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ORIGINAL RESEARCH

Skew derivations on partially ordered sets

Ahmed Y. Abdelwanis · Shakir Ali

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Abstract Let P be a poset and $\alpha : P \to P$ be a function. The aim of this paper is to introduce and study the notion of skew derivations on P. We prove some fundamental properties of posets involving skew derivations. In particular, apart from proving the other results, we prove that if d and g are two skew derivations of P associated with an automorphism α such that $d\alpha = \alpha d$ and $g\alpha = \alpha g$, then $d \leq g$ if and only if $gd = \alpha d$. Also, we prove that $Fix_{\alpha,d}(P) \cap l(\alpha(x)) = l(d(x))$ for all $x \in P$. Furthermore, we give some examples to demonstrate that various restrictions imposed in the hypotheses of our results are not superfluous.

Keywords Derivation · α-fixed point · Ideal · Poset · Skew derivation

Mathematics Subject Classification 06E20 · 13N15 · 06Axx

1 Introduction

Throughout the present paper, $(P, \leq)$ always denotes a partially ordered set (poset). We additionally utilize the shorthand P to indicate a poset. For $y \in P$, we write as in [1], $\downarrow y = \{p \in P : p \leq y\}$ and $\uparrow y = \{p \in P : y \leq p\}$. For $B \subseteq P$, we denote $l(B) = \{p \in P : p \leq b \text{ for all } b \in B\}$ the lower cone of B and $u(B) = \{p \in P : b \leq p, \text{ for all } b \in B\}$ the upper cone of A dually. It is straightforward to see that both are antitone and their compositions $l(u())$ and $u(l())$ are monotone. Following [2], we write $l(u(l())) = l()$ and $u(l(u())) = u()$. If $B = \{b_1, b_2, ..., b_n\}$ is a finite subset, then we denote $l(B) = l(b_1, b_2, ..., b_n)$ and $u(B) = u(b_1, b_2, ..., b_n)$, respectively. Also, for $A \subseteq P$ and $B \subseteq P$, we write $l(A, B)$ for $l(A \cup B)$ and $u(A, B)$ for $u(A \cup B)$. For $A \subseteq P$, we write $\downarrow A = \{p \in P : p \leq a, \text{ for some } a \in A\}$. From [3], we find that if $A = \downarrow A$, then A is said to be a lower set. A is directed if it is nonempty and every finite subset of A has an upper bound in A. From nonemptiness, it is adequate to expect each combine of components in A has an upper bound in A. A subset J of P is called an ideal of P if it is directed lower set.

Roused by fruitful utilization of ordinary derivative in various branches of mathematical science, the notion of derivation was introduced in rings and algebras long back. But the study of derivations in algebras

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A. Y. Abdelwanis (✉)
Department of Mathematics, Faculty of Science, Cairo University, Giza 12613, Egypt
E-mail: ayunis@sci.cu.edu.eg

S. Ali
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail: shakir.ali.mm@amu.ac.in; shakir50@rediffmail.com

got an impetus soon after Herstein [4] and Posner [5] simultaneously obtained some remarkable results, particularly for prime rings in the year 1957. Several algebraists, for example, Bell, Bergen, Herstein, Martindale, Posner, and Ashraf etcetera have made remarkable contributions to the theory of derivation in rings and algebras(see for example [6, 7], ([8, 9] and [10] where further references can be found).

Recently some researchers have verified a lot of meaningful conclusions by applying derivations and its generalized forms to lattices (viz.; [11, 12, 13, 14] and references therein). In the year 2012, Xin et al. [15] examined the thought of fixed sets of derivations on lattices and obtained some remarkable results. In particular, they proved that if d is an isotone derivation of a lattice L, then the fixed set $Fix_d(L)$ is an ideal of L. Also, they established that $D(L)$ (the set of derivations on L) is isomorphic to L in a distributive lattice L. Numerous specialists have examined systematic and arithmetical properties of lattices (see for example [11] and [6] for detailes). Very recently, Zhang and Li [1] initiated the study of derivations on partially ordered sets and established some fundamental properties of ideals and operations related to derivations.

