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ORIGINAL RESEARCH

Spectra of *M*-edge rooted product of graphs

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Abstract In this paper, we define a graph operation, namely, *M*-edge rooted product of graphs. This generalizes the existing graph operation called graphs with edge pockets. Also we introduce a matrix invariant, namely, coronal of a matrix constrained by the index sets. We compute this value for some class of matrices with respect to some index sets. We obtain the generalized characteristic polynomial of the graph obtained by *M*-edge rooted product with a help of this invariant. Consequently, we deduce the characteristic polynomial of the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix of this graph. Using these results, we derive the *L*-spectrum of several families of *M*-edge rooted product of graphs and deduce several existing results on the spectra of graphs with edge pockets in the literature. As applications, we obtain infinitely many *L*-cospectral graphs and construct *A*-integral graphs, *L*-integral graphs.

Keywords Graph products · Adjacency spectrum · Laplacian spectrum · Signless Laplacian spectrum · Cospectral graphs · Integral graphs

Mathematics Subject Classification 05C50 · 05C76

1 Introduction

All the graphs considered in this paper are finite, simple and undirected. Let G be a graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$. The *adjacency matrix of* G is the $n \times n$ $0-1$ matrix $A(G) = (a_{ij})$, with $a_{ij} = 1$ if and only if u_i and u_j are adjacent in G. The *degree matrix* of G is the diagonal matrix $D(G)$, whose diagonal entries are the degree of vertices of G. The *Laplacian matrix* $L(G)$ (resp. *signless Laplacian matrix* $Q(G)$) of G is defined as $L(G) = D(G) - A(G)$ (resp. $Q(G) = D(G) + A(G)$). The multi set of eigenvalues of $L(G)$ (resp. $A(G)$, $Q(G)$) are known as the *L-spectrum* (resp. *A-spectrum*, *Q-spectrum*) of G. Let $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ (resp. $\mu_1(G) = 0, \mu_2(G), \ldots, \mu_n(G)$ and $\gamma_1(G)$, $\gamma_2(G), \ldots, \gamma_n(G)$) denote the *A*-spectrum (resp. *L*-spectrum and *Q*-spectrum) of G. The *vertex-edge incidence matrix* $B(G) = (b_{ij})$ of G is the $0-1$ matrix with rows and columns are indexed by the vertices and edges of G, respectively, defined by $b_{ij} = 1$ if and only if the vertex u_i is incident with j-th edge in G.

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Two graphs are said to be *A-cospectral* (resp. *L-cospectral*, *$\mathcal{Q}$-cospectral*) if they have the same A-spectrum (resp. L-spectrum, $\mathcal{Q}$-spectrum). A graph is said to be *A-integral* (resp. *L-integral*, *$\mathcal{Q}$-integral)* if all the eigenvalues of $A(G)$ (resp.$L(G)$, $\mathcal{Q}(G)$) are integers. The characteristic polynomial of a square matrix M is denoted by $\Phi_M(x)$. In [11], Cvetkoić et al. defined the generalized characteristic polynomial of a graph G as $\phi_G(x,t) = det(xI - A(G) + tD(G))$. Observe that $\Phi_{A(G)}(x) = \phi_G(x,0)$, $\Phi_{L(G)}(x) = (-1)^{|V(G)|}$ $\phi_G(-x,1)$ and $\Phi_{\mathcal{Q}(G)}(x) = \phi_G(x,-1)$.

Eigenvalues and eigenvectors of various graph matrices have been found several applications within the sciences and many other fields (see, [12–14, 29]). Consequently, a lot of research has been done on obtaining the spectra of the graph matrices. In this direction, it is natural to ask the following: 'how far the spectra of the given graph can be described in terms of the spectra of some other graphs by using graph operations ?'. In the literature several papers have been appeared for answering this question, by expressing the given graph in terms of some other graphs by using graph operations such as union, join, corona, NEPS, coalescence, vertex deletion, edge deletion, etc and obtaining the spectra of the given graphs by using the spectra of the constituting graphs (see, [8, 13, 16, 26, 27, 33] and the references there in).

In [22], Hedetniemi defined a generalization of join of graphs as follows: Let G and H be graphs on m and n vertices, respectively. Let π be a (symmetric) binary relation on $V(G) \times V(H)$, that is $\pi \subseteq V(G) \times V(H)$. The *$\pi-$graph of G and H*, is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \pi$. An equivalent formulation of this definition called *N-join of G and H* was given by Gayathri and Rajkumar [17] as follows: View the binary relation π as a $0-1$ matrix of size $m \times n$, say $N = (n_{ij})$ such that $n_{ij} = 1$ if and only if the i-th vertex of G and the j-th vertex of H are related with respect to π. Then the $\pi-$graph of G and H is the graph obtained by taking one copy of each of G and H, and joining the i-th vertex of G to the j-th vertex of H if and only if $n_{ij} = 1$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. This graph is denoted by $G \vee_N H$. Therein, the spectral properties of these graphs were studied. Notice that $G \vee_{J_{m\times n}} H$ is the usual join $G \vee H$ of G and H.

In [18], Godsil and Mckay introduced the graph operation called the *rooted product of graphs* and determined the characteristic polynomial of the adjacency matrix of this graph. Several particular cases of rooted product of graphs have been considered in the literature, which includes, *the comb product* [1, 25], *the graph with pockets* [5, 6, 10], *the hierarchical product of two graphs* [7], graphs constructed from bethe trees [3, 35, 36] and their spectral properties have been discussed. The authors of this paper defined a graph operation called the *M-rooted product of graphs* [34], which is a generalization of rooted product of graphs, and obtained the characteristic polynomial of its adjacency, Laplacian and signless Laplacian matrices.

In [31], Nath and Paul defined the following graph operation.

Definition 1.1 ([31]) Let G be a graph on $n \geq 2$ vertices, m edges and let $S = \{x_1, x_2, \ldots, x_h\}$ be a subset of $E(G)$. Let H be a graph on $p \geq 3$ vertices such that it has a specified edge (u, v) with $H - u$ and $H - v$ are isomorphic. The graph $G(S, H)$ is obtained from G and h vertex disjoint copies of H by identifying the edge (u, v) in the i-th copy of H with $x_i \in S$ for $i = 1, 2, \ldots, h$. This graph is called the *graph with edge pockets*.

Therein, they obtained the characteristic polynomial of $A(G(S, H))$ and $L(G(S, H))$, when the degree of u and v is $p-1$, and S induces a regular spanning subgraph of G. Also they determined A-spectrum and L-spectrum for some particular graph structures. In [10], Cui and Tian discussed the characteristic polynomial of $\mathcal{Q}(G(S,H))$ and described the $\mathcal{Q}$-spectrum for some particular cases.

Motivated by these we define the following graph operation.

Definition 1.2 Let G be a graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and edge set $E(G) = \{x_1, x_2, \ldots, x_m\}$. Let $\mathcal{H}_k$ be a sequence of graphs H_i with $V(H_i) = \{u_{i1}, u_{i2}, \ldots, u_{in_i}\}$, $n_i \geq 3$ and $|E(H_i)| = m_i \geq 1$ for $i = 1, 2, \ldots, k$. Let (u_{in_i-1}, u_{in_i}) be the specified edge of H_i such that the graphs $H_i - u_{in_i}$ and $H_i - u_{in_i-1}$ are isomorphic. Let $M = (m_{ij})$ be a non-zero matrix of size $m \times k$ whose entries are non-negative integers. Then the *M-edge rooted product of G with $\mathcal{H}_k$*, denoted by $G \boxplus_M \mathcal{H}_k$, is the graph obtained by taking one copy of G, and m_{ij} vertex disjoint copies of H_j for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, k$ and identifying the edge (u_{jn_j-1}, u_{jn_j}) of each copy of H_j with the edge x_i of G. We call the matrix M as the *generating matrix* of $G \boxplus_M \mathcal{H}_k$.

Notice that the generating matrix M of $G \boxplus_M \mathcal{H}_k$ has the following features: It describes how the specified edges of the graphs $H_1, H_2, \ldots, H_k$ are identified with the edges of G. By varying M over all $m \times k$ non-zero matrices whose entries are non-negative integers, we can get a family of graphs for a fixed G and $\mathcal{H}_k$.

Further, notice that Definition 1.2 generalizes Definition 1.1, since by taking $H_i = H$, for $i = 1, 2, \ldots, k$, with $k \le m$ and $M = [\,I_k \quad \mathbf{0}\,]^T$ with rows of I_k are indexed by S, then we get $G \boxplus_M \mathcal{H}_k = G(S, H)$.

Example 1.1 Consider the graphs G, H_1, H_2, H_3, H_4 and H_5 as shown in Figures 1(a)-1(f), respectively, where the edge of H_i for $i = 1, 2, \ldots, 5$ colored with red represents its specified edge. Let $\mathcal{H}_5$ be the sequence H_1, H_2, H_3, H_4, H_5, and let

$$M = \begin{bmatrix} 2 & 0 & 1 & 1 & 4 \\ 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the graph $G \boxplus_M \mathcal{H}_5$ is as shown in Figure 1(g).

Let A be a $0-1$ matrix of size $n \times m$. Let $\mathcal{P}(A)$ denote the set of all matrices of size $n \times m$ whose entries are non-negative integers such that the (i, j)-th entry of a matrix in $\mathcal{P}(A)$ is a positive integer if and only if the (i, j)-th entry of A is 1. The line graph $\mathcal{L}(G)$ of a graph G is the graph with the edges of G as its vertices, and where two edges of G are adjacent in $\mathcal{L}(G)$ if and only if they are adjacent in G. Let $\overline{G}$ denote the complement graph of G. For an edge x in a graph G, the neighborhood of x (closed neighborhood of x) is the set of all edges adjacent to x (adjacent to x including x itself).

The way in which the specified edges of copies of H_i ($i = 1, 2, \ldots, k$) are identified with the edges of G (i.e., by the nature of the generating matrix M) in the construction of $G \boxplus_M \mathcal{H}_k$ enable us to define the following special M-edge rooted product of graphs: Let $M \in \mathcal{P}(A(\mathcal{L}(G)))$. In this case, the specified edges of copies of H_i are identified with the edges in the neighborhood of i-th edge of G for $i = 1, 2, \ldots, m$ in $G \boxplus_M \mathcal{H}_m$ and so we call this graph as the *M-neighborhood edge rooted product of G with $\mathcal{H}_m$*. Similarly, if M belongs to any one of $\mathcal{P}(J - A(\mathcal{L}(G)))$, $\mathcal{P}(I + A(\mathcal{L}(G)))$, $\mathcal{P}(J - I - A(\mathcal{L}(G)))$, $\mathcal{P}(B(\mathcal{L}(G)))$, $\mathcal{P}(J - B(\mathcal{L}(G)))$, $\mathcal{P}(B(\overline{\mathcal{L}(G)}))$, $\mathcal{P}(J - B(\overline{\mathcal{L}(G)}))$, then we call $G \boxplus_M \mathcal{H}_k$ as *the M-neighborhood complement edge rooted product, the M-closed neighborhood edge rooted product, the M-closed neighborhood complement edge rooted product, the M-adjacent edge rooted product, the M-adjacent edges complement edge rooted product, the M-non-adjacent edge rooted product, or the M-non-adjacent edges complement edge rooted product, respectively of G with $\mathcal{H}_k$*.

