Indian J Pure Appl Math (2021) 52:1263–1275
https://doi.org/10.1007/s13226-021-00012-z

ORIGINAL RESEARCH

Cohomology of $\mathfrak{sl}(2)$ acting on the space of n-ary differential operators on $\mathbb{R}$

Mabrouk Ben Ammar · Rabeb Sidaoui

Received: 9 October 2018 / Accepted: 5 January 2021 / Published online: 12 July 2021

Abstract We consider the spaces $\mathcal{F}_\mu$ of polynomial μ-densities on the line as $\mathfrak{sl}(2)$-modules and then we compute the cohomological spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$, where $\mu \in \mathbb{R}$, $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $\mathcal{D}_{\bar{\lambda},\mu}$ is the space of n-ary differential operators from $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_\mu$.

Keywords Cohomology · Weighted Densities

Mathematics Subject Classification (2010) 17B56

1 Introduction

Consider the space of polynomial μ-densities:

$$\mathcal{F}_\mu = \{ f dx^\mu, f \in \mathbb{R}[x] \}, \quad \mu \in \mathbb{R}.$$

The Lie algebra $\mathrm{Vect}(\mathbb{R})$ of polynomial vector fields $X_h = h\frac{d}{dx}$, where $h \in \mathbb{R}[x]$, acts on $\mathcal{F}_\mu$ by the *Lie derivative* L^μ:

$$X_h \cdot (f dx^\mu) = L^\mu_{X_h}(f dx^\mu) := (hf' + \mu h' f) dx^\mu. \tag{1.1}$$

For $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ we denote by $\mathcal{D}_{\bar{\lambda},\mu}$ the space of n-ary differential operators A from $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ to $\mathcal{F}_\mu$. The Lie algebra $\mathrm{Vect}(\mathbb{R})$ acts on the space $\mathcal{D}_{\bar{\lambda},\mu}$ of these differential operators by:

$$X_h \cdot A := L^{\bar{\lambda},\mu}_{X_h}(A) = L^\mu_{X_h} \circ A - A \circ L^{\bar{\lambda}}_{X_h} \tag{1.2}$$

where $L^{\bar{\lambda}}_{X_h}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_n}$ defined by the Leibnitz rule. The spaces $\mathcal{F}_\mu$ and $\mathcal{D}_{\bar{\lambda},\mu}$ can be also viewed as $\mathfrak{sl}(2)$-modules, where $\mathfrak{sl}(2)$ is realized as a subalgebra of $\mathrm{Vect}(\mathbb{R})$:

$$\mathfrak{sl}(2) = \mathrm{Span}(X_1, X_x, X_{x^2}).$$

According to Nijenhuis-Richardson [8], the space $\mathrm{H}^1(\mathfrak{g}, \mathrm{End}(V))$ classifies the infinitesimal deformations of a $\mathfrak{g}$-module V and the obstructions to integrability of a given infinitesimal deformation of V are elements of $\mathrm{H}^2(\mathfrak{g}, \mathrm{End}(V))$.

Communicated by BEN AMMAR Mabrouk.

M. B. Ammar (✉) · R. Sidaoui
Faculté des Sciences, Département de Mathématiques, Laboratoire d'Algèbre, Géométrie et Théorie Spectrale (AGTS) LR11ES53, Université de Sfax, BP 802, 3038 Sfax, Tunisie
E-mail: mabrouk.benammar@fss.rnu.tn

For $\bar{\lambda} \in \mathbb{R}$ the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$ are computed by Gargoubi [6] and Lecomte [7], the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$ are computed by Bouarroudj and Ovsienko [3] and the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\bar{\lambda},\mu})$ are computed by Feigen and Fuchs [4]. For $\bar{\lambda} \in \mathbb{R}^2$ the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$ are computed by Bouarroudj [2]. For $\bar{\lambda} \in \mathbb{R}^3$ the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$ are computed by O. Basdouri and N. Elamine [1]. In this paper we are interested to compute the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$ for $\bar{\lambda} \in \mathbb{R}^n$.

2 Cohomology

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [5]). Let $\mathfrak{g}$ be a Lie algebra acting on a space V and let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{h}$ is omitted it assumed to be $\{0\}$). The space of $\mathfrak{h}$-relative n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g}, \mathfrak{h}, V) := \mathrm{Hom}_{\mathfrak{h}}(\Lambda^n(\mathfrak{g}/\mathfrak{h}), V).$$

The *coboundary operator* $\partial : C^n(\mathfrak{g}, \mathfrak{h}, V) \longrightarrow C^{n+1}(\mathfrak{g}, \mathfrak{h}, V)$ is a $\mathfrak{g}$-map satisfying $\partial^2 = 0$. The operator ∂ is defined by

$$\begin{aligned}(\partial f)(u_0, \ldots, u_n) = &\sum_{i=0}^{n} (-1)^i u_i f(u_0, \ldots, \hat{\imath}, \ldots, u_n) + \\ &+ \sum_{0 \le i < j \le n} (-1)^{i+j} f([u_i, u_j], u_0, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, u_n).\end{aligned}$$

The kernel of $\partial|_{C^n}$, denoted $Z^n(\mathfrak{g}, \mathfrak{h}, V)$, is the space of $\mathfrak{h}$-relative *n-cocycles*, among them, the elements of $\partial(C^{n-1}(\mathfrak{g}, \mathfrak{h}, V))$ are called $\mathfrak{h}$-relative *n-coboundaries*. We denote $B^n(\mathfrak{g}, \mathfrak{h}, V)$ the space of n-coboundaries.