In this paper, we introduce the notion of skew derivations on partially ordered sets (posets) and study some properties of skew derivation on partially ordered sets. In particular, apart from proving the other classical results, we prove that if d and g are two skew derivations of P associated with an automorphism α such that $d\alpha = \alpha d$ and $g\alpha = \alpha g$, then $d \leq g$ if and only if $gd = \alpha d$. These results generalize the case of derivations by Zhang and Li [1].

The paper is sorted out as takes after. In Section 2, we introduce and initiate the study of skew derivations of partially ordered sets and establish some important properties involving newly defined notions. Further in Section 3, we examine the α-fixed sets with skew derivations. Finally, in Section 4, we discuss the structural properties of posets involving skew derivations.

2 Skew derivations of posets

This section deals with study of skew derivations on posets and motivated by the work's of Kharchenko [16] . In the year 2017, Zhang and Li [1] initiated the study of derivations on a poset P and defined as follows:

Definition 2.1 Let P be a poset. A mapping $d : P \to P$ is said to a derivation if it satisfies the following conditions:

(i) $d(l(x, y)) = l(u(l(d(x), y), l(x, d(y))))$ for all $x, y \in P$;
(ii) $l(d(u(x, y))) = l(u(d(x), d(y)))$ for all $x, y \in P$.

Inspired by the definition of skew derivations in rings [16], we introduce the notion of skew derivations on posets as follows:

Definition 2.2 Let P be a poset and $d, \alpha : P \to P$ are functions. We call d a skew derivation on P associated with α, if it satisfies the following assertions:

(i) $d(l(x, y)) = l(u(l(d(x), \alpha(y)), l(x, d(y))))$ for all $x, y \in P$;
(ii) $l(d(u(x, y))) = l(u(d(x), d(y)))$ for all $x, y \in P$.

Remark 2.3 For brevity, we also call it α-derivation. Let I_P be the identity mapping on P. Then, clearly the mapping $\alpha - I_P$ is the simplest example of skew derivations and I_P-derivations are just ordinary derivations. Hence, the concept of skew derivations can be regarded as a generalization of derivations.

Remark 2.4 Suppose $P = (P, \leq, \wedge, \vee)$ is a lattice and $\alpha : P \to P$ is a function. Easily, we can prove that if d is a skew derivation on $(P, \leq)$, then d is a skew derivation on lattice $(P, \wedge, \vee)$.

Remark 2.5 Obviously, every derivation is a skew derivation on P with $\alpha = I_P$. But the converse need not be true in general. The following example justifies this fact:

Example 2.6 Let $(P, \leq) = (\mathbb{R}, \leq)$ be a poset. Define the functions $d, \alpha : \mathbb{R} \to \mathbb{R}$ by $d(y) = \alpha(y) = 2y$ for all $y \in P$. Then, it is straightforward to check that d is a skew derivation associated with α on $\mathbb{R}$. However, d is not a derivation on P.

We begin with following definitions (see [17] for more details).

Definition 2.7 Let $(P, \leq)$ be a poset and $\alpha : P \to P$ be a function, we say that α increasing if the following hold for all $x, y \in P$ if $x \leq y$, then $\alpha(x) \leq \alpha(y)$.

Definition 2.8 Let $(P, \leq)$ be a poset and $\alpha : P \to P$ be a function, we say that α is an automorphism if α is increasing, bijective and its inverse map α^{-1} is also increasing.

Remark 2.9 Note that if $(P, \leq)$ is a poset with the least element 0 and $\alpha : P \to P$ is an automorphism, then $\alpha(0) = 0$

The first main result of this section is the following theorem.

Theorem 2.10 *Let P be a poset and $d, \alpha : P \to P$ are functions such that α is an automorphism of P and $x \leq d(x) \leq \alpha(x)$ for all $x \in P$. Then d is a skew derivation on P if and only if*

(i) $d(l(x,y)) = l(d(x), \alpha(y)))$ for all $x, y \in P$;
(ii) $l(d(u(x,y))) = l(u(d(x), d(y)))$ for all $x, y \in P$.

In order to prove above theorem, we need some auxiliaries results which will be used in the sequel. We begin with following.