Example 1.2 Consider the graphs G, H_1, H_2, H_3 and H_4 as shown in Figures 1(a)-1(e), respectively, where the edge of H_i for $i = 1, 2, \ldots, 4$ colored with red represents its specified edge. Let $\mathcal{H}_4$ be the sequence H_1, H_2, H_3, H_4, and $\mathcal{H}_2$ be the sequence H_1, H_2. Let

$$M_1 = \begin{bmatrix} 0 & 2 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 1 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix}, \quad M_7 = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad M_8 = \begin{bmatrix} 0 & 4 \\ 2 & 0 \\ 0 & 3 \\ 3 & 0 \end{bmatrix}.$$

Then the M_1-neighborhood edge rooted product, M_2-neighborhood complement edge rooted product, the M_3-closed neighborhood edge rooted product, the M_4-closed neighborhood complement edge rooted product, the M_5-adjacent edge rooted product, the M_6-adjacent edges complement edge rooted product of G with $\mathcal{H}_4$, the M_7-non-adjacent egde rooted product, the M_8-non-adjacent edges complement edge rooted product of G with $\mathcal{H}_2$, are shown in Figure 2(f)-2(m), respectively.

The rest of this paper is compiled as follows: In Section 2, we introduce a new invariant of a matrix which is used to compute the spectra of graphs, and determine this value for some particular cases.

As mentioned above, it is natural to ask the following question: 'To what extent the spectrum of G, H_i's and the properties of M can be used to describe the spectrum of $G \boxplus_M H_k$?'. Sections 3 and 4 are devoted to answer this question.

In Section 3, we obtain the generalized characteristic polynomial of the M-edge rooted product of graphs by using the newly defined invariant (see, Theorem 3.1). Using this we derive the characteristic polynomial of the adjacency, the Laplacian and the signless Laplacian matrices of this graph (see, Corollary 3.1). Moreover, we deduce several existing results on spectra of graphs with edge pockets (see, Remark 3.1).

In Section 4, we determine the L-spectra of several families of M-edge rooted product of graphs, including the constructions in which H_i's are obtained from N-join of two graphs (see, Corollaries 4.1, 4.2, 4.3 and 4.4). Also we describe the method of deriving the A-spectra and Q-spectra of the special classes of this graph constructions (see, Remark 4.1).

Van Dam and Haemers [38] posed the following, which is one of the fundamental questions in spectral graph theory: "Which graphs are determined by their spectra ?". Subsequently, in literature, researchers constructed many families of cospectral graphs by using several methods (see, [19, 20, 37]). As applications of the proved results in this paper, we obtain several infinite families of cospectral graphs in subsection 4.1.

In 1973, Harary and Schwenk [21] initiated the research on integral graphs by raising the following question: Which graphs have integral (adjacency) spectra ?. Since then, in the literature, several special types of integral graphs are constructed (see, for example [4] and its references therein). As a contribution to this direction, in subsection 4.2, we construct several families of A-integral graphs and L-integral graphs.

2 Coronal of a matrix constrained by index sets

In this section, we introduce a new invariant of a square matrix and determine this value for some special cases.

Let M be a matrix of size $n \times n$ and let $S, T \subseteq \{1, 2, \ldots, n\}$. Then $M[S|T]$ denotes the submatrix of M determined by the rows corresponding to S and the columns corresponding to T. For a subset S of an ordered set $A = \{a_1, a_2, \ldots, a_n\}$, the *indicator vector of S (with respect to A)* is the vector $\mathbf{r}_S = (r_1, r_2, \ldots, r_n)$ in which $r_i = 1$ or 0, according as $a_i \in S$ or $a_i \notin S$. Let $R_S := diag(r_1, r_2, \ldots, r_n)$.

We recall the following two basic results which are used in the subsequent results.

Theorem 2.1 ([24]) *Let*

$$A = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

be a partitioned matrix of size $n \times n$, with M_1, M_4 are invertible matrices.
Then

$$\det A = \det(M_4)\det(M_1 - M_2M_4^{-1}M_3) = \det(M_1)\det(M_4 - M_3M_1^{-1}M_2) \tag{2.1}$$

and

$$A^{-1} = \begin{bmatrix} (M_1 - M_2M_4^{-1}M_3)^{-1} & -M_1^{-1}M_2(M_4 - M_3M_1^{-1}M_2)^{-1} \\ -M_4^{-1}M_3(M_1 - M_2M_4^{-1}M_3)^{-1} & (M_4 - M_3M_1^{-1}M_2)^{-1} \end{bmatrix}. \tag{2.2}$$

Theorem 2.2 ([39]) *Let A be a non-singular matrix. If the sum of the elements in each row of A is t, then the sum of the elements in each row of A^{-1} is $1/t$.*

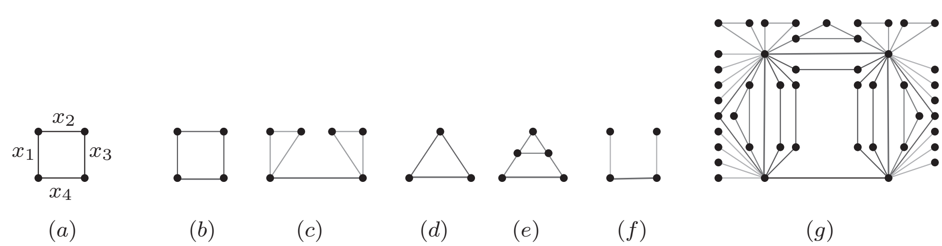

Fig. 1 The graphs **(a)** G, **(b)** H_1, **(c)** H_2, **(d)** H_3, **(e)** H_4, **(f)** H_5, and **(g)** $G \boxplus_M \mathcal{H}_5$

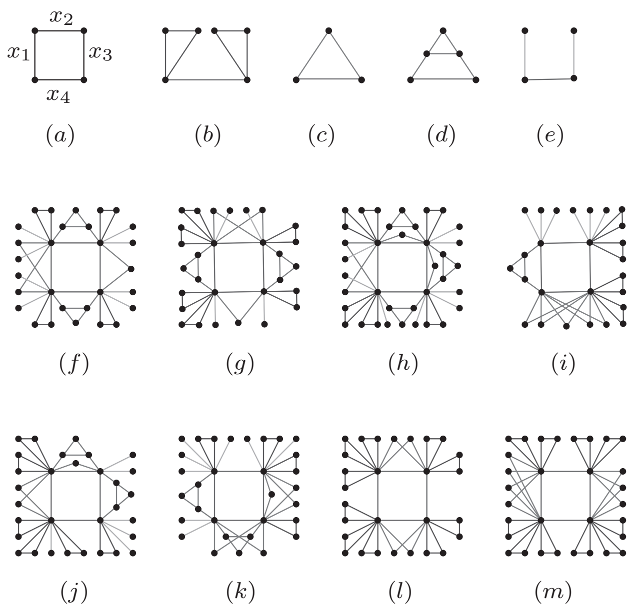

Fig. 2 The graphs (**a**) G, (**b**) H_1, (**c**) H_2, (**d**) H_3, (**e**) H_4, **f**) $G\boxplus_{M_1}\mathcal{H}_4$, (**g**) $G\boxplus_{M_2}\mathcal{H}_4$, (**h**) $G\boxplus_{M_3}\mathcal{H}_4$, (**i**) $G\boxplus_{M_4}\mathcal{H}_4$, (**j**) $G\boxplus_{M_5}\mathcal{H}_4$, (**k**) $G\boxplus_{M_6}\mathcal{H}_4$, (**l**) $G\boxplus_{M_7}\mathcal{H}_2$ and (**m**) $G\boxplus_{M_8}\mathcal{H}_2$

McLeman and McNicholas introduced the following invariant.

Definition 2.1 ([28]) Let G be a graph on n vertices. The *coronal* $\Gamma_G(x)$ *of* G is defined to be the sum of the entries of the matrix $(xI_n - A(G))^{-1}$, that is

$$\Gamma_G(x) = J_{1\times n}(xI_n - A(G))^{-1}J_{n\times 1}.$$

Therein they used it to compute the spectrum of the corona of two graphs. Subsequently, this invariant has been used to compute the spectra of graphs obtained from several graph operations in the literature.

Cui and Tian generalized the above concept as follows.

Definition 2.2 ([9]) Let G be a graph on n vertices and let M be a graph matrix of G. The $M-$*coronal* $\Gamma_M(x)$ *of* G is defined to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is

$$\Gamma_M(x) = J_{1\times n}(xI_n - M)^{-1}J_{n\times 1}.$$

Rajkumar and Gayathri defined the following invariant which is a generalization of Definition 2.2 and they used it to compute the spectra of some graph constructions.

Definition 2.3 ([32]) Let M be a real matrix of size $n\times n$ and let $\alpha \subseteq \{1,2,\ldots,n\}$ be an index set. Then *the coronal of* M *constrained by* α, denoted by $\Gamma_M^{\alpha}(x)$, is defined as the sum of all entries in the matrix $(xI_n - M)^{-1}[\alpha|\alpha]$. That is

$$\Gamma_M^{\alpha}(x) = \mathbf{r}_\alpha(xI_n - M)^{-1}\mathbf{r}_\alpha^T.$$

We generalize the above concept by defining as follows.

Definition 2.4 Let M be a real matrix of size $n\times n$ and let $\alpha,\beta \subseteq \{1,2,\ldots,n\}$ be index sets. Then *the coronal of* M *constrained by* α *and* β, denoted by $\Gamma_M^{(\alpha,\beta)}(x)$, is defined to be the sum of all entries in the matrix $(xI_n - M)^{-1}[\alpha|\beta]$, that is,

$$\Gamma_M^{(\alpha,\beta)}(x) = \mathbf{r}_\alpha(xI_n - M)^{-1}\mathbf{r}_\beta^{\mathbf{T}}.$$

Notice that if $\alpha = \beta$ then $\Gamma_M^{(\alpha,\alpha)}(x) = \Gamma_M^{\alpha}(x)$.

In the next result, we obtain coronal of a matrix M constrained by some classes of index sets.