By definition, the n^{th} $\mathfrak{h}$-relative cohomology space is the quotient space

$$\mathrm{H}^n(\mathfrak{g}, \mathfrak{h}, V) = Z^n(\mathfrak{g}, \mathfrak{h}, V)/B^n(\mathfrak{g}, \mathfrak{h}, V).$$

Here we consider $\mathrm{H}^n(\mathfrak{g}, V)$ where $\mathfrak{g} = \mathfrak{sl}(2)$ and $V = \mathcal{D}_{\bar{\lambda},\mu}$. In this paper we are interested to the differential cohomology $\mathrm{H}^1_{\mathrm{diff}}$ (i.e., we consider only cochains that are given by differential operators).

3 The spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$

Consider $\mu \in \mathbb{R}$, $\alpha = (\alpha_1 \ldots, \alpha_n) \in \mathbb{N}^n$ and $\bar{\lambda} = (\lambda_1 \ldots, \lambda_n) \in \mathbb{R}^n$, we define

$$|\bar{\lambda}| = \sum_{i=1}^{n} \lambda_i, \quad \delta = \mu - |\bar{\lambda}| \quad \text{and} \quad |\alpha| = \sum_{i=1}^{n} \alpha_i.$$

We also consider the canonical basis $(\varepsilon_1, \ldots, \varepsilon_n)$ of $\mathbb{R}^n$ where ε_i is the n-tuple $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the i^{th} position). The space $\mathcal{D}_{\bar{\lambda},\mu}$ is spanned by the operators Ω^α defined by

$$\Omega^\alpha(f_1 dx^{\lambda_1} \otimes \cdots \otimes f_n dx^{\lambda_n}) = f_1^{(\alpha_1)} \ldots f_n^{(\alpha_n)} dx^\mu.$$

Lemma 3.1 *The action of* X_x *on* $\mathcal{D}_{\bar{\lambda},\mu}$ *is diagonalizable. The operators* Ω^α *are eigenvectors*:

$$X_x \cdot \Omega^\alpha = (\mu - |\bar{\lambda}| - |\alpha|)\Omega^\alpha = (\delta - |\alpha|)\Omega^\alpha.$$

The following lemma gives a first reduction of any 1-cocycle.

Lemma 3.2 *We have*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), X_1, \mathcal{D}_{\bar{\lambda},\mu}).$$

Moreover, up to a coboundary, any 1*-cocycle* $f \in \mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu})$ *can be expressed as follows*:

$$f(X_h)=\sum_\alpha B_\alpha h'\Omega^\alpha+\sum_\alpha C_\alpha h''\Omega^\alpha,\qquad X_h\in\mathfrak{sl}(2),$$

where, for any α, *the coefficients* $B_{\alpha+\varepsilon_i}$ *and* C_α *are constants satisfying*:

$$2(\delta-|\alpha|-1)C_\alpha+\sum_i(\alpha_i+1)(\alpha_i+2\lambda_i)B_{\alpha+\varepsilon_i}=0. \tag{3.1}$$

Proof. Any 1-cocycle on $\mathfrak{sl}(2)$ should retains the following general form:

$$f(X_h)=\sum_\alpha U_\alpha h\Omega^\alpha+\sum_\alpha V_\alpha h'\Omega^\alpha+\sum_\alpha W_\alpha h''\Omega^\alpha,$$

where U_α, V_α and W_α are, a priori, functions. First, we prove that the terms in h can be annihilated by adding a coboundary. Consider the n-ary differential operator

$$g:\mathcal{F}_{\lambda_1}\otimes\cdots\otimes\mathcal{F}_{\lambda_n}\to\mathcal{F}_\mu,\quad g=\sum_\alpha D_\alpha\Omega^\alpha,\quad D_\alpha\in\mathbb{R}[x].$$

We have

$$\begin{aligned}\partial g(X_h)&=hg'+\mu h'g-g\circ L_{X_h}^{\bar\lambda}\\&=\sum_\alpha D'_\alpha h\Omega^\alpha+\sum_\alpha(\delta-|\alpha|)D_\alpha h'\Omega^\alpha\\&\quad-\tfrac12\sum_\alpha\sum_{i=1}^n\alpha_i(\alpha_i+2\lambda_i-1)D_\alpha h''\Omega^{\alpha-\varepsilon_i}.\end{aligned}\tag{3.2}$$

Thus, if $D'_\alpha=U_\alpha$ then $f-\partial g$ does not contain terms in h. So, we can replace f by $f-\partial g$. That is, up to a coboundary, any 1-cocycle on $\mathfrak{sl}(2)$ can be expressed as follows:

$$f(X_h)=\sum_\alpha B_\alpha h'\Omega^\alpha+\sum_\alpha C_\alpha h''\Omega^\alpha.$$

Now, for $X_{h_1}, X_{h_2}\in\mathfrak{sl}(2)$, consider the 1-cocycle condition:

$$f([X_{h_1},X_{h_2}])-X_{h_1}\cdot f(X_{h_2})+X_{h_2}\cdot f(X_{h_1})=0,$$

which can be expressed as follows:

$$\begin{aligned}&\sum_\alpha B'_\alpha(h_1h'_2-h'_1h_2)\Omega^\alpha+\sum_\alpha C'_\alpha(h_1h''_2-h''_1h_2)\Omega^\alpha\\&+\tfrac12\sum_\alpha\left(2(\delta-|\alpha|-1)C_\alpha+\sum_{i=1}^n(\alpha_i+1)(\alpha_i+2\lambda_i)B_{\alpha+\varepsilon_i}\right)(h'_1h''_2-h''_1h'_2)\Omega^\alpha=0.\end{aligned}$$