Lemma 2.11 *Let d be a skew derivation associated with an automorphism $\alpha : P \to P$ and $x \leq \alpha(x)$ for all $x \in P$. Then the following statements hold*:

(i) $d(x) \leq \alpha(x)$ for all $x \in P$;
(ii) If $x \leq y$, then $d(x) \leq d(y)$;
(iii) If I is an ideal of P and $\alpha(I) \subseteq I$, then $d(I) \subseteq I$;
(iv) If P has the least element 0, then $d(0) = 0$.

Proof

(i) We are given that $d : P \to P$ is a skew derivation and $\alpha : P \to P$ is a function such that $x \leq \alpha(x)$ for all $x \in P$. By the assumption, we have

$$\begin{aligned} d(l(x,x)) &= l(u(l(d(x), \alpha(x)), l(x, d(x)))) \\ &= l(u(l(d(x), \alpha(x)))) \\ &= l(d(x), \alpha(x)) \text{for all} x \in P \end{aligned}$$

Therefore, we obtain $d(l(x)) = d(l(x,x)) = l(d(x), \alpha(x))$ for all $x \in P$. Also we have, $d(x) \in d(l(x))$, and hence we get $d(x) \in l(d(x), \alpha(x))$. Thus $d(x) \leq \alpha(x)$ for all $x \in P$.

(ii) By the assumption, we have $x \leq y$. Then, we obtain $l(d(u(x,y))) = l(d(u(y))) = l(u(d(x), d(y)))$. Since $d(x) \in l(u(d(x), d(y)))$, so we find that $d(x) \in l(d(u(y)))$. Hence $d(x) \leq d(y)$ for all $x, y \in P$.
(iii) Assume that I is an ideal of P. If $x \in d(I)$, then there exist $z \in I$ such that $d(z) = x$. By part (*i*) we conclude that $d(z) \leq \alpha(z)$. Hence, last relation gives $x \leq \alpha(z)$, but $\alpha(I) \subseteq I$ and I is an ideal of P. Therefore, $x \in I$. Hence, we conclude that $d(I) \subseteq I$.
(iv) Suppose P has the least element 0 and α be an automorphism , then from Remark 2.9 we obtain $\alpha(0) = 0$. Now in view of part (*i*), we are force to conclude that $0 \leq d(0) \leq \alpha(0) = 0$. Hence $d(0) = 0$. This prove the lemma. □

Lemma 2.12 *Let d be a skew derivation on P associative with an automorphism α. Then the following statements hold*:

(i) If $d(l(x)) = l(y)$, then $d(x) = y$ for all $x, y \in P$;
(ii) If $d(u(x)) = u(y)$, then $d(x) = y$ for all $x, y \in P$.

Proof

(i) Assume that $d(l(x)) = l(y)$. Then, by the definition $y \in l(y)$ and therefore we have $y \in d(l(x))$. Hence, there exist $z \in l(x)$ such that $d(z) = y$. Since $d(z) \leq d(x)$, the above relation yields that $y \leq d(x)$. On the other hand $d(x) \in d(l(x)) = l(y)$, so $d(x) \leq y$. Hence $d(x) = y$ for all $x, y \in P$.
(ii) Using similar approach with necessary variation, we can prove (*ii*). □

Now we are ready to prove our first theorem.

Proof of Theorem 2.7 We just need to demonstrate that the condition (i) in definition 2.2 is equivalent to the one (i) in this theorem. First assume that the condition in this theorem is satisfied, then

$$\begin{aligned} d(l(x,y)) &= l(d(x),\alpha(y)) \\ &= l(u(l(d(x),\alpha(y)))) \\ &= l(u(l(d(x),\alpha(y)),l(x,d(y)))). \end{aligned}$$

Secondly, we assume that d is a skew derivation of P. Then, we have

$$\begin{aligned} l(d(x),\alpha(y)) = l(u(l(d(x),\alpha(y)))) &\subseteq l(u(l(d(x),\alpha(y)),l(x,d(y)))) \\ &= d(l(x,y)) \text{for all} x,y \in P. \end{aligned}$$

This implies that

$$l(d(x),\alpha(y)) \subseteq d(l(x,y)) \text{for all} x,y \in P. \tag{2.1}$$

On the other hand, we assume that $z \in d(l(x,y))$, then there exists $v \in l(x,y)$ satisfying $d(v) = z$. By Lemma 2.11 (i) and (ii), we have $d(v) \le d(x)$, $d(v) \le d(y) \le \alpha(y)$. This shows that $z = d(v) \in l(d(x),\alpha(y))$. Thus, we find that

$$d(l(x,y)) \subseteq l(d(x),\alpha(y)) \text{for all} x,y \in P. \tag{2.2}$$

On combining (2.1) and (2.2), we obtain $d(l(x,y)) = l(d(x),\alpha(y))$. This completes the proof the theorem.