Theorem 2.3 *Let M be a $n \times n$ matrix and let $S = \{s_1, s_2, \ldots, s_{n_1}\} \subset \{1, 2, \ldots, n\}$. If each row sum of $M[S|S]$, $M[S|S^c]$, $M[S^c|S]$ and $M[S^c|S^c]$ equals a constant t_1, t_2, t_3 and t_4, respectively, then*

$$\Gamma_M^{(S,S^c)}(x) = \frac{n_1 t_2}{(x - t_4)(x - t_1) - t_2 t_3}. \tag{2.3}$$

Proof Without loss of generality, we assume that $s_1 < s_2 < \cdots < s_{n_1}$. Let π be a permutation on $\{1, 2, \ldots, n\}$ such that $\pi(1) = s_1, \pi(2) = s_2, \ldots, \pi(n_1) = s_{n_1}$ and let P_π be the permutation matrix corresponding to π. Then we get $P_\pi M P_\pi^T = M'$, where

$$M' = \begin{bmatrix} M[S|S] & M[S|S^c] \\ M[S^c|S] & M[S^c|S^c] \end{bmatrix}.$$

Now

$$\begin{aligned}
\Gamma_M^{(S,S^c)}(x) &= \mathbf{r}_S (xI_n - M)^{-1} \mathbf{r}_{S^c}^T \\
&= \mathbf{r}_S (xI_n - P_\pi^T M' P_\pi)^{-1} \mathbf{r}_{S^c}^T \\
&= \mathbf{r}_S P_\pi^T (xI_n - M')^{-1} P_\pi \mathbf{r}_{S^c}^T \\
&= \begin{bmatrix} J_{1\times n_1} & \mathbf{0}_{1\times(n-n_1)} \end{bmatrix} (xI_n - M')^{-1} \begin{bmatrix} \mathbf{0}_{1\times n_1} & J_{1\times(n-n_1)} \end{bmatrix}^T.
\end{aligned}$$

By using (2.2) in the above equation, we get

$$\Gamma_M^{(S,S^c)}(x) = J_{1\times n_1} \Big((xI - M[S|S])^{-1} M[S|S^c] (xI - M[S^c|S^c] \tag{2.4}$$

$$-M[S^c|S](xI - M[S|S])^{-1} M[S|S^c] \Big)^{-1} \Big) J_{1\times(n-n_1)}^T$$

$$= \frac{n_1 t_2}{(x - t_4)(x - t_1) - t_2 t_3} \text{ (by Theorem 2.2).} \tag{2.5}$$

This completes the proof. □

The following existing result can be proved by taking $\mathbf{r}_S^T$ instead of $\mathbf{r}_{S^c}^T$ in the proof of the previous theorem.

Proposition 2.1 ([32]) *Let M be a $n \times n$ matrix and let $S = \{s_1, s_2, \ldots, s_{n_1}\} \subset \{1, 2, \ldots, n\}$. If each row sum of $M[S|S]$, $M[S|S^c]$, $M[S^c|S]$ and $M[S^c|S^c]$ equals a constant t_1, t_2, t_3 and t_4, respectively, then*

$$\Gamma_M^S(x) = \frac{n_1(x - t_4)}{(x - t_4)(x - t_1) - t_2 t_3}. \tag{2.6}$$

We use the above mentioned results to determine the spectra of graphs in the upcoming sections.

3 Generalized characteristic polynomial of $\mathbf{G} \boxplus_{\mathbf{M}} \mathcal{H}_{\mathbf{k}}$

In this section, we determine the generalized characteristic polynomial of $G \boxplus_M \mathcal{H}_k$. As a consequence, we deduce the characteristic polynomial of the adjacency matrix, the Laplacian matrix, the signless Laplacian matrix of $G \boxplus_M \mathcal{H}_k$ and the existing results on spectra of graphs with edge pockets.

The following assumptions are made and notations are used in the rest of the paper.

Let G be a graph with vertex set $V(G) = \{u_1, u_2, \ldots, u_n\}$ and edge set $E(G) = \{x_1, x_2, \ldots, x_m\}$. Let $\mathcal{H}_k$ be a sequence of graphs H_i with $V(H_i) = \{u_{i1}, u_{i2}, \ldots, u_{in_i}\}$, $n_i \geq 3$ and $|E(H_i)| = m_i \geq 1$ for $i = 1, 2, \ldots, k$. Let (u_{in_i-1}, u_{in_i}) be the specified edge of H_i such that the graphs $H_i - u_{in_i}$ and $H_i - u_{in_i-1}$ are isomorphic. Let $M = (m_{ij})$ be a non-zero matrix of size $m \times k$ whose entries are non-negative integers.

(i) For $i = 1, 2, \ldots, k$, let c_i denote the sum of all entries in the i-th column of M.
(ii) For $i = 1, 2, \ldots, k$, let H_i' denote the graph $H_i - \{u_{in_i-1}, u_{in_i}\}$ obtained from H_i by deleting the vertices u_{in_i-1} and u_{in_i}.

(iii) For $i = 1, 2, \ldots, k$, let $\mathbf{r}_{S_i}$ and $\mathbf{r}_{T_i}$ be the indicator vectors of S_i and T_i with respect to $V(H_i')$, respectively and let $D_i := R_{S_i} + R_{T_i}$.

(iv) For each $t \in \mathbb{R}$, we denote $A(H_i') - t\big(D_i + D(H_i')\big)$ by $M_t(H_i')$ for $i = 1, 2, \ldots, k$.

(v) Let $N_G(u)$ denote the set of all neighbors of the vertex u in G.

(vi) For $i = 1, 2, \ldots, k$, let $S_i := N_{H_i - (u_{in_i-1}, u_{in_i})}(u_{in_i-1})$ and $T_i := N_{H_i - (u_{in_i-1}, u_{in_i})}(u_{in_i})$, where $H_i - (u_{in_i-1}, u_{in_i})$ is the graph obtained from H_i by deleting the edge (u_{in_i-1}, u_{in_i}).

(vii) For $i = 1, 2, \ldots, k$, let $d_i := |S_i| = |T_i|$. Notice that in Definition 1.2, $H_i - u_{in_i} \cong H_i - u_{in_i-1}$ implies that $|S_i| = |T_i|$.

(viii) Let h be the number of non-zero rows of M. For $i = 1, 2, \ldots, h$, let the α_i-th row of M be non-zero with $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_h \le m$.

(ix) For $i = 1, 2, \ldots, h$, let l_i be the number of non-zero entries in the α_i-th row of M. For each $i = 1, 2, \ldots, h$, let the sequence $(g_{\alpha_i 1}, g_{\alpha_i 2}, \ldots, g_{\alpha_i l_i})$ with $g_{\alpha_i 1} < g_{\alpha_i 2} < \cdots < g_{\alpha_i l_i}$ denote the column position of non zero entries in α_i-th row of M.

(x) Let G' denote the subgraph of G induced by the edge set $\{x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_h}\}$.

(xi) For $i = 1, 2, \ldots, n$, e_i is the column vector of size n whose i-th entry is 1 and 0 otherwise.

(xii) For $i = 1, 2, \ldots, m$, let $x_i := (u_{k_i}, u_{h_i})$, that is, u_{k_i} and u_{h_i} are the end vertices of x_i, where $k_i, h_i \in \{1, 2, \ldots, n\}$ and so $b_{k_i i} = 1$ and $b_{h_i i} = 1$ with $k_i < h_i$.

(xiii) In the construction of $G \boxplus_M \mathcal{H}_k$, without loss of generality, we assume that $u_{k_{\alpha_i}}$ and $u_{h_{\alpha_i}}$ are identified with $u_{g_{\alpha_i l} n_{g_{\alpha_i l}} - 1}$ and $u_{g_{\alpha_i l} n_{g_{\alpha_i l}}}$, respectively for $i = 1, 2, \ldots, h$, $l = 1, 2, \ldots, l_i$.

(xiv) Throughout this paper, the rows of M and the columns of $B(G)$ are indexed by $E(G)$ with the following order $x_1, x_2, \ldots, x_m$.

The *Kronecker product* $A \otimes B$ of two matrices $A = (a_{ij})$ of size $n \times m$ and $B = (b_{ij})$ of size $p \times q$ is the matrix obtained by replacing each entry a_{ij} by $a_{ij}B$. Let I_n denote the identity matrix of size $n \times n$ and let $J_{n \times m}$ denote the matrix of size $n \times m$ with all its entries are 1. We denote these two matrices simply by I and J when their sizes can be inferred from the context. $\mathbf{0}$ denotes the zero matrix of appropriate size.

The following is one of the main results of this paper, which describes the generalized characteristic polynomial of $G \boxplus_M \mathcal{H}_k$.

Theorem 3.1 *The generalized characteristic polynomial of $G \boxplus_M \mathcal{H}_k$ is*

$$\left(\prod_{i=1}^{k} \left(\Phi_{M_t(H_i')}(x)\right)^{c_i}\right) det(xI - (A(G) - tN - tD(G)) - C), \tag{3.1}$$

where

$$N = diag\left(\sum_{j=1}^{m}\left(b_{1j}\sum_{i=1}^{k} m_{ji} d_i\right), \ldots, \sum_{j=1}^{m}\left(b_{nj}\sum_{i=1}^{k} m_{ji} d_i\right)\right) \tag{3.2}$$

and

$$\begin{aligned} C = \sum_{i=1}^{m}\Bigg(&\left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H_l')}^{(S_l,S_l)}(x)\right)\left(\mathbf{e}_{k_i}\mathbf{e}_{k_i}^T\right) + \left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H_l')}^{(S_l,T_l)}(x)\right)\left(\mathbf{e}_{k_i}\mathbf{e}_{h_i}^T\right) \\ &+ \left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H_l')}^{(T_l,T_l)}(x)\right)\left(\mathbf{e}_{h_i}\mathbf{e}_{h_i}^T\right) + \left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H_l')}^{(T_l,S_l)}(x)\right)\left(\mathbf{e}_{h_i}\mathbf{e}_{k_i}^T\right)\Bigg). \end{aligned} \tag{3.3}$$

Proof With a suitable ordering of vertices of $G \boxplus_M \mathcal{H}_k$, we can write

$$A(G \boxplus_M \mathcal{H}_k) = \begin{bmatrix} A(G) & X \\ X^T & Y \end{bmatrix},$$

where

$$X = [X_1 \quad X_2 \quad \cdots \quad X_h], \tag{3.4}$$

$$Y = \begin{bmatrix} Y_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Y_2 & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Y_h \end{bmatrix}, \tag{3.5}$$

$$\begin{aligned} X_i = &\left(e_{k_{\alpha_i}} \otimes \left[J_{1\times m_{\alpha_i g_{\alpha_i 1}}} \otimes \mathbf{r}_{S_{g_{\alpha_i 1}}} \quad \cdots \quad J_{1\times m_{\alpha_i g_{\alpha_i l_i}}} \otimes \mathbf{r}_{S_{g_{\alpha_i l_i}}}\right]\right) \\ &+ \left(e_{h_{\alpha_i}} \otimes \left[J_{1\times m_{\alpha_i g_{\alpha_i 1}}} \otimes \mathbf{r}_{T_{g_{\alpha_i 1}}} \quad \cdots \quad J_{1\times m_{\alpha_i g_{\alpha_i l_i}}} \otimes \mathbf{r}_{T_{g_{\alpha_i l_i}}}\right]\right), \end{aligned} \tag{3.6}$$

$$Y_i = \begin{bmatrix} I_{m_{\alpha_i g_{\alpha_i 1}}} \otimes A(H'_{g_{\alpha_i 1}}) & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & I_{m_{\alpha_i g_{\alpha_i l_i}}} \otimes A(H'_{g_{\alpha_i l_i}}) \end{bmatrix} \tag{3.7}$$

for $i = 1, 2, \dots, h$. Next, the degree matrix of the graph $G \boxplus_M \mathcal{H}_k$ can be written as

$$\begin{bmatrix} D(G) + N & \mathbf{0} \\ \mathbf{0}^T & Z \end{bmatrix},$$

where

$$N = diag\left(\sum_{j=1}^{m}\left(b_{1j}\sum_{i=1}^{k} m_{ji}d_i\right), \dots, \sum_{j=1}^{m}\left(b_{nj}\sum_{i=1}^{k} m_{ji}d_i\right)\right), \tag{3.8}$$

$$Z = \begin{bmatrix} Z_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & Z_2 & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & Z_h \end{bmatrix}, \tag{3.9}$$

with

$$Z_i = \begin{bmatrix} I_{m_{\alpha_i g_{\alpha_i 1}}} \otimes (D_{g_{\alpha_i 1}} + D(H'_{g_{\alpha_i 1}})) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & I_{m_{\alpha_i g_{\alpha_i l_i}}} \otimes (D_{g_{\alpha_i l_i}} + D(H'_{g_{\alpha_i l_i}})) \end{bmatrix}, \tag{3.10}$$

for $i = 1, 2, \dots, h$. Then

$$A(G \boxplus_M \mathcal{H}_k) - tD(G \boxplus_M \mathcal{H}_k) = \begin{bmatrix} A(G) - tN - tD(G) & X \\ X^T & Y - tZ \end{bmatrix}. \tag{3.11}$$