Thus, for all α, we have $B'_\alpha=C'_\alpha=0$, and, moreover, the $B_{\alpha+\varepsilon_i}$ and C_α satisfy (3.1). □

Theorem 3.3

1) *If* $\delta\notin\mathbb{N}$ *then* $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})=0$.
2) *If* $\delta=k\in\mathbb{N}$ *then, up to a coboundary, any* 1-*cocycle* $f\in\mathrm{Z}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})$ *can be expressed as follows*:

$$f(X_h)=\sum_{|\alpha|=k}B_\alpha h'\Omega^\alpha+\sum_{|\beta|=k-1}C_\beta h''\Omega^\beta,\qquad X_h\in\mathfrak{sl}(2),\tag{3.3}$$

where, for any β *such that* $|\beta|=k-1$, *the* $B_{\beta+\varepsilon_i}$ *are constants satisfying*:

$$\sum_i(\beta_i+1)(\beta_i+2\lambda_i)B_{\beta+\varepsilon_i}=0.\tag{3.4}$$

3) *If the* B_α *are not all zero then the cocycles* (3.3) *are nontrivial.*

Proof. Consider the 1-cocycle f defined by (3.2) and consider the operator ∂g where

$$g=\sum_{|\alpha|\neq\delta}\frac{1}{\delta-|\alpha|}B_\alpha\Omega^\alpha.$$

We have

$$\begin{aligned}\partial g(X_h) &= \sum_{|\alpha|\neq\delta}B_\alpha h'\Omega^\alpha-\frac{1}{2}\sum_{|\alpha|\neq\delta}\sum_{i=1}^{n}\frac{\alpha_i(\alpha_i+2\lambda_i-1)}{\delta-|\alpha|}B_\alpha h''\Omega^{\alpha-\varepsilon_i}\\ &= \sum_{|\alpha|\neq\delta}B_\alpha h'\Omega^\alpha-\frac{1}{2}\sum_{|\beta|\neq\delta-1}\sum_{i=1}^{n}\frac{(\beta_i+1)(\beta_i+2\lambda_i)}{\delta-|\beta|-1}B_{\beta+\varepsilon_i}h''\Omega^\beta.\end{aligned}$$

According to (3.1) we have

$$-\frac{1}{2}\sum_{i=1}^{n}\frac{(\beta_i+1)(\beta_i+2\lambda_i)}{\delta-|\beta|-1}B_{\beta+\varepsilon_i}=C_\beta.$$

Therefore, we have

$$\partial g(X_h)=\sum_{|\alpha|\neq\delta}B_\alpha h'\Omega^\alpha+\sum_{|\beta|\neq\delta-1}C_\beta h''\Omega^\beta$$

and

$$(f-\partial g)(X_h)=\sum_{|\alpha|=\delta}B_\alpha h'\Omega^\alpha+\sum_{|\beta|=\delta-1}C_\beta h''\Omega^\beta.$$

Thus, if $\delta\notin\mathbb{N}$ then $f-\partial g=0$ (since $|\alpha|\in\mathbb{N}$ then $|\alpha|$ can not be equal to δ), therefore $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})=0$. If $\delta\in\mathbb{N}$ then the condition (3.4) is coming from (3.1), since, in (3.1) we have $\delta-|\beta|-1=0$. Moreover, if $\delta=\mu-|\lambda|=|\alpha|=k$, then there are no terms in $h'\Omega^\alpha$ in the expression of $\partial g(X_h)$ for any $g\in\mathrm{D}_{\bar\lambda,\mu}$ (see (3.2)), therefore, the non vanishing cocycle $f(X_h)=\sum_{|\alpha|=k}B_\alpha h'\Omega^\alpha$ are non-trivial. □

Now, we prove that, generically, we can annihilate the term in h'' in the expression of the 1-cocycle (3.5) by adding a coboundary and we describe completely the space $\mathrm{H}^1(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})$ for generic $\bar\lambda$.

Theorem 3.4 *If $\delta=k\in\mathbb{N}$ and $-2\bar\lambda\notin\{0,\ldots,k-1\}^n$ then, we have*

$$\dim\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})=\binom{n+k-2}{k}.$$

These spaces are spanned by the cocycles:

$$f(X_h)=\sum_{|\alpha|=k}B_\alpha h'\Omega^\alpha,$$

where the B_α are constants satisfying the conditions (3.4).