Theorem 2.13 *Let d be a skew derivation of P with an automorphism $\alpha : P \to P$ such that $x \le d(x)$ for all $x \in P$. Then $d(l(x,y)) = l(d(x),d(y))$ for all $x,y \in P$.*

Proof First assume that $t \in l(d(x),d(y))$. Then, $t \le d(x)$, $t \le d(y)$. By Lemma 2.11 (i), $t \le \alpha(y)$. Then we have $t \in l(d(x),\alpha(y))$. By Theorem 2.10(i), $l(d(x),d(y)) \subseteq d(l(x,y))$. On the other hand, we suppose $t \in d(l(x,y))$, so there exists $z \in l(x,y)$ such that $d(z) = t$. Thus by Lemma 2.11(ii), we get $d(z) \le d(x), d(z) \le d(y)$. Thus $t \in l(d(x),d(y))$. Hence $d(l(x,y)) \subseteq l(d(x),d(y))$ for all $x,y \in R$. This implies that $d(l(x,y)) = l(d(x),d(y))$ for all $x,y \in P$. □

3 The α-fixed points of a skew derivations

In this section, we would like to describe the general form of fixed points of skew derivations of a poset P. For a given function $\alpha : P \to P$, d is a skew derivation of P associated with α such that $x \le d(x)$ for all $x \in P$. We call x a α-fixed point of d in P if $d(x) = \alpha(x)$. We denote $Fix_{\alpha,d}(P) = \{x \in P : d(x) = \alpha(x)\}$, the set of all α-fixed points of d in P, and $d(P) = \{d(x) : x \in P\}$. Clearly, $Fix_{I_P,d}(P) = Fix_d(P) = \{x \in P : d(x) = x\}$.

Now we prove our first result of the present section which generalizes Theorem 3.1 in [1].

Theorem 3.1 *Let d be a skew derivation on P such that $x \le d(x)$, $\alpha : P \to P$ is a bijective map on P such that $\alpha d = d\alpha$. Then the following statements hold*:

(i) $d(x) \in Fix_{\alpha,d}(P)$ for all $x \in P$;
(ii) $Fix_{\alpha,d}(P) = d(P)$.

Proof (i). In order to prove that $d(x) \in Fix_{\alpha,d}(P)$ for all $x \in P$. It is sufficient to show that $d(d(x)) = \alpha(d((x)))$ for all $x \in P$. In view of Lemma 2.11 and Theorem 2.10 (i), we have

$$\begin{aligned} d(l(d(x))) &= d(l(\alpha(x),d(x))) \\ &= l(d(\alpha(x)),\alpha(d(x))) \\ &= l(\alpha(d(x)),\alpha(d(x))) \text{ since } \alpha d = d\alpha \\ &= l(\alpha(d(x))) \text{for all } x \in P. \end{aligned}$$

Application of Lemma 2.12 yields that $d(d(x)) = \alpha(d((x)))$ for all $x \in P$. This implies that $d(x) \in Fix_{\alpha,d}(P)$ for all $x \in P$.

To prove (*ii*). By part (*i*) we have $d(x) \in Fix_{\alpha,d}(P)$ for all $x \in P$. This implies that $d(P) \subseteq Fix_{\alpha,d}(P)$. On the other hand, we assume that $x \in Fix_{\alpha,d}(P)$, so we have $\alpha(x) = d(x)$ for all $x \in P$. Since α is bijective map on P and $\alpha d = d\alpha$. This also implies $\alpha^{-1}d = d\alpha^{-1}$. Thus, we obtain, $x = \alpha^{-1}(d(x)) = d(\alpha^{-1}(x))$for all $x \in P$. Henceforward, we conclude that $Fix_{\alpha,d}(P) \subseteq d(P)$. Therefore, we find that $Fix_{\alpha,d}(P) = d(P)$. This proves the theorem completely. □

The next example shows that the conditions of α is an automorphism and commute with d are necessary in Theorem 3.1.