So

$$\phi_{G \boxplus_M \mathcal{H}_k}(x,t) = det\left(\begin{bmatrix} xI - (A(G) - tN - tD(G)) & -X \\ -X^T & xI - (Y - tZ) \end{bmatrix}\right). \tag{3.12}$$

By using (2.1) in (3.12), we get

$$\begin{aligned} \phi_{G \boxplus_M \mathcal{H}_k}(x,t) = &\, det(xI - (Y - tZ))det(xI - (A(G) - tN - tD(G)) \\ &- X(xI - (Y - tZ))^{-1}X^T\Big). \end{aligned} \tag{3.13}$$

From (3.5) and (3.9), we get

$$det(xI-(Y-tZ))=\prod_{i=1}^{h}(\Phi_{Y_i-tZ_i}(x)). \tag{3.14}$$

From (3.7) and (3.10) and by using the properties of Kronecker product of matrices, we get

$$\begin{aligned}\Phi_{Y_i-tZ_i}(x)&=det\left(\begin{bmatrix} I_{m_{\alpha_i g_{\alpha_i 1}}}\otimes(xI-M_t(H'_{g_{\alpha_i 1}})) & \dots & \mathbf{0}\\ \vdots & \ddots & \vdots\\ \mathbf{0} & \dots & I_{m_{\alpha_i g_{\alpha_i l_i}}}\otimes(xI-M_t(H'_{g_{\alpha_i l_i}}))\end{bmatrix}\right)\\ &=\prod_{l=1}^{l_i}\Big(det(I_{m_{\alpha_i g_{\alpha_i l}}}\otimes(xI-M_t(H'_{g_{\alpha_i l}})))\Big)\\ &=\prod_{l=1}^{l_i}\Big(\Phi_{M_t(H'_{g_{\alpha_i l}})}(x)\Big)^{m_{\alpha_i g_{\alpha_i l}}}\end{aligned} \tag{3.15}$$

for $i=1,2,\dots,h$. Substituting (3.15) in (3.14), we get

$$\begin{aligned}det(xI-(Y-tZ))&=\prod_{i=1}^{h}\prod_{l=1}^{l_i}\Phi_{M_t(H'_{g_{\alpha_i l}})}(x)^{m_{\alpha_i g_{\alpha_i l}}}\\ &=\prod_{i=1}^{m}\prod_{j=1}^{k}\Phi_{M_t(H'_j)}(x)^{m_{ij}}\\ &=\prod_{j=1}^{k}\Phi_{M_t(H'_j)}(x)^{m_{1j}+m_{2j}+\dots+m_{mj}}\\ &=\prod_{j=1}^{k}\Phi_{M_t(H'_j)}(x)^{c_j}.\end{aligned} \tag{3.16}$$

From (3.4)-(3.10), we get

$$\begin{aligned}X(xI-(Y-tZ))^{-1}X^T&=\sum_{i=1}^{h}X_i(xI-(Y_i-tZ_i))^{-1}X_i^T\\ &=\sum_{i=1}^{h}\Big(A_i\Big(xI-Y_i+tZ_i\Big)^{-1}A_i^T+A_i\Big(xI-Y_i+tZ_i\Big)^{-1}B_i^T\\ &\quad+B_i\Big(xI-Y_i+tZ_i\Big)^{-1}A_i^T+B_i\Big(xI-Y_i+tZ_i\Big)^{-1}B_i^T\Big),\end{aligned} \tag{3.17}$$

where

$$A_i=e_{k_{\alpha_i}}\otimes\Big[J_{1\times m_{\alpha_i g_{\alpha_i 1}}}\otimes\mathbf{r}_{S_{g_{\alpha_i 1}}}\quad\cdots\quad J_{1\times m_{\alpha_i g_{\alpha_i l_i}}}\otimes\mathbf{r}_{S_{g_{\alpha_i l_i}}}\Big] \tag{3.18}$$

and

$$B_i=e_{h_{\alpha_i}}\otimes\Big[J_{1\times m_{\alpha_i g_{\alpha_i 1}}}\otimes\mathbf{r}_{T_{g_{\alpha_i 1}}}\quad\cdots\quad J_{1\times m_{\alpha_i g_{\alpha_i l_i}}}\otimes\mathbf{r}_{T_{g_{\alpha_i l_i}}}\Big] \tag{3.19}$$

for $i=1,2,\dots,h$. From (3.7), (3.10), (3.18), (3.19) and by using the properties of Kronecker product of matrices, we get

$$\begin{aligned}
A_i(xI - Y_i + tZ_i)^{-1}A_i^T &= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{k_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{l_i} J_{1\times m_{\alpha_i g_{\alpha_i l}}} J_{m_{\alpha_i g_{\alpha_i l}}\times 1} \otimes \mathbf{r}_{S_{g_{\alpha_i l}}} \left(xI - M_t(H'_{g_{\alpha_i l}})\right)^{-1} \mathbf{r}_{S_{g_{\alpha_i l}}}^T\right) \\
&= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{k_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{l_i} m_{\alpha_i g_{\alpha_i l}} \otimes \Gamma_{M_t(H'_{g_{\alpha_i l}})}^{(S_{g_{\alpha_i l}}, S_{g_{\alpha_i l}})}(x)\right) (\text{by Definition 2.4}) \\
&= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{k_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{l_i} m_{\alpha_i g_{\alpha_i l}} \Gamma_{M_t(H'_{g_{\alpha_i l}})}^{(S_{g_{\alpha_i l}}, S_{g_{\alpha_i l}})}(x)\right) \\
&= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{k_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{k} m_{\alpha_i l} \Gamma_{M_t(H'_l)}^{(S_l, S_l)}(x)\right) \\
&= \left(\sum_{l=1}^{k} m_{\alpha_i l} \Gamma_{M_t(H'_l)}^{(S_l, S_l)}(x)\right) \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{k_{\alpha_i}}^T\right)
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
A_i(xI - Y_i + tZ_i)^{-1}B_i^T &= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{h_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{l_i} J_{1\times m_{\alpha_i g_{\alpha_i l}}} J_{m_{\alpha_i g_{\alpha_i l}}\times 1} \otimes \mathbf{r}_{S_{g_{\alpha_i l}}} \left(xI - M_t(H'_{g_{\alpha_i l}})\right)^{-1} \mathbf{r}_{T_{g_{\alpha_i l}}}^T\right) \\
&= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{h_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{l_i} m_{\alpha_i g_{\alpha_i l}} \otimes \Gamma_{M_t(H'_{g_{\alpha_i l}})}^{(S_{g_{\alpha_i l}}, T_{g_{\alpha_i l}})}(x)\right) (\text{by Definition 2.4}) \\
&= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{h_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{l_i} m_{\alpha_i g_{\alpha_i l}} \Gamma_{M_t(H'_{g_{\alpha_i l}})}^{(S_{g_{\alpha_i l}}, T_{g_{\alpha_i l}})}(x)\right) \\
&= \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{h_{\alpha_i}}^T\right) \otimes \left(\sum_{l=1}^{k} m_{\alpha_i l} \Gamma_{M_t(H'_l)}^{(S_l, T_l)}(x)\right) \\
&= \left(\sum_{l=1}^{k} m_{\alpha_i l} \Gamma_{M_t(H'_l)}^{(S_l, T_l)}(x)\right) \left(\mathbf{e}_{k_{\alpha_i}}\mathbf{e}_{h_{\alpha_i}}^T\right).
\end{aligned} \tag{3.21}$$

Similarly, we get

$$B_i(xI - Y_i + tZ_i)^{-1}B_i^T = \left(\sum_{l=1}^{k} m_{\alpha_i l} \Gamma_{M_t(H'_l)}^{(T_l, T_l)}(x)\right) \left(\mathbf{e}_{h_{\alpha_i}}\mathbf{e}_{h_{\alpha_i}}^T\right) \tag{3.22}$$

and

$$B_i(xI - Y_i + tZ_i)^{-1}A_i^T = \left(\sum_{l=1}^{k} m_{\alpha_i l} \Gamma_{M_t(H'_l)}^{(T_l, S_l)}(x)\right) \left(\mathbf{e}_{h_{\alpha_i}}\mathbf{e}_{k_{\alpha_i}}^T\right). \tag{3.23}$$

Substituting (3.20)-(3.23) in (3.17), we get

$$\begin{aligned}
X(xI - (Y - tZ))^{-1}X^T = \sum_{i=1}^{m} \Bigg(&\left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H'_l)}^{(S_l, S_l)}(x)\right) \left(\mathbf{e}_{k_i}\mathbf{e}_{k_i}^T\right) + \left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H'_l)}^{(S_l, T_l)}(x)\right) \left(\mathbf{e}_{k_i}\mathbf{e}_{h_i}^T\right) \\
&+ \left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H'_l)}^{(T_l, T_l)}(x)\right) \left(\mathbf{e}_{h_i}\mathbf{e}_{h_i}^T\right) + \left(\sum_{l=1}^{k} m_{il} \Gamma_{M_t(H'_l)}^{(T_l, S_l)}(x)\right) \left(\mathbf{e}_{h_i}\mathbf{e}_{k_i}^T\right) \Bigg).
\end{aligned} \tag{3.24}$$

Substituting (3.16), (3.24) in (3.13), we get the result. □

As a consequence of Theorem 3.1, we have the following result.