Proof. By Theorem 3.3, for $\delta=k\geq1$, any 1-cocycle f can be expressed as follows:

$$f(X_h)=\sum_{|\alpha|=k}B_\alpha h'\Omega^\alpha+\sum_{|\beta|=k-1}C_\beta h''\Omega^\beta. \tag{3.5}$$

Consider

$$g=\sum_{|\alpha|=k}D_\alpha\Omega^\alpha.$$

We have

$$
\begin{aligned}
\partial g(X_h) &= -\tfrac{1}{2}\sum_{|\alpha|=k}\sum_{i=1}^{n}\alpha_i(\alpha_i+2\lambda_i-1)D_\alpha h''\Omega^{\alpha-\varepsilon_i} \\
&= -\tfrac{1}{2}\sum_{|\beta|=k-1}\sum_{i=1}^{n}(\beta_i+1)(\beta_i+2\lambda_i)D_{\beta+\varepsilon_i}h''\Omega^{\beta}.
\end{aligned}
\tag{3.6}
$$

Now, consider the linear system

$$
\frac{1}{2}\sum_{i=1}^{n}(\beta_i+1)(\beta_i+2\lambda_i)D_{\beta+\varepsilon_i}=C_\beta,\quad |\beta|=k-1 \tag{3.7}
$$

which express that

$$
\sum_{|\beta|=k-1}C_\beta h''\Omega^\beta=-\partial g(X_h).
$$

Without loss of generality, assume that $-2\lambda_1\notin\{0,\ldots,k-1\}$. Choose arbitrarily the D_α where $\alpha=(0,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}^n$ with $|\alpha|=k$. Now, for any $\alpha=(1,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}^n$ with $|\alpha|=k$, consider $\beta=(0,\alpha_2,\ldots,\alpha_n)$ (then $|\beta|=k-1$). The coefficient D_α is uniquely defined by (3.7), in function of C_β and the $D_{\beta+\varepsilon_i}$, for $i\geq 2$, (indeed, $D_\alpha=D_{\beta+\varepsilon_1}$, $\beta_1=0$ and $\lambda_1\neq 0$). Similarly, by (3.7), we define the coefficients D_α with $\alpha_1=2$, in function of those with $\alpha_1=1$ and $C_{(1,\alpha_2,\ldots,\alpha_n)}$. So, step by step, we define all the coefficients D_α so that $(f+\partial g)(X_h)$ does not contain terms in h''.

The non vanishing 1-cocycles defined be (3.4) are non trivial, since any coboundary does not contain terms in h' (see (3.6)). Thus, the space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})$ is isomorphic to the space of solutions of the system of linear equations (3.4). As before, we assume that $-2\lambda_1\notin\{0,\ldots,k-1\}$, the space of solutions of the system of linear equations (3.4) is generated by the arbitrary coefficients B_α where $\alpha=(0,\alpha_2,\ldots,\alpha_n)\in\mathbb{N}^n$ with $|\alpha|=k$. The number of such B_α is the well known binomial coefficient with repetition

$$
\Gamma^k_{n-1}=\binom{n+k-2}{k},
$$

which is equal to the dimension of $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})$. □

4 Singular cases

Now assume that

$$
-2\bar\lambda=(t_1,t_2,\ldots,t_n)\in\{0,1,\ldots,k-1\}^n.
$$

We cut the system of equations (3.4) into the two following subsystems (S_1) and (S_2):

$$
(S_1):\quad \sum_{i=1}^{n}(\alpha_i+1)(\alpha_i+2\lambda_i)B_{\alpha+\varepsilon_i}=0,\quad \alpha_1\neq t_1. \tag{4.1}
$$

$$
(S_2):\quad \sum_{i=1}^{n}(\alpha_i+1)(\alpha_i+2\lambda_i)B_{\alpha+\varepsilon_i}=0,\quad \alpha_1=t_1. \tag{4.2}
$$

From (S_1) we extract the following system (S_1')

$$
(S_1'):\quad \sum_{i=2}^{n}(\alpha_i+1)(\alpha_i+2\lambda_i)B_{\alpha+\varepsilon_i}=0,\quad \alpha_1=t_1-1. \tag{4.3}
$$

The space $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})$ is managed by the system (3.4), which is a system with Γ^k_n unknowns and Γ^{k-1}_n equations. Any equation is coming from a given $\alpha=(\alpha_1,\ldots,\alpha_n)$ with $|\alpha|=k-1$. The rank of (3.4) is then less or equal to Γ^{k-1}_n. The cocycles generating $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar\lambda,\mu})$ are of two types:

$$\text{type } 1: \quad \sum_{|\alpha|=k} B_\alpha h' \Omega^\alpha \quad \text{or} \quad \text{type } 2: \quad \sum_{|\beta|=k-1} C_\beta h'' \Omega^\beta.$$

If the rank of (3.4) is $\Gamma_n^{k-1} - \ell$ then the dimension of the space of cocycle of type 1 is

$$\Gamma_n^k - \Gamma_n^{k-1} + \ell = \Gamma_{n-1}^k + \ell.$$

The dimension of the space of classes of cocycles of type 2 is equal to the dimension of the space of all C_β, with $|\beta| = k-1$, from which we subtract the rank of (3.7) (or (3.4)). Thus, the dimension of the space of classes of cocycles of type 2 is ℓ since the space of parameters C_β, with $|\beta| = k-1$, is Γ_n^{k-1}-dimensional. Thus,

$$\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \Gamma_{n-1}^k + 2\ell = \binom{n+k-2}{k} + 2\ell.$$

Theorem 4.1 *If $\sigma_n < k-1$, then*

$$\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \binom{n+k-2}{k}.$$

Proof. This is equivalent to the fact that, in this case, the system (3.4) is of maximal rank. We proceed by recurrence to prove this result. It is true for $n = 2$ (see [2]). Assume that the result is true for $n-1$. Consider the subsystems (S_1) and (S_2), it is easy to prove that (S_1) is of maximal rank, while, by the recurrence hypothesis, the subsystem (S_2) is also of maximal rank. Indeed, the subsystem (S_2) can be considered as a system of equations related to the $(n-1)$-tuple $(\alpha_2, \ldots, \alpha_n)$. So, we are in the case $n-1$ with

$$\alpha_2 + \cdots + \alpha_n = k - 1 - t_1 = k' - 1 \quad \text{and} \quad \sigma_{n-1} = t_2 + \cdots + t_n < k - t_1 - 1 = k' - 1.$$

If $t_1 = 0$ then, combining the two subsystems (S_1) and (S_2), we get a system with maximal rank, since there are no common unknowns. Indeed, the unknowns $B_{(\alpha_1,\ldots,\alpha_n)}$ of (S_1) are all with $\alpha_1 \neq 0$, while those of (S_2) are all with $\alpha_1 = 0$.