Example 3.2 Let $(P, \leq) = (\mathbb{R} - (-3, 3), \leq)$ be a poset. Define the functions $d, \alpha : P \to P$ by $d(y) = 3y$ and $\alpha(y) = y^2$ for all $y \in P$. Then, it is straightforward to check that d is a skew derivation associated with α on P. Also $d\alpha \neq \alpha d$, but we find that $d(d(y)) = 9y \neq 9y^2 = \alpha(d(y))$ for every $y \in P$ i.e., $d(y) \notin Fix_{\alpha,d}(P)$. Hence, the assumed restrictions in the hypothesis of Theorem 3.1 can not be relaxed.

Proposition 3.3 *Let d, g are two skew derivations on P associated with a bijective map α such that $d\alpha = \alpha d$ and $g\alpha = \alpha g$. Then $d = g$ if and only if $Fix_{\alpha,d}(P) = Fix_{\alpha,g}(P)$.*

Proof One way is obvious if $d = g$. On the other hand, we assume that $Fix_{\alpha,d}(P) = Fix_{\alpha,g}(P)$ and $x \in P$. Then, we want to prove that $d = g$. In view of Theorem 3.1(i), we have $d(x) \in Fix_{\alpha,d}(P) = Fix_{\alpha,g}(P)$. This forces that $g(d(x)) = \alpha(d(x))$. By the assumption, we obtain $g(d(x)) = d(\alpha(x))$. In similar way, we get $d(gx) = g(\alpha(x))$. By Lemma 2.11 (i), (ii) we obtain $d(\alpha(x)) \leq g(\alpha(x))$ and $g(\alpha(x)) \leq d(\alpha(x))$. From last two relations, we conclude that $d(\alpha(x)) = g(\alpha(x))$ and hence $\alpha(d(x)) = \alpha(g(x))$ for all $x \in P$. Since α is bijective, so the expression yields that $d(x) = g(x)$ for all $x \in P i.e., d = g$. This proves the result.

Proposition 3.4 *Let d be a skew derivation on P with the least element 0, α is an automorphism such that $d\alpha = \alpha d$. Then the following statements hold:*

(i) $0 \in Fix_{\alpha,d}(P)$;
(ii) If $x \in Fix_{\alpha,d}(P)$, and $y \leq x$ then $y \in Fix_{\alpha,d}(P)$;
(iii) If P is directed, then, for any $x, y \in Fix_{\alpha,d}(P)$, there exists $z \in Fix_{\alpha,d}(P)$ such that $x \leq z, y \leq z$.

Proof

(i) By Lemma 2.11 (iv), we have $d(0) = \alpha(0) = 0$, since α is an automorphism. This implies that $0 \in Fix_{\alpha,d}(P)$.
(ii) Assume that $x \in Fix_{\alpha,d}(P)$, and $y \leq x$ then $d(x) = \alpha(x)$. Then by Theorem 2.10(i) we get $d(l(y)) = d(l(x, y)) = l(d(x), \alpha(y)) = l(\alpha(x), \alpha(y)) = l(\alpha(y))$ for all $x, y \in P$. Application of Lemma 2.12(i) gives $d(y) = \alpha(y)$ for all $\in y$ P and henceforward we conclude that $y \in Fix_{\alpha,d}(P)$.
(iii) By the assumption, P is directed. Then for any $x, y \in P$, there exist $v \in P$ such that $x \leq v$ and $y \leq v$. Since $x, y \in Fix_{\alpha,d}(P)$, then we write $d(x) = \alpha(x)$ and $d(y) = \alpha(y)$. But $d(x) = \alpha(x) \leq d(v)$ and $d(y) = \alpha(y) \leq d(v)$. We are given that α is an automorphism such that $d\alpha = \alpha d$ and therefore, we obtain $x \leq d(\alpha^{-1}(v))$ and $y \leq d(\alpha^{-1}(v))$. Now taking $z = d(\alpha^{-1}(v))$, hence using Theorem 3.1(ii), we obtain $z \in Fix_{\alpha,d}(P)$. This proves the result. □