Corollary 3.1

(i) *The characteristic polynomial of* $A(G \boxplus_M \mathcal{H}_k)$ *is*

$$\left(\prod_{i=1}^{k}\left(\Phi_{A(H_i')}(x)\right)^{c_i}\right)det(xI - A(G) - C_1), \tag{3.25}$$

where

$$\begin{aligned} C_1 = \sum_{i=1}^{m}\Bigg(&\left(\sum_{l=1}^{k} m_{il}\Gamma_{A(H_l')}^{(S_l,S_l)}(x)\right)\left(e_{k_i}e_{k_i}^T\right) + \left(\sum_{l=1}^{k} m_{il}\Gamma_{A(H_l')}^{(S_l,T_l)}(x)\right)\left(e_{k_i}e_{h_i}^T\right) \\ &+\left(\sum_{l=1}^{k} m_{il}\Gamma_{A(H_l')}^{(T_l,T_l)}(x)\right)\left(e_{h_i}e_{h_i}^T\right) + \left(\sum_{l=1}^{k} m_{il}\Gamma_{A(H_l')}^{(T_l,S_l)}(x)\right)\left(e_{h_i}e_{k_i}^T\right)\Bigg). \end{aligned} \tag{3.26}$$

(ii) *The characteristic polynomial of $L(G \boxplus_M \mathcal{H}_k)$ is*

$$\left(\prod_{i=1}^{k}\left(\Phi_{L(H_i')+D_i}(x)\right)^{c_i}\right)det(xI - N - L(G) - C_2), \tag{3.27}$$

where

$$\begin{aligned} C_2 = \sum_{i=1}^{m}\Bigg(&\left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+L(H_l')}^{(S_l,S_l)}(x)\right)\left(e_{k_i}e_{k_i}^T\right) + \left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+L(H_l')}^{(S_l,T_l)}(x)\right)\left(e_{k_i}e_{h_i}^T\right) \\ &+\left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+L(H_l')}^{(T_l,T_l)}(x)\right)\left(e_{h_i}e_{h_i}^T\right) + \left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+L(H_l')}^{(T_l,S_l)}(x)\right)\left(e_{h_i}e_{k_i}^T\right)\Bigg). \end{aligned} \tag{3.28}$$

and N is given in (3.2).

(iii) *The characteristic polynomial of $\mathcal{Q}(G \boxplus_M \mathcal{H}_k)$ is*

$$\left(\prod_{i=1}^{k}\left(\Phi_{\mathcal{Q}(H_i')+D_i}(x)\right)^{c_i}\right)det(xI - N - \mathcal{Q}(G) - C_3),$$

where

$$\begin{aligned} C_3 = \sum_{i=1}^{m}\Bigg(&\left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+\mathcal{Q}(H_l')}^{(S_l,S_l)}(x)\right)\left(e_{k_i}e_{k_i}^T\right) + \left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+\mathcal{Q}(H_l')}^{(S_l,T_l)}(x)\right)\left(e_{k_i}e_{h_i}^T\right) \\ &+\left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+\mathcal{Q}(H_l')}^{(T_l,T_l)}(x)\right)\left(e_{h_i}e_{h_i}^T\right) + \left(\sum_{l=1}^{k} m_{il}\Gamma_{D_l+\mathcal{Q}(H_l')}^{(T_l,S_l)}(x)\right)\left(e_{h_i}e_{k_i}^T\right)\Bigg) \end{aligned}$$

and N is given in (3.2).

As a direct consequence of Theorem 3.1 we have the following result.

Corollary 3.2 *Let H be a graph on p vertices such that the end vertices of its specified edge are adjacent to all other vertices. Let $H_i = H$ for $i = 1, 2, \ldots, h$ with $h \le m$. Let*

$$M = \begin{bmatrix} I_h \\ \mathbf{0}_{(m-h)\times h} \end{bmatrix},$$

and let G' be a r-regular graph on q vertices. Then

$$\phi_{G \boxplus_M \mathcal{H}_h}(x,t) = \left(\Phi_{M_t(H')}(x)\right)^h det(xI - A(G) + tD(G)) + P), \tag{3.29}$$

where

$$P = \begin{bmatrix} (tr(p-2) - r\Gamma_{M_t(H')}(x))I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} - \Gamma_{M_t(H')}(x)\begin{bmatrix} A(G') & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

and H' is a graph obtained from H by deleting the end vertices of its specified edge.

Proof Since

$$M = \begin{bmatrix} I_h \\ \mathbf{0}_{(m-h)\times h} \end{bmatrix},$$

and $d_i = p-2$ for $i = 1, 2, \ldots, h$. Then from (3.2), we get

$$N = \begin{bmatrix} r(p-2)I_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-q} \end{bmatrix}. \tag{3.30}$$

For $i = 1, 2, \ldots, h$, since $H_i = H$ and $S_i = T_i = V(H')$. Then by (3.3), we get

$$C = \Gamma_{M_i(H')}(x) \begin{bmatrix} r(p-2)I_q + A(G') & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-q} \end{bmatrix}. \tag{3.31}$$

Now substituting (3.30) and (3.31) in (3.1), we get the result. □

Remark 3.1 From Corollary 3.2, we deduce the following existing main results on the spectra of the graph with edge pockets. Let S be a subset of $E(G)$ with $|S| = h$. Let H be a graph on p vertices such that the end vertices of its specified edge are adjacent to all other vertices. Let G' be a r-regular graph on q vertices. If the rows of the block I_h in the matrix M mentioned in Corollary 3.2 are indexed by S, then from (3.29) we get the following.

(1) ([31, Theorem 4.4]) $\phi_{G\boxplus_M\mathcal{H}_h}(x, o)$ is the characteristic polynomial of $A(G(S, H))$;
(2) ([31, Theorem 3.10]) $(-1)^{|V(G\boxplus_M\mathcal{H}_h)|}\phi_{G\boxplus_M\mathcal{H}_h}(-x, 1)$ is the characteristic polynomial of $L(G(S, H))$;
(3) ([10, Proposition 4.1]) $\phi_{G\boxplus_M\mathcal{H}_h}(x, -1)$ is the characteristic polynomial of $\mathcal{Q}(G(S, H))$.

4 Some particular constructions

In this section, we derive the L-spectra of some particular constructions of M-edge rooted product of graphs from Corollary 3.1(ii). As a consequence, we get some infinite families of L-cospectral graphs and integral graphs.

Throughout this section, we assume that G is a connected r-regular graph on n vertices, $m \geq 1$ edges with $\lambda_1(G) = r$ and M is a non-zero matrix of size $m \times k$ whose entries are non-negative integers such that sum of all entries in each of its row is b, unless we specifically mentioned otherwise.

Corollary 4.1 *For $i = 1, 2, \ldots, k$, let H_i be a graph on $p+2$ vertices such that the end vertices of its specified edge are adjacent to all other vertices. Then the L-spectrum of $G\boxplus_M\mathcal{H}_k$ is*

(i) 2 *with multiplicity* $bm - n$,
(ii) $2 + \mu_j(H_i')$ *with multiplicity* c_i *for* $i = 1, 2, \ldots, k$, $j = 2, \ldots, p$,
(iii) $\frac{1}{2}\left(pbr + \mu_i(G) + 2 \pm \sqrt{(pbr + \mu_i(G) + 2)^2 - 4bp\mu_i(G) - 8\mu_i(G)}\right)$ *with multiplicity* 1 *for* $i = 1, 2, \ldots, n$,

Proof For $i = 1, 2, \ldots, k$, since $S_i = T_i = V(H_i')$ and so $D_i = 2I_p$. Then by Theorem 2.2, we get

$$\Gamma^{(S_i,S_i)}_{L(H_i')+D_i}(x) = \Gamma^{(S_i,T_i)}_{L(H_i')+D_i}(x) = \Gamma^{(T_i,S_i)}_{L(H_i')+D_i}(x) = \Gamma^{(T_i,T_i)}_{L(H_i')+D_i}(x) = \frac{p}{x-2}. \tag{4.1}$$

Notice that each row sum of M is b, G is r-regular and $d_i = p$ for $i = 1, 2, \ldots, p$. Using these facts and applying (4.1) in (3.28) and (3.2), we get

$$C_2 = \left(\frac{bp}{x-2}\right)\mathcal{Q}(G)$$

and $N = bprI_n$. From the above equations, we have

$$
\begin{aligned}
det(xI_n - N - L(G) - C_2) &= det\left(\left(x - r - brp - \frac{bpr}{x-2}\right)I_n - \left(\frac{bp}{x-2} - 1\right)A(G)\right) \\
&= \frac{1}{(x-2)^n}\prod_{i=1}^{n}\big(x^2 - (bpr + 2 + \mu_i(G))x + (2+bp)\mu_i(G)\big).
\end{aligned}
\tag{4.2}
$$

It is not hard to obtain that

$$
\Phi_{L(H_i')+D_i}(x) = (x-2)\prod_{j=2}^{p}(x - 2 - \mu_j(H_i')) \tag{4.3}
$$

for $i = 1, 2, \ldots, k$. Applying equations (4.2) and (4.3) in (3.27) we get the result. □

Next we present an existing result and give its proof for the sake of completeness.

Proposition 4.1 ([34]) *Let A_1, A_2 and A_3 be real matrices of size $p \times p$, $p \times q$ and $q \times q$, respectively, where $p \le q$, with the following condition: For $i = 1, 2, \ldots, p$, $A_1\mathbf{x}_i = \delta_i\mathbf{x}_i$, $A_3\mathbf{y}_i = \zeta_i\mathbf{y}_i$, $A_2\mathbf{y}_i = \eta_i\mathbf{x}_i$, $A_2^T\mathbf{x}_i = \eta_i\mathbf{y}_i$, and for $i = p+1, p+2, \ldots, q$, $A_2\mathbf{y}_i = \mathbf{0}$ and $A_3\mathbf{y}_i = \zeta_i\mathbf{y}_i$ where δ_i, ζ_i are eigenvalues of A_1 and A_3 corresponding to the eigenvectors $\mathbf{x}_i$ and $\mathbf{y}_i$ respectively and η_i is a scalar. Let*

$$
A = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{bmatrix} \tag{4.4}
$$

be a block matrix and A_1, A_3 are symmetric, then the eigenvalues of A are

(i) $\frac{1}{2}\left(\delta_i + \zeta_i \pm \sqrt{(\delta_i - \zeta_i)^2 + 4\eta_i^2}\right)$ *with multiplicity* 1 *for $i = 1, 2, \ldots, p$;*

(ii) ζ_i *with multiplicity* 1 *for $i = p+1, p+2, \ldots, q$.*

Proof For $i = 1, 2, \ldots, p$, consider the vector

$$
\mathbf{z}_i = \begin{pmatrix} k\mathbf{x}_i \\ \mathbf{y}_i \end{pmatrix}. \tag{4.5}
$$

where k is unknown constant. For $i = 1, 2, \ldots, p$, from $A\mathbf{z}_i = \lambda_i\mathbf{z}_i$ by eliminating k, we get

$$
\lambda_i = \frac{1}{2}\left(\delta_i + \zeta_i \pm \sqrt{(\delta_i - \zeta_i)^2 + 4\eta_i^2}\right).
$$

Also for $i = p+1, p+2, \ldots, q$,

$$
A\begin{pmatrix} \mathbf{0}_p \\ \mathbf{y}_i \end{pmatrix} = \zeta_i\begin{pmatrix} \mathbf{0}_p \\ \mathbf{y}_i \end{pmatrix}. \tag{4.6}
$$