If $t_1 > 0$ then the unknowns $B_{(\alpha_1,\ldots,\alpha_n)}$ of (S_2) are all with $\alpha_1 = t_1$. The unknowns $B_{(\alpha_1,\ldots,\alpha_n)}$ with $\alpha_1 = t_1$ can also appear in (S_1), but they appear only in the equations corresponding to $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 = t_1 - 1$. Therefore, we consider the system (S_1') which can be also considered as a system of equations related to $(\alpha_2, \ldots, \alpha_n)$, but, here we have

$$\alpha_2 + \cdots + \alpha_n = k - t_1 = k' - 1 \quad \text{and} \quad \sigma_{n-1} = t_2 + \cdots + t_n < k - t_1 - 1 = k' - 2 < k' - 1.$$

For (S_1') we are in the case $n-1$, therefore, by the recurrence hypothesis, the system (S_1') is of maximal rank, therefore, there are no nontrivial combination of some equations of (S_1) belonging to (S_2), since there are no nontrivial combination of some equations of (S_1) killing the B_α with $\alpha_1 = t_1 - 1$. That is, the spaces of the equations respectively of (S_1) and (S_2) are supplementary. Thus, combining the two subsystems (S_1) and (S_2), we get a system of maximal rank which is the system (3.4). Theorem 4.1 is proved. □

Theorem 4.2 *If $\sigma_n = k-1$ then*

$$\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \binom{n+k-2}{k} + 2.$$

Proof. This is equivalent to the fact that, in this case, the system (3.4) is of rank $\Gamma_n^{k-1} - 1$. We proceed by recurrence to prove this fact. This is true for $n = 2$ (see [2]). Assume that the result is true for $n-1$. Consider the subsystems (S_1) and (S_2). As before (S_1) is of maximal rank, it is of rank $\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1}$. The subsystem (S_2) is of rank $\Gamma_{n-1}^{k-t_1-1} - 1$ (by the recurrence hypothesis).

If $t_1 = 0$ then, as before, we see that the subsystems (S_1) and (S_2) are independent of each other. Therefore, combining the two subsystems (S_1) and (S_2), we get a system of rank

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} - 1 = \Gamma_n^{k-1} - 1.$$

Indeed, the unknowns $B_{(\alpha_1,\ldots,\alpha_n)}$ of (S_1) are all with $\alpha_1 \neq 0$, while those of (S_2) are all with $\alpha_1 = 0$ (there are common unknowns).

If $t_1 > 0$ then the unknowns $B_{(\alpha_1,\ldots,\alpha_n)}$ of (S_2) are all with $\alpha_1 = t_1$. The unknowns $B_{(\alpha_1,\ldots,\alpha_n)}$ with $\alpha_1 = t_1$ appear in (S_1) only in the equations corresponding to $(\alpha_1,\ldots,\alpha_n)$ with $\alpha_1 = t_1 - 1$. As before, we consider the system (S_1'). We are in the case $n-1$ with

$$\alpha_2 + \cdots + \alpha_n = k - t_1 = k' - 1 \quad \text{and} \quad \sigma_{n-1} = t_2 + \cdots + t_n = k - t_1 - 1 = k' - 2 < k' - 1.$$

By the recurrence hypothesis, the system (S_1') is of maximal rank, therefore, there are no nontrivial combination of some equations of (S_1) killing the B_α with $\alpha_1 = t_1 - 1$, therefore, combining the two subsystems (S_1) and (S_2), we get a system with rank

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} - 1 = \Gamma_n^{k-1} - 1.$$

Theorem 4.2 is proved. □

Remark 4.3 If $\sigma_n = k-1$, we see that the equation corresponding to $(t_1,\ldots,t_n)$ is trivial, so, it's enough to prove that, if we subtract this equation from (3.4), we get a maximal rank system. In this case the space of cocycles of type 2 is one dimensional, spanned by

$$\omega(X_h) = h''\Omega^{(t_1,\ldots,t_n)}.$$

Theorem 4.4 *If $\sigma_n = k$ then*

$$\dim \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \binom{n+k-2}{k} + 2(s-1),$$

where s is the number of $t_i \geq 1$.

