (2)* **21** (1919), 81–93.
20. C. A. Noble, Note on loxodromes, *Bull. Am. Math. Soc.,* **1-2** (1905), 116–119.
21. B. O' Neill, *Semi-Riemannian Geometry. With Applications to Relativity,* Academic Press, New York (1983).
22. M. Petrovic, Differential Equation of a Loxodrome on the Spheroid, *Nase More,* **54** (2007), 87–89.
23. J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds,* Springer, Graduate Texts in Mathematics, 149, Second Edition (2006).
24. H. Şimşek and M. Özdemir, On Conformal Curves in 2-Dimensional de Sitter Space, *Adv. Appl. Clifford Algebr.,* **26** (2016), 757–770.
25. D. W. Yoon, Rotation Surfaces with Finite Type Gauss Map in $\mathbb{E}^4$, *Indian J. Pure Appl. Math.,* **32** (2001), 1803–1808.
26. D. W. Yoon, Some Properties of the Clifford Torus as Rotational Surfaces, *Indian J. Pure Appl. Math.,* **34** (2003), 907–915.