This completes the proof. □

Corollary 4.2 *For $i = 1, 2, \ldots, k$, let F_i^1 and F_i^2 be graphs on p, q vertices, respectively with $p \le q$. Let N be a $0-1$ matrix of size $p \times q$ such that its row sum and column sum is τ_1 and τ_2, respectively. For $i = 1, 2, \ldots, k$, let H_i be a graph obtained from $F_i^1 \vee_N F_i^2$ by adding a new edge and joining its end vertices to all the vertices of F_i^1. Let this new edge be the specified edge of H_i. If $L(F_i^1)$, N and $L(F_i^2)$ satisfies the conditions of A_1, A_2 and A_3, respectively, given in Proposition* 4.1, *then the L-spectrum of $G \boxplus_M \mathcal{H}_k$ is*

(i) $\frac{1}{2}\left(\tau + \mu_j(F_i^1) + \mu_j(F_i^2) \pm \sqrt{(\tau + \mu_j(F_i^1) + \mu_j(F_i^2))^2 - 4(\tau_1 + 2 + \mu_j(F_i^1))(\tau_2 + \mu_j(F_i^2)) + 4\eta_j^2}\right)$, *with multiplicity c_i for $i = 1, 2, \ldots, k$, $j = 2, 3, \ldots, p$, where $\tau = \tau_1 + \tau_2 + 2$,*

(ii) $\tau_2 + \mu_j(F_i^2)$ *with multiplicity c_i for $i = 1, 2, \ldots, k$, $j = p+1, \ldots, q$,*

(iii) *roots of the polynomial*

$$
\begin{aligned}
&x^3 - \Big(\tau_1 + \tau_2 + 2 + bpr + \mu_i(G)\Big)x^2 + \Big(2\tau_2 + (bp+2)\mu_i(G) + (\tau_1 + \tau_2)\big(bpr + \mu_i(G)\big)\Big)x \\
&\quad - \tau_2\mu_i(G)(2 + bp)
\end{aligned}
\tag{4.7}
$$

with multiplicity 1 *for $i = 1, 2, \ldots, n$.*

Proof For $i = 1, 2, \ldots, k$, with a suitable ordering of vertices of H_i', we get

$$L(H_i') = \begin{bmatrix} L(F_i^1) + \tau_1 I_p & -N \\ -N^T & L(F_i^2) + \tau_2 I_q \end{bmatrix}. \tag{4.8}$$

For $i = 1, 2, \ldots, k$, since the end vertices of the specified edge of H_i are adjacent to all the vertices of F_i^1, we get $S_i = T_i = V(F_i^1)$ and

$$D_i = \begin{bmatrix} 2I_p & \mathbf{0}_{p\times q} \\ \mathbf{0}_{q\times p} & \mathbf{0}_q \end{bmatrix}. \tag{4.9}$$

So from (4.8) and (4.9), we get

$$L(H_i') + D_i = \begin{bmatrix} (\tau_1 + 2)I_p + L(F_i^1) & -N \\ -N^T & \tau_2 I_q + L(F_i^2) \end{bmatrix}. \tag{4.10}$$

By using Proposition 4.1 in (4.10), we get

$$\begin{aligned} \Phi_{L(H_i')+D_i}(x) &= \left(\prod_{j=1}^{p} (x - \tau_1 - 2 - \mu_j(F_i^1))(x - \tau_2 - \mu_j(F_i^2)) - \eta_j^2 \right) \\ &\quad \times \left(\prod_{j=p+1}^{q} (x - \tau_2 - \mu_j(F_i^2)) \right) \end{aligned} \tag{4.11}$$

for $i = 1, 2, \ldots, k$. Let $\mathbf{x}_1 = \tau_1 \mathbf{1}_p$ and $\mathbf{y}_1 = \sqrt{\tau_1\tau_2}\mathbf{1}_q$. Then $\mathbf{x}_1$ and $\mathbf{y}_1$ are eigenvectors of $L(F_i^1)$ and $L(F_i^2)$ corresponding to the eigenvalue 0 and $N\mathbf{y}_1 = \sqrt{\tau_1\tau_2}\mathbf{x}_1$, $N^T\mathbf{x}_1 = \sqrt{\tau_1\tau_2}\mathbf{y}_1$. So for $i = 1, 2, \ldots, k$, (4.11) becomes

$$\begin{aligned} \Phi_{L(H_i')+D_i}(x) &= \left(\prod_{j=2}^{p} (x - \tau_1 - 2 - \mu_j(F_i^1))(x - \tau_2 - \mu_j(F_i^2)) - \eta_j^2 \right) \\ &\quad \times \Big((x - \tau_1 - 2)(x - \tau_2) - \tau_1\tau_2 \Big) \left(\prod_{j=p+1}^{q} (x - \tau_2 - \mu_j(F_i^2)) \right). \end{aligned} \tag{4.12}$$

For $i = 1, 2, \ldots, k$, since $S_i = T_i = V(F_i^1)$, row sum and column sum of N is τ_1 and τ_2, respectively, so by applying Proposition 2.1 in (4.10), we get

$$\begin{aligned} \Gamma_{L(H_i')+D_i}^{(S_i,S_i)}(x) = \Gamma_{L(H_i')+D_i}^{(T_i,T_i)}(x) &= \Gamma_{L(H_i')+D_i}^{(S_i,T_i)}(x) = \Gamma_{L(H_i')+D_i}^{(T_i,S_i)}(x) \\ &= \frac{p(x - \tau_2)}{(x - \tau_1 - 2)(x - \tau_2) - \tau_1\tau_2}. \end{aligned} \tag{4.13}$$

Notice that G is r-regular, sum of all entries in each row of M is b and $d_i = p$ for $i = 1, 2, \ldots, k$. Using these facts and (4.13) in (3.28) and (3.2), we get

$$C_2 = \frac{bp(x - \tau_2)}{(x - \tau_1 - 2)(x - \tau_2) - \tau_1\tau_2} \mathcal{Q}(G) \tag{4.14}$$

and $N = bprI_n$. Therefore

$$\begin{aligned} &det(xI - N - L(G) - C_2) \\ &= det\left(\left(x - bpr - r - \frac{rbp(x - \tau_2)}{(x - \tau_1 - 2)(x - \tau_2) - \tau_1\tau_2} \right) I - \left(\frac{bp(x - \tau_2)}{(x - \tau_1 - 2)(x - \tau_2) - \tau_1\tau_2} - 1 \right) A(G) \right) \\ &= \prod_{i=1}^{n} \left(\frac{(x - bpr - r + \lambda_i(G))((x - \tau_1 - 2)(x - \tau_2) - \tau_1\tau_2) - bpr(x - \tau_2) - \lambda_i(G)bp(x - \tau_2)}{(x - \tau_1 - 2)(x - \tau_2) - \tau_1\tau_2} \right) \end{aligned} \tag{4.15}$$

Now substituting (4.12) and (4.15) in (3.27) we get the result. □

The following result is used in the subsequent results.

Proposition 4.2 ([15], Kapitel 8.5, Folgerung 3) *Let $M_1, M_2, \ldots, M_p$ be real symmetric matrices of size n. Then the following statements are equivalent:*

(1) $M_iM_j = M_jM_i$ for all $i, j \in \{1, 2, \ldots, p\}$.
(2) *There exists an orthonormal basis $\{y_1, y_2, \ldots, y_n\}$ of $\mathbb{R}^n$ such that $y_1, y_2, \ldots, y_n$ are eigenvectors of M_i for $i = 1, 2, \ldots, p$.*

Note 4.1 Let F_1 and F_2 be graphs on p and q vertices, respectively and let N be a $0-1$ matrix of size $p \times q$.

Some particular ways of choosing the matrix N to construct $F_1 \vee_N F_2$ are given below so that these can be used in Corollary 4.2.

(i) I_p, $J_{p\times p} - I_p$, if F_1 and F_2 are graphs on p vertices and $L(F_1)$ commutes with $L(F_2)$.
(ii) $A(F)$, $A(F) + I$, $J - A(F)$, $J - I - A(F)$, where F is any regular graph on p vertices, if F_1 and F_2 are graphs on p vertices and $L(F_1)$, $L(F_2)$, $A(F)$ are commuting matrices.
(iii) Any $0-1$ matrix of size $p \times q$ with constant row sum and column sum if F_1 and F_2 are totally disconnected.
(iv) $J_{p\times q}$,
(v) $B(F)$, $J - B(F)$, where F is any r-regular graph on p vertices and q edges, if $F_1 = \overline{K}_p$ or F and $F_2 = \overline{K}_q$ or $\mathcal{L}(F)$.

The above Cases (i) and (ii) can be verified easily by using Proposition 4.2. Remaining cases can be verified by using the facts that $B(F)(B(F))^T = A(F) + rI_p$, $(B(F))^T B(F) = A(\mathcal{L}(F)) + 2I_q$, and the properties of singular values and singular vectors. The details are omitted.

Further, some constructions of families of non-isomorphic regular graphs with commuting adjacency matrices were given in [23, Section 2.3]. If the adjacency matrices of two regular graphs commutes, then their Laplacian matrices are so. These facts guarantees the existence of non-isomorphic graphs satisfying Cases (i) and (ii).

Corollary 4.3 *For $i = 1, 2, \ldots, k$, let F_i be a graph on p vertices and let F_i^1, F_i^2 be two copies of F_i. Let N be a symmetric $0-1$ matrix of size $p \times p$ such that its row sum is τ. For $i = 1, 2, \ldots, k$, let H_i be a graph obtained from $F_i^1 \vee_N F_i^2$ by adding a new edge and joining its one end to all the vertices of F_i^1 and another end to all vertices of F_i^2. Let this new edge be the specified edge of H_i. If $L(F_i)$ and N are commuting matrices for $i = 1, 2, \ldots, k$, then the L-spectrum of $G \boxplus_M \mathcal{H}_k$ is*

(i) 1 *with multiplicity $mb - n$,*
(ii) $1 + 2\tau$ *with multiplicity $mb - n$,*
(iii) $1 + \tau + \mu_j(F_i) - \delta_j(N)$ *with multiplicity c_i for $i = 1, 2, \ldots, k$, $j = 2, 3, \ldots, p$,*
(iv) $1 + \tau + \mu_j(F_i) + \delta_j(N)$ *with multiplicity c_i for $i = 1, 2, \ldots, k$, $j = 2, 3, \ldots, p$,*
(v) *roots of the polynomial*

$$\begin{aligned} &x^3 - \Big(2\tau + 2 + bpr + \mu_i(G)\Big)x^2 + \Big(2\tau + 1 + (2\tau + 2)\big(bpr + \mu_i(G)\big) - bpr\Big)x \\ &- (2\tau + 1 + bp\tau)\mu_i(G) \end{aligned}$$

with multiplicity 1 for $i = 1, 2, \ldots, n$.