Proof. We proceed by recurrence to prove that the rank of (3.4) is

$$\Gamma_n^{k-1} - (s-1) = \binom{n+k-2}{k-1} - (s-1).$$

This is true for $n = 2$ (see [2]), indeed, for $n = 2$ we have necessarily $s = 2$. Assume that the result is true for $n-1$. As before, if $t_1 = 0$ then the subsystems (S_1) and (S_2) are independent of each other. The subsystem (S_1) is of rank $\Gamma_n^{k-1} - \Gamma_{n-1}^{k-1}$, while, according to the recurrence hypothesis, the subsystem (S_2) is of rank $\Gamma_{n-1}^{k-1} - (s-1)$. Therefore, the rank of (3.4) is

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-1} + \Gamma_{n-1}^{k-1} - (s-1) = \Gamma_n^{k-1} - (s-1) = \binom{n+k-2}{k-1} - (s-1).$$

Now, for $t_1 > 0$, the subsystem (S_1) is of rank $\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1}$, while, according to the recurrence hypothesis, the subsystem (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1} - (s-2).$$

Consider the subsystem (S_1'). We are in the case $n-1$ with

$$\alpha_2 + \cdots + \alpha_n = k - t_1 = k' - 1 \quad \text{and} \quad \sigma_{n-1} = t_2 + \cdots + t_n = k - t_1 = k' - 1.$$

Therefore, the subsystem (S_1') is of rank

$$\Gamma_{n-1}^{k-t_1} - 1.$$

The equation corresponding to $(t_1 - 1, t_2, \ldots, t_n)$ appear as a trivial equation in (S_1'), corresponding in (S_1) to the equation

$$B_{(t_1,t_2,\ldots,t_n)} = 0. \tag{4.4}$$

The equation (4.4) appears also in (S_2) corresponding to any $(t_1, t_2, \ldots, t_n) - \varepsilon_i$ for $i \geq 2$.

Obviously, if we subtract the trivial equation from (S_1') we obtain a system of maximal rank. Thus, the system (3.4) is of rank

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} - (s-2) - 1 = \Gamma_n^{k-1} - (s-1).$$

Theorem 4.4 is proved. □

Remark 4.5 The result of the previous theorem can be explained by the fact that the equations (3.4) corresponding to $(t_1, \ldots, t_n) - \varepsilon_i$, for $t_i > 0$, are the same. They are all equivalent to (4.4). So, it's enough to prove that, if we subtract $(s-1)$ equations from these s equivalent equations we get a maximal rank system. The space of cocycles of type 2 is spanned by

$$\Gamma_i(X_h) = h''\Omega^{\gamma_i}, \quad \gamma_i = (t_1, t_2, \ldots, t_n) - \varepsilon_i, \quad t_i \geq 1.$$

Theorem 4.6 *If $\sigma_n = k+1$, then*

$$\dim \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \begin{cases} \Gamma_n^{k-1} + s(s-1) - 2r & \text{if} \quad \max t_i \geq 2 \\ \Gamma_n^{k-1} & \text{if} \quad \max t_i = 1. \end{cases}$$

where s is the number of $t_i \geq 1$ and r is the number of $t_i = 1$.

Proof. Assume that $t_1 \geq 1$ and consider the system (S_1'). We are in the case $n-1$ with

$$\alpha_2 + \cdots + \alpha_n = k - t_1 = k' - 1 \quad \text{and} \quad \sigma_{n-1} = t_2 + \cdots + t_n = k + 1 - t_1 = k'.$$

Therefore, it was proved in the proof of Theorem 4.4 that the system (S_1') is of rank

$$\Gamma_{n-1}^{k-t_1} - (s-2).$$

In (S_1') the equations corresponding to $(t_1 - 1, t_2, \ldots, t_n) - \varepsilon_i$, for $t_i \geq 1$, are equivalent to

$$B_{(t_1-1,t_2,\ldots,t_n)} = 0.$$

But, the correspondent equations in (S_1) are

$$B_{(t_1,t_2,\ldots,t_n)-\varepsilon_i} + B_{(t_1-1,t_2,\ldots,t_n)} = 0, \quad t_i \geq 1. \tag{4.5}$$

Case 1: $\max t_i \geq 2$. Assume that $t_2 \geq 2$. We proceed by recurrence to prove that the rank of (3.4) is

$$\Gamma_n^{k-1} - \frac{1}{2}s(s-1) + r.$$

This is true for $n = 2$ (see [2]), indeed, for $n = 2$ we have necessarily $s = 2$ and $r = 0$. Assume that the result is true for $n - 1$.

Assume that $t_1 = 1$. By the recurrence hypothesis, the subsystem (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}(s-2)(s-1) + r - 1.$$

We have $t_2 \geq 2$, then, in (S_2) the equation corresponding to $(t_1, t_2 - 2, \ldots, t_n)$ gives

$$B_{(t_1,t_2-1,\ldots,t_n)} = 0.$$

Therefore, the equation corresponding to $(t_1, t_2 - 1, \ldots, t_n) - \varepsilon_i$, for $t_i \geq 1$, gives

$$B_{(t_1,t_2,\ldots,t_n)-\varepsilon_i} = 0, \quad t_i \geq 1, \tag{4.6}$$

and the $(s-2)$ correspondent equations (4.5) in (S_1), for $i \geq 3$, become trivial (we can also say that the equations (4.5) of (S_1), for $i \geq 3$, are combination of some equations of (S_2)). If we subtract the equations (4.6), for $i \geq 3$, from (S_1'), we obtain a maximal rank system. Therefore, the rank of (3.4) is

$$\begin{aligned} &\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}(s-2)(s-1) + r - 1 - (s-2) \\ &= \binom{n+k-2}{k} - \frac{1}{2}s(s-1) + r. \end{aligned}$$

If $t_1 \geq 2$, then the subsystem (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}(s-2)(s-1) + r.$$