Corollary 3.5 *If P is a directed poset with the least element* 0, *then $Fix_{\alpha,d}(P)$ is an ideal of P.*

Theorem 3.6 *Let d be a skew derivation on P and $\alpha : P \to P$ is an automorphism such that $d\alpha = \alpha d$. Then, $Fix_{\alpha,d}(P) \cap l(\alpha(x)) = l(d(x))$ for all $x \in P$.*

Proof First we assume that $y \in Fix_{\alpha,d}(P) \cap l(\alpha(x))$. Then, we get $\alpha(y) = d(y)$ and $y \leq \alpha(x)$. This implies that $d(y) \leq d(\alpha(x)) = \alpha(d(x))$, i.e.; $\alpha(y) \leq \alpha(d(x))$. But α is an automorphism, henceforward we conclude $y \leq d(x)$. Thus $y \in l(d(x))$, and therefore we obtain $Fix_{\alpha,d}(P) \cap l(\alpha(x)) \subseteq l(d(x))$. On the other hand we have $d(x) \leq \alpha(x)$ and $d(x) \in Fix_{\alpha,d}(P)$. This yields $d(x) \in Fix_{\alpha,d}(P) \cap l(\alpha(x))$. Thus by Proposition 2.12(*ii*), we obtain $l(d(x)) \subseteq Fix_{\alpha,d}(P) \cap l(\alpha(x))$ for all $x \in P$. Hence $Fix_{\alpha,d}(P) \cap l(\alpha(x)) = l(d(x))$for all $x \in P$. □

Following is an immediate consequence of the above theorem.

Corollary 3.7 ([1], *Theorem* 3.4) Let d be a derivation on P. Then, $Fix_d(P) \cap l(x) = l(d(x))$ for all $x \in P$.

4 Structural properties of posets involving skew derivations

The present section deals with the study of some structural properties of posets involving skew derivations. Apart from proving some fundamental results related to ideals of P and kernel of skew derivations, we prove that two skew derivations on posets always commute. Henceforward, we assume that P is a poset with the least element 0 and $\alpha : P \to P$ is an automorphism. We begin with the following result.

Theorem 4.1 *Let d be a skew derivation on P and $\alpha : P \to P$ be an automorphism such that commute $d\alpha = \alpha d$. Next, let I and J be two ideals of P. Then the following statements hold*:

(i) if $I \subseteq J$, then $d(I) \subseteq d(J)$,
(ii) $d(I \cap J) = d(I) \cap d(J)$.

Proof

(i) We are given that d is a skew derivation and $\alpha : P \to P$ be an automorphism of P. Let $x \in d(I)$. Then there exist $y \in I \subseteq J$ such that $x = d(y)$. This implies that $x \in d(J)$. Henceforward, we conclude that $d(I) \subseteq d(J)$.
(ii) Since d is a skew derivation and α is an automorphism of P. One way is obvious, $d(I \cap J) \subseteq d(I) \cap d(J)$. On the other hand, let $x \in d(I) \cap d(J)$. Then there exist $a \in I, b \in J$ such that $d(a) = x$ and $d(b) = x$. Thus, application of theorem 2.10 (i) yields that $d(l(a, \alpha^{-1}(d(b)))) = l(d(a), d(b)) = l(x, x) = l(x)$. But $x \in l(x)$, and hence we are force to conclude that $x \in d(l(a, \alpha^{-1}(d(b))))$. Therefore, there exist $z \in l(a, \alpha^{-1}(d(b)))$ such that $d(z) = x$. By $z \leq a$ and $z \leq \alpha^{-1}(d(b)) = d(\alpha^{-1}(b)) \leq b$ so we have $z \in I \cap J$. Therefore, $x \in d(I \cap J)$. This implies that $d(I) \cap d(J) \subseteq d(I \cap J)$. Hence, $d(I \cap J) = d(I) \cap d(J)$. This completes the proof. □

A careful scrutiny of the proofs of our next two results shows that the proofs runs on parallel lines to Theorem 4.1 and Proposition 4.2 in [1] and hence we state these two results without proof just to avoid repetition.