Proof For $i = 1, 2, \ldots, k$, with a suitable ordering of vertices of H_i', we get

$$L(H_i') = \begin{bmatrix} L(F_i) + \tau I_p & -N \\ -N & L(F_i) + \tau I_p \end{bmatrix}. \tag{4.16}$$

For $i = 1, 2, \ldots, k$, since one end vertex of the specified edge of H_i is adjacent to all the vertices of F_i^1 and the another end vertex is adjacent to all the vertices of F_i^2, we have $S_i = V(F_i^1)$ and $T_i = V(F_i^2)$ and so $D_i = I_{2p}$. So from (4.16), we get

$$L(H_i') + D_i = \begin{bmatrix} (\tau+1)I_p + L(F_i) & -N \\ -N & (\tau+1)I_p + L(F_i) \end{bmatrix}. \tag{4.17}$$

For $i = 1, 2, \ldots, k$, since $L(F_i)$ and N are commuting matrices, Proposition 4.2 assures that there exist orthonormal vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ such that $(L(F_i) + (\tau+1)I_p)\mathbf{x}_j = (\mu_j(F_i) + 1 + \tau)\mathbf{x}_j$ and $N\mathbf{x}_j = \delta_j(N)\mathbf{x}_j$, where $\delta_j(N)$ is an eigenvalue of N for $j = 1, 2, \ldots, p$. Then by applying Proposition 4.1, we get

$$\Phi_{L(H_i')+D_i}(x) = \prod_{j=1}^{p} \big((x - \tau - 1 - \mu_j(F_i) - \delta_j(N))(x - \tau - 1 - \mu_j(F_i) + \delta_j(N))\big).$$

For $i = 1, 2, \ldots, k$, since $\mu_1(F_i) = 0$ and $\delta_1(N) = \tau$ corresponding to the eigenvector $\frac{1}{\sqrt{p}}\mathbf{1}_p$, so the above equation becomes

$$\Phi_{L(H_i')+D_i}(x) = \Big((x - \tau - 1)^2 - \tau^2\Big) \prod_{j=2}^{p} \Big((x - \tau - 1 - \mu_j(F_i))^2 - \delta_j(N)^2\Big). \tag{4.18}$$

Since $S_i = V(F_i^1)$, $T_i = S_i^c$ for $i = 1, 2, \ldots, k$ and row sum of the symmetric matrix N is τ, by using Proposition 2.1 and Theorem 2.3 in (4.17), we get

$$\Gamma_{L(H_i')+D_i}^{(S_i,S_i)}(x) = \Gamma_{L(H_i')+D_i}^{(T_i,T_i)}(x) = \frac{p(x - \tau - 1)}{(x - \tau - 1)^2 - \tau^2} \tag{4.19}$$

and

$$\Gamma_{L(H_i')+D_i}^{(S_i,T_i)}(x) = \Gamma_{L(H_i')+D_i}^{(T_i,S_i)}(x) = \frac{-p\tau}{(x - \tau - 1)^2 - \tau^2} \tag{4.20}$$

Notice that the sum of all entries in each row of M is b, G is r-regular and $d_i = p$ for $i = 1, 2, \ldots, k$. Using these facts and substituting (4.19), (4.20) in (3.28) and (3.2), we get

$$C_2 = \left(\frac{bp(x - \tau - 1)}{(x - \tau - 1)^2 - \tau^2}\right) D(G) + \left(\frac{-bp\tau}{(x - \tau - 1)^2 - \tau^2}\right) A(G) \tag{4.21}$$

and $N = bprI_n$. Therefore,

$$\begin{aligned} &det(xI - N - L(G) - C_2) \\ &= det\left(\left(x - bpr - r - \frac{bpr(x - \tau - 1)}{(x - 1 - \tau)^2 - \tau^2}\right) I - \left(\frac{-bp\tau}{(x - \tau - 1)^2 - \tau^2} - 1\right) A(G)\right) \\ &= \prod_{i=1}^{n} \left(\frac{(x - bp\tau - \tau + \lambda_i(G))((x - \tau - 1)^2 - \tau^2) - bpr(x - \tau - 1) + \lambda_i(G) bp\tau}{(x - \tau - 1)^2 - \tau^2}\right). \end{aligned} \tag{4.22}$$

Now substituting (4.22) and (4.18) in (3.27) we get the result. □

Note 4.2 The following are some particular ways of choosing the matrix N to construct $F_i^1 \vee_N F_i^2$ which can be used in Corollary 4.3: I_p, $J_{p\times p}$, $J_{p\times p} - I_p$, $A(F)$, $A(F) + I$, $J - A(F)$, $J - I - A(F)$, where F is any regular graph on p vertices, if $L(F)$ and $A(F)$ are commuting matrices.

Corollary 4.4 *For $i = 1, 2, \ldots, k$, let F_i be a graph on p vertices and F, a r_2-regular graph on $q \geq 2$ vertices. Let N be a $0 - 1$ matrix of size $p \times q$ whose entries in the first column are 1 and the remaining entries are 0. For $i = 1, 2, \ldots, k$, let H_i be the graph obtained from $F_i \vee_N F$ by adding a new edge and joining its end vertices to all vertices of F_i. Let this new edge be the specified edge of H_i for all $i = 1, 2, \ldots, k$. Let u be a vertex of F which is adjacent to all the vertices of F_i in $F_i \vee_N F$. Then the L-spectrum of $G \boxplus_M \mathcal{H}_k$ is*

(i) $3 + \mu_j(F_i)$ *with multiplicity c_i for $i = 1, 2, \ldots, k$, $j = 2, 3, \ldots, p$,*
(ii) *roots of the polynomial*

$$(x - 3)\phi_{L(F)}(x) - p(x - 2)\phi_{A(F-u)}(x - r_2)$$

with multiplicity $mb-n$,
(iii) *roots of the polynomial*

$$\begin{aligned}&(x-r-bpr+\lambda_i(G))\Big((x-3)\phi_{L(F)}(x)-p(x-2)\phi_{A(F-u)}(x-r_2)\Big)\\&\quad-bp(r+\lambda_i(G))\Big(\phi_{L(F)}(x)-p\phi_{A(F-u)}(x-r_2)\Big)\end{aligned}$$

with multiplicity 1 *for* $i=1,2,\ldots,n$.

Proof For $i=1,2,\ldots,k$, with a suitable ordering of vertices of H_i', we get

$$L(H_i')=\begin{bmatrix}L(F_i)+I_p & -N\\ -N^T & L(F)+N'\end{bmatrix}. \tag{4.23}$$

where $N'=diag(p,0,\ldots,0)$ of size q. For $i=1,2,\ldots,k$, since the end vertices of the specified edge of H_i are adjacent to all the vertices of F_i, we get $S_i=T_i=V(F_i)$ and

$$D_i=\begin{bmatrix}2I_p & \mathbf{0}_{p\times q}\\ \mathbf{0}_{q\times p} & \mathbf{0}_q\end{bmatrix}. \tag{4.24}$$

So from (4.23) and (4.24), we get

$$L(H_i')+D_i=\begin{bmatrix}L(F_i)+3I_p & -N\\ -N^T & L(F)+N'\end{bmatrix} \tag{4.25}$$

for $i=1,2,\ldots,k$. By using (2.1) and [30, Exercise 6.2.7, pp. 483] in the above equation, we get

$$\begin{aligned}\Phi_{L(H_i')+D_i}(x)&=det\big(xI_q-L(F)-N'\big)det\big((x-3)I_p-L(F_i)-a_{11}J_p\big)\\&=\big(\Phi_{L(F)}(x)-p\Phi_{A(F-u)}(x-r_2)\big)det\Big((x-3)I_p-L(F_i)-a_{11}J_{p\times 1}J_{p\times 1}^T\Big)\\&=\big(\Phi_{L(F)}(x)-p\Phi_{A(F-u)}(x-r_2)\big)det((x-3)I_p-L(F_i))\\&\quad\times det(1+J_{p\times 1}^T((x-3)I_p-L(F_i))^{-1}(-a_{11}J_{p\times 1}))\\&=\big(\Phi_{L(F)}(x)-p\Phi_{A(F-u)}(x-r_2)\big)\left[\prod_{j=1}^{p}(x-3-\mu_j(F_i))\right]\left[1-\frac{a_{11}p}{x-3}\right]\\&=\big(\Phi_{L(F)}(x)-p\Phi_{A(F-u)}(x-r_2)\big)(x-3-pa_{11})\prod_{j=2}^{p}(x-3-\mu_j(F_i)),\end{aligned} \tag{4.26}$$

where $F-u$ is the subgraph of F which is obtained by deleting the vertex u and a_{11} is (1,1)-th entry of $(xI_q-L(F)-N')^{-1}$. Notice that

$$a_{11}=\frac{\Phi_{A(F-u)}(x-r_2)}{\Phi_{L(F)}(x)-p\Phi_{A(F-u)}(x-r_2)}. \tag{4.27}$$

For $i=1,2,\ldots,k$, since $S_i=T_i=V(F_i)$ and by using (2.2), Theorem 2.2 in (4.25), we get

$$\Gamma_{L(H_i')+D_i}^{(S_i,S_i)}(x)=\Gamma_{L(H_i')+D_i}^{(S_i,T_i)}(x)=\Gamma_{L(H_i')+D_i}^{(T_i,S_i)}(x)=\Gamma_{L(H_i')+D_i}^{(T_i,T_i)}(x)=\frac{p}{x-3-pa_{11}}. \tag{4.28}$$

Notice that each row sum of M is b, G is r-regular and $d_i=p$ for $i=1,2,\ldots,k$. Using these facts and (4.28) in (3.28), (3.2), we get

$$C_2=\left(\frac{bp}{x-3-pa_{11}}\right)\mathcal{Q}(G)$$

and $N=bprI_n$. Therefore,

$$
\begin{aligned}
&det(xI_n - N - L(G) - C_2)\\
&= det\left(\left(x - r - bpr - \frac{bpr}{x-3-pa_{11}}\right)I_n - \left(\frac{bp}{x-3-pa_{11}} - 1\right)A(G)\right)\\
&= \prod_{i=1}^{n} \frac{(x - bpr - r + \lambda_i(G))(x-3-pa_{11}) - bp(r+\lambda_i(G))}{x-3-pa_{11}}.
\end{aligned} \tag{4.29}
$$

Applying equations (4.29), (4.27), (4.26) in (3.27) we get the result. □

Remark 4.1 Let F be any regular graph on n vertices, m edges. Then the row sum of each of the following matrix is equal to a constant: $\alpha A(\mathcal{L}(F))$, $\alpha \mathcal{Q}(\mathcal{L}(F))$, αI_m, $\alpha\overline{(J - A(\mathcal{L}(F)))}$, $\alpha\overline{(I + A(\mathcal{L}(F)))}$, $\alpha(J - I - A(\mathcal{L}(F)))$, $\alpha(J_{m\times m} - I_{m\times m})$, $\alpha B(\mathcal{L}(F))$, $\alpha(J - B(\mathcal{L}(F)))$, $\alpha B(\overline{\mathcal{L}(F)})$, $\alpha(J - B(\overline{\mathcal{L}(F)}))$, where α is a positive integer. Consequently, from the Corollaries proved so far in this section, we can obtain the L-spectrum of the graph $G \boxplus_M \mathcal{H}_k$ when M is any one of the above mentioned matrices. In particular, if $F = G$, a regular graph on n vertices, then we obtain the L-spectrum of the special M-edge rooted product of the graphs defined in Section 1.