But, here we have $B_{(t_1-1,t_2,\ldots,t_n)} = 0$ as equation corresponding to $(t_1-2, t_2, \ldots, t_n)$. Therefore, the equation corresponding to $(t_1-1, t_2, \ldots, t_n) - \varepsilon_i$ gives $B_{(t_1,t_2,\ldots,t_n)-\varepsilon_i} = 0$. So, the $(s-2)$ correspondent equations (4.5) in (S_1) (for $i \geq 3$) become trivial, but, the equation in (S_1): $B_{(t_1,t_2-1,\ldots,t_n)} = 0$, appear also in (S_2) as equation corresponding to $(t_1, t_2-2, \ldots, t_n)$. Thus, the rank of (3.4) is

$$\begin{aligned}\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}(s-2)(s-1) + r - (s-2) - 1 \\ = \binom{n+k-2}{k} - \frac{1}{2}s(s-1) + r.\end{aligned}$$

Case 2: $\max t_i = 1$. In this case we prove the rank of (3.4) is

$$\Gamma_n^{k-1} = \binom{n+k-2}{k}.$$

By the recurrence hypothesis, the subsystem (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1}.$$

Assume that $t_1 = t_2 = 1$. Therefore, the $(s-2)$ equations, in (S_1'), corresponding to $(t_1-1, t_2, \ldots, t_n) - \varepsilon_i$, $t_i = 1$ for $i \geq 3$, are equivalent to $B_{(t_1-1,t_2,\ldots,t_n)} = 0$. Obviously, if we subtract these equations from (S_1'), we get a maximal rank system. Therefore, the rank of (3.4) is

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} = \binom{n+k-2}{k}.$$

Theorem 4.6 is proved. □

Remark 4.7 In the previous theorem, if there exist some $t_i \geq 2$, then, for any $t_i \geq 2$, (3.4) applied to $(t_1, t_2, \ldots, t_n) - 2\varepsilon_i$ gives $B_{(t_1,t_2,\ldots,t_n)-\varepsilon_i} = 0$. We use one of $t_i \geq 2$ to prove that $B_{(t_1,t_2,\ldots,t_n)-\varepsilon_j} = 0$ for $t_j = 1$. That is, we have r equations: $B_{(t_1,t_2,\ldots,t_n)-\varepsilon_j} = 0$, corresponding to $(t_1, t_2, \ldots, t_n) - \varepsilon_{i_0} - \varepsilon_j$ for a fixed $t_{i_0} \geq 2$ and $t_j = 1$. Therefore, all other equations corresponding to $(t_1, t_2, \ldots, t_n) - \varepsilon_i - \varepsilon_j$, for $i<j$ and $t_i,\ t_j \geq 1$, become trivial. We prove that if we subtract these trivial equations we get a maximal rank system. Thus, the rank of (3.4) is

$$\Gamma_n^{k-1} - \frac{1}{2}s(s-1) + r = \binom{n+k-2}{k} - \frac{1}{2}s(s-1) + r.$$

Theorem 4.8 *If* $\sigma_n = k+2$, *then*

$$\dim \mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \binom{n+k-2}{k} + s(s-1),$$

where s *is the number of* $t_i \geq 3$.

Proof. We proceed by recurrence to prove that the rank of (3.4) is

$$\Gamma_n^{k-1} - \frac{1}{2}s(s-1) = \binom{n+k-2}{k} - \frac{1}{2}s(s-1).$$

This is true for $n=2$. Assume that it is true for $n-1$. As usually, the result is true if $t_1 = 0$.
If $\max t_i = 1$ then (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1},$$

and the system (S_1') is of maximal rank. Therefore, the rank of (3.4) is

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} = \binom{n+k-2}{k}.$$

If $\max t_i = 2$ then we can assume that $t_1 = 2$. In this case (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1}.$$

For (S_1') we distinguish two cases. $\max_{i>1} t_i = 1$ or $\max_{i>1} t_i = 2$. In the first case, the system (S_1') is of maximal rank. Therefore, the rank of (3.4) is

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} = \binom{n+k-2}{k}.$$

In the second case, the system (S_1') is of rank

$$\Gamma_{n-1}^{k-t_1} - \frac{1}{2}(s'-2)(s'-1) + r',$$

where s' is the number of $t_i \geq 1$ and r' is the number of $t_i = 1$. Indeed, for the system (S_1'), we are in the case $n-1$ with

$$\alpha_2 + \cdots + \alpha_n = k - t_1 = k' - 1 \quad \text{and} \quad \sigma_{n-1} = t_2 + \cdots + t_n = k + 2 - t_1 = k' + 1.$$

Assume that $t_2 = 2$. In the system (S_1'), for $i \geq 2$ such that $t_i \geq 2$, the equations corresponding to $(t_1 - 1, t_2, \ldots, t_n) - 2\varepsilon_i$ are

$$B_{(t_1-1,t_2,\ldots,t_n)-\varepsilon_i} = 0, \quad t_i \geq 2, \quad i \geq 2. \tag{4.7}$$

In particular, we have $B_{(t_1-1,t_2-1,\ldots,t_n)} = 0$. Therefore, there are r' equations

$$B_{(t_1-1,t_2,\ldots,t_n)-\varepsilon_i} = 0, \quad t_i = 1, \quad i \geq 3, \tag{4.8}$$

corresponding to $(t_1 - 1, t_2 - 1, \ldots, t_n) - \varepsilon_i$, for $i \geq 3$ and $t_i = 1$. Therefore, all other equations corresponding to $(t_1 - 1, t_2, \ldots, t_n) - \varepsilon_i - \varepsilon_j$, for $i < j$ and $t_i,\, t_j \geq 1$, become trivial in (S_1'). The number of these trivial equations is $\frac{1}{2}(s'-2)(s'-1) - r'$. Of course, if we subtract these trivial equations from (S_1') we obtain a maximal rank system. Therefore, the rank of (3.4) is

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} = \binom{n+k-2}{k}.$$

Indeed, these trivial equations of (S_1') are associated in (S_1) to

$$B_{(t_1,t_2,\ldots,t_n)-\varepsilon_i-\varepsilon_j} = 0. \tag{4.9}$$

But, (4.9) appears also in (S_2) as equation corresponding to $(t_1, t_2 - 1, \ldots, t_n) - \varepsilon_i - \varepsilon_j$, since we have also $B_{(t_1,t_2-1,\ldots,t_n)-\varepsilon_i} = 0$, for $t_i \geq 2$.