Proposition 4.2 *Let d be a skew derivation on P associated with an automorphism α. Then, $kerd = \{x \in P : d(x) = 0\}$ is a nonempty lower set of P.*

Proposition 4.3 *Let d be a skew derivation on P associated with an automorphism α of P and I be an ideal of P. Then $d^{-1}(I)$ is a nonempty lower set of P such that $kerd \subseteq d^{-1}(I)$.*

Theorem 4.4 *Let d and g be two skew derivations on P associated with an automorphism α of P such that $\alpha d = d\alpha$ and $\alpha g = g\alpha$. Then, d and g must be commute.*

Proof We assume that d and g are two skew derivations on P associated with an automorphism α of P. Thus, for any $x \in P$, we have

$$\begin{aligned} d(l(g(x))) &= d(l(\alpha(x), g(x))) \\ &= l(d(\alpha(x)), \alpha(g(x))) \\ &= l(\alpha(d(x)), \alpha(g(x))) \text{ for all } x, y \in P, \end{aligned}$$

and

$$\begin{aligned} g(l(d(x))) &= g(l(\alpha(x), d(x))) \\ &= l(g(\alpha(x)), \alpha(dx)) \\ &= l(\alpha(d(x)), \alpha(g(x))) \text{ for all } x, y \in P. \end{aligned}$$

This implies that $d(l(g(x))) = g(l(d(x)))$. But $dg(x) = d(g(x)) \in d(l(g(x)))$, thus $dg(x) \in g(l(d(x)))$. Then there exists $y \in l(d(x))$ such that $dg(x) = g(y)$. By Lemma 2.11 (ii), $g(y) \leq g(d(x))$, and therefore $dg(x) \leq gd(x)$. Similarly, we can prove that $gd(x) \leq dg(x)$. Henceforth, we conclude that $dg(x) = gd(x)$ for all $x \in P$. □

Example 4.5 Let $(P, \leq) = (\mathbb{R} - (-1, 1), \leq)$ be a poset. Define the functions $d, g, \alpha : P \to P$ by $d(y) = 2y + 2$, $g(y) = 2y + 1$ and $\alpha(y) = 2y + 3$ for all $y \in P$. Then, it is straightforward to check that d, g are skew derivations associated with α on P. But we find that $dg \neq gd$ i.e., d, g not commutes. Hence, the condition α to commutes with d, g in Theorem 4.4 is an essential one.

Theorem 4.6 *Let d and g be two skew derivations of P associated with an automorphism α such that $\alpha d = d\alpha$ and $\alpha g = g\alpha$. Then $d \leq g$ if and only if $gd = \alpha d$.*

Proof First we assume that d and g skew derivations on P associated with an automorphism α such that $d \leq g$. Then, we have $d(x) \in Fix_{\alpha,d}(P)$ i.e., $\alpha d(x) = d(d(x)) \leq g(d(x))$ for all $x \in P$. Application of Lemma 2.11 (i) yields $gd(x) \leq \alpha d(x)$ for all $x \in P$. Thus $gd(x) = \alpha d(x)$ for all $x \in P$. This shows that $gd = \alpha d(x)$. On the other hand, we assume that $\alpha d(x) = gd(x) = dg(x) \leq \alpha g(x)$ for all $x \in P$. This implies that $\alpha d(x) \leq \alpha g(x)$ for all $x \in P$. Multiplying by α^{-1} from left both sides to the last relation, we obtain $d(x) \leq g(x)$ for all $x \in P$. Hence, we conclude that $d \leq g$. This completes the proof of the theorem. ☐

As an immediate consequence of Theorem 2.12, we get the following corollary which generalizes [[1], Theorem 4.5] .

Corollary 4.7 *Let d and g be two skew derivations of P. Then $d \leq g$ if and only if $gd = d$.*

Our last example prove that the conditions α to be an automorphism and commutes with d, g are necessary in Theorem 2.12.

Example 4.8 Let $(P, \leq) = (\mathbb{R} - (-2, 2), \leq)$ be a poset. Define the functions $d, g, \alpha : P \to P$ by $d(y) = g(y) = 2y$ and $\alpha(y) = y^2$ for all $y \in P$. Then, it is straightforward to check that $d = g$ is a skew derivation associated with α on P. Also $d \leq g$, but we find that $dg(x) = 4x \neq 4x^2 = \alpha d(x)$ for all $x \in P$. Hence, the assumed restriction can not be omitted.

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