Remark 4.2 From parts (i) and (iii) of Corollary 3.1, by the analogues method used in the proofs of Corollaries 4.1 - 4.4, we can obtain the characteristic polynomials of the adjacency and the signless Laplacian matrices of the graphs considered in these results, with the following additional assumptions:

(i) Taking H_i' as q-regular for all $i = 1, 2, \ldots, k$ in Corollary 4.1.
(ii) Taking F_{i1} and F_{i2} are q_1, q_2 regular, respectively for all $i = 1, 2, \ldots, k$ in Corollary 4.2 with $A(F_{i1})$, $A(F_{i2})$ and N satisfies the conditions of A_1, A_3 and A_2, respectively given in Proposition 4.1 .
(iii) Taking F_i as q-regular for all $i = 1, 2, \ldots, k$ in Corollary 4.3 with $A(F_i)$ and N are commuting matrices.
(iv) Taking F_i as q-regular for all $i = 1, 2, \ldots, k$ in Corollary 4.4.

4.1 Cospectral graphs

As an application of the results obtained so far in this section, we present some ways to construct cospectral graphs.

One of the significance of this graph construction is that it enable us to construct infinite families of L-cospectral graphs without assuming any additional cospectrality condition on the constituting graphs, which is described below.

Let $\mathcal{H}_k$ be a sequence of k graphs H_i for $i = 1, 2, \ldots, k$, on $p + 2$ vertices with specified edges. Assume that the end vertices of the specified edge of H_i are adjacent to all other vertices. For each integer $b \geq 1$, let $\mathcal{M}_b$ be the set of all non-zero matrices of size $m \times k$ whose entries are non-negative integers such that each matrix in $\mathcal{M}_b$ has row sum is b and i-th column sum c_i for $i = 1, 2, \ldots, k$. Let

$$\mathcal{G}_b = \{G \boxplus_M \mathcal{H}_k | M \in \mathcal{M}_b\}.$$

Clearly, for each b, Corollary 4.1 assures that any pair of graphs in $\mathcal{G}_b$ are L-cospectral. Analogously, we can obtain the L-cospectral graphs by using Corollaries 4.3, 4.2 and 4.4.

In the following results, let $\widehat{G}$ be a connected r-regular graph on n vertices and m edges. Let $\widehat{\mathcal{H}}_k$ be a sequence of k graphs $\widehat{H}_i$ for $i = 1, 2, \ldots, k$ with $V(\widehat{H}_i) = \{\widehat{u}_{i1}, \widehat{u}_{i2}, \ldots, \widehat{u}_{in_i}\}$, $n_i \geq 3$ and $m_i \geq 1$. Let $(\widehat{u}_{in_i-1}, \widehat{u}_{in_i})$ be the specified edge of $\widehat{H}_i$ such that the graphs $\widehat{H}_i - \widehat{u}_{in_i}$ and $\widehat{H}_i - \widehat{u}_{in_i-1}$ are isomorphic. Let $\widehat{M} = (\widehat{m}_{ij})$ be a non-zero matrix of size $m \times k$ whose entries are non-negative integers with its each row sum and i-th column sum is $b \geq 1$ and c_i, respectively, for $i = 1, 2, \ldots, k$.

Corollary 4.5 *The graphs $G \boxplus_M \mathcal{H}_k$ and $\widehat{G} \boxplus_{\widehat{M}} \widehat{\mathcal{H}}_k$ considered in Corollary 4.1 are L -cospectral, if the pairs $(G, \widehat{G})$ and $(H_i', \widehat{H}_i')$ for $i = 1, 2, \ldots, k$ are L-cospectral.*

Corollary 4.6 *For $i = 1, 2, \ldots, k$, let $\widehat{F}_i^1$ and $\widehat{F}_i^2$ be graphs on p and q vertices, respectively with $p \leq q$ and let $\widehat{N} = N$ be a $0-1$ matrix of size $p \times q$ such that its row sum and column sum is τ_1 and τ_2, respectively. Then the graphs $G \boxplus_M \mathcal{H}_k$ and $\widehat{G} \boxplus_{\widehat{M}} \widehat{\mathcal{H}}_k$ considered in Corollary 4.2 are L-cospectral, if the pairs $(G, \widehat{G})$,*

$(F_i^1, \widehat{F}_i^1)$ *are L-cospectral and one of the following hold:*

(i) $N = J_{p\times q}$ *and* $(F_i^2, \widehat{F}_i^2)$ *are L-cospectral,*
(ii) $N = I_p$ *and* $F_i^2 = \widehat{F}_i^2 = K_p$ or $\overline{K}_p$,
(iii) $N = J_p - I_p$ *and* $F_i^2 = \widehat{F}_i^2 = K_p$ or $\overline{K}_p$.

Corollary 4.7 *For* $i = 1, 2, \ldots, k$, *let* $\widehat{F}_i$ *be a graph on p vertices. Let* $\widehat{N} = N$ *be a* $0-1$ *symmetric matrix of size* $p \times p$ *such that its row sum is* τ. *Then the graphs* $G \boxplus_M \mathcal{H}_k$ *and* $\widehat{G} \boxplus_{\widehat{M}} \widehat{\mathcal{H}}_k$ *considered in Corollary* 4.3 *are L-cospectral, if the pairs* $(G, \widehat{G})$, $(F_i, \widehat{F}_i)$ *are L-cospectral and* $N = J_{p\times p}, I_p$ *or* $J_p - I_p$.

Corollary 4.8 *Let* $\widehat{F}$ *be a* r_2*-regular graph on* $q \geq 2$ *vertices and let* $\widehat{F}_i$ *be a graph on p vertices for* $i = 1, 2, \ldots, k$. *Let* $\widehat{N} = N$ *be a* $0-1$ *matrix of size* $p \times q$ *whose entries in the first column are* 1 *and the remaining are* 0. *Let* $\widehat{u}$ *be a vertex of* $\widehat{F}$ *which is adjacent to all the vertices of* $\widehat{F}_i$ *in* $\widehat{F}_i \vee_N \widehat{F}$. *The graphs* $G \boxplus_M \mathcal{H}_k$ *and* $\widehat{G} \boxplus_{\widehat{M}} \widehat{\mathcal{H}}_k$ *considered in Corollary* 4.4 *are L-cospectral, if the pairs* $(F_i, \widehat{F}_i)$ *and* $(F, \widehat{F})$ *are L-cospectral, the pairs* $(G, \widehat{G})$ *and* $(F - u, \widehat{F} - \widehat{u})$ *are A-cospectral.*

In a similar way to the above and from the information given in Remark 4.2, we can construct A-cospectral and $\mathcal{Q}$-cospectral graphs.

4.2 A-integral graphs and L-integral graphs

As another application, in this section, we construct infinite number of A-integral graphs and L-integral graphs.

Corollary 4.9

(1) *For* $i = 1, 2, \ldots, m$, *where* $m = \frac{n(n+1)}{2}$ *for some positive integer n, let* F_i *be a* $(n + r)$*-regular A-integral graph on* $2n + r$ *vertices with* $n + r \geq 0$. *Let* H_i *be the graph obtained from* F_i *by adding a new edge and joining its end vertices to all vertices of* F_i. *Let this new edge be the specified edge of* H_i *for* $i = 1, 2, \ldots, m$. *Then the graph* $K_{n+1} \boxplus_{I_m} \mathcal{H}_m$ *is A-integral.*
(2) *For* $i = 1, 2, \ldots, m$, *where* $m = n^2$ *for some positive integer n, let* F_i *be a* $(n + r)$*-regular A-integral graph on* $2n + r$ *vertices, where r is an integer with* $n + r \geq 0$. *Let* H_i *be the graph obtained from* F_i *by adding a new edge and joining its end vertices to all vertices of* F_i. *Let this new edge be the specified edge of* H_i *for* $i = 1, 2, \ldots, m$. *Then the graph* $K_{n,n} \boxplus_{I_m} \mathcal{H}_m$ *is A-integral.*
(3) *For* $i = 1, 2, \ldots, 4$, *Let* H_i *be the graph obtained from* $\overline{K}_n$ *by adding a new edge and joining its end vertices to all vertices of* $\overline{K}_n$. *Let this new edge be the specified edge of* H_i. *Then the graph* $C_4 \boxplus_{I_4} \mathcal{H}_4$ *is A-integral if* $\sqrt{2n}$ *and* $\sqrt{1 + 4n}$ *are integers.*
(4) *Let G be the* 4*-regular graph on* 6 *vertices. For* $i = 1, 2, \ldots, 12$, *Let* H_i *be the graph obtained from* $\overline{K}_n$ *by adding a new edge and joining its end vertices to all vertices of* $\overline{K}_n$. *Let this new edge be the specified edge of* H_i. *Then the graph* $G \boxplus_{I_{12}} \mathcal{H}_{12}$ *is A-integral if* $\sqrt{n}$ *and* $\sqrt{1 + 2n}$ *are integers.*

Proof We prove part (1) by using Corollary 3.1(i). For $i = 1, 2, \ldots, m$, since $S_i = T_i = V(H_i')$, by Theorem 2.2, we get

$$\Gamma_{A(H_i')}^{(S_i,S_i)}(x) = \frac{2n + r}{x - n - r}. \tag{4.30}$$

Since $M = I$ and applying (4.30) in (3.26), we get

$$C_1 = \left(\frac{2n + r}{x - n - r}\right) \mathcal{Q}(K_{n+1}).$$

Substituting the above equation in (3.25), we get

$$\begin{aligned}\Phi_{A(K_{n+1}\boxplus_{I_m}\mathcal{H}_m)}(x) &= det\left(\left(\frac{x^2-x(n+r)-2n^2-rn}{x-n-r}\right)I-\left(\frac{x+n}{x-n-r}\right)A(K_{n+1})\right)\\ &\quad\times\prod_{i=1}^{m}\Phi_{A(H_i')}(x)\\ &=\left(\prod_{i=1}^{m}\prod_{j=2}^{2n+r}\left(x-\lambda_j(H_i')\right)\right)(x^2-x(2n+r)-2n^2-rn-n^2)\\ &\quad\times(x-n-r)^{m-n-1}(x^2-x(r+n-1)-2n^2-rn+n)^n.\end{aligned}$$

Notice that for $i=1,2,\ldots,m$; $j=2,3,\ldots,2n+r$, $\lambda_j(H_i')$ are integers, since H_i' is A-integral. Also the roots of the polynomials $x^2-x(2n+r)-2n^2-rn-n^2$ and $x^2-x(r+n-1)-2n^2-rn+n$ are integers. So the proof follows.

Similarly, we can prove the parts (2)-(4). □

5 Concluding remarks

In this paper, we obtained the generalized characteristic polynomial of the graphs obtained from the M-edge rooted product, by using the coronal of a matrix constrained by index sets. As a consequence, several existing results on the spectra of graphs with edge pockets are deduced and complete spectra of some special constructions of graphs are obtained. As applications, several infinite classes of cospectral graphs are obtained and certain class of integral graphs were constructed.

Determining the complete spectra of M-edge rooted product of graphs which are not considered in this paper is a problem with much attraction. Moreover, studying the graph theoretic properties and the spectra of other graph matrices of the graphs constructed from this graph operation are further research problems.

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