Now, if $t_1 \geq 3$, then (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}(s-2)(s-1).$$

The rank of (S_1') is

$$\Gamma_{n-1}^{k-t_1} - \frac{1}{2}(s'-2)(s'-1) + r'.$$

In (S_1'), for $i \geq 2$ such that $t_i \geq 2$, the equations corresponding to $(t_1 - 1, t_2, \ldots, t_n) - 2\varepsilon_i$ are

$$B_{(t_1-1,t_2,\ldots,t_n)-\varepsilon_i} = 0, \quad t_i \geq 2, \quad i \geq 2. \tag{4.10}$$

Moreover, there are r' equations

$$B_{(t_1-1,t_2,\ldots,t_n)-\varepsilon_i} = 0, \quad t_i = 1, \quad i \geq 2, \tag{4.11}$$

corresponding to $(t_1 - 1, t_2, \ldots, t_n) - \varepsilon_{i_0} - \varepsilon_i$, for a fixed $i_0 \geq 2$ such that $t_{i_0} \geq 2$ and $i \geq 2$ such that $t_i = 1$.

All other equations corresponding to $(t_1-1,t_2,\ldots,t_n)-\varepsilon_i-\varepsilon_j$, for $i<j$ such that $t_i,\ t_j\geq 1$, become trivial in (S_1'). The number of these trivial equations is $\frac{1}{2}(s'-2)(s'-1)-r'$. These trivial equations in (S_1') appear in (S_1) as

$$B_{(t_1,t_2,\ldots,t_n)-\varepsilon_i-\varepsilon_j}=0. \tag{4.12}$$

For any $t_i\geq 3$, the equation corresponding to $(t_1,t_2,\ldots,t_n)-3\varepsilon_i$ gives $B_{(t_1,t_2,\ldots,t_n)-2\varepsilon_i}=0$. Therefore, the equation (4.12) appear in (S_2) as equation corresponding to $(t_1,t_2,\ldots,t_n)-3\varepsilon_i$ only for $i\geq 2$ such that $t_i\geq 3$. Thus, the rank of (3.4) is

$$\Gamma_n^{k-1}-\Gamma_{n-1}^{k-t_1-1}+\Gamma_{n-1}^{k-t_1-1}-\frac{1}{2}(s-2)(s-1)-(s-1)=\binom{n+k-2}{k}-\frac{1}{2}s(s-1).$$

Theorem 4.8 is proved. □

Theorem 4.9 *If $\sigma_n=k+m$, with $m\geq 2$, then*

$$\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2),\mathcal{D}_{\bar{\lambda},\mu})=\binom{n+k-2}{k}+s(s-1),$$

where s is the number of $t_i>m$.

Proof. Assume that $m\geq 3$, since the case $m=2$ was treated in the previous theorem. We proceed by recurrence to prove that the rank of (3.4) is

$$\Gamma_n^{k-1}-\frac{1}{2}s(s-1).$$

This is true for $n=2$. Assume that it is true for $n-1$.

4.1 If $t_1\leq m$.

In this case the system (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1}-\frac{1}{2}s(s-1).$$

The system (S_1') is of rank

$$\Gamma_{n-1}^{k-t_1}-\frac{1}{2}s'(s'-1)$$

where s' is the number of $t_i\geq m$ for $i\geq 2$. Indeed, for the system (S_1'), we are in the case $n-1$ with

$$\alpha_2+\cdots+\alpha_n=k-t_1=k'-1\quad\text{and}\quad\sigma_{n-1}=t_2+\cdots+t_n=k+m-t_1=k'+m-1.$$

In (S_1'), for any $t_i\geq m$, the equation corresponding to $\alpha=(t_1-1,t_2,\ldots,t_n)-m\varepsilon_i$ is

$$B_{(t_1-1,t_2,\ldots,t_n)-(m-1)\varepsilon_i}=0. \tag{4.13}$$

Case 1: $m=2h+1$ with $h\geq 1$. In (S_1'), for $i\neq j$, $t_i\geq 2h+1$ and $t_j\geq 1$, according to (4.13), the equation corresponding to $(t_1-1,t_2,\ldots,t_n)-2h\varepsilon_i-\varepsilon_j$ gives

$$B_{(t_1-1,t_2,\ldots,t_n)-(2h-1)\varepsilon_i-\varepsilon_j}=0.$$

Step by step, for $t_i\geq(2h+1)$ and $t_j\geq(2h+1)$, $i\neq j$, the equations corresponding respectively to $(t_1-1,t_2,\ldots,t_n)-(h+1)\varepsilon_i-h\varepsilon_j$ and $(t_1-1,t_2,\ldots,t_n)-h\varepsilon_i-(h+1)\varepsilon_j$ are equivalent to the same equation which is

$$B_{(t_1-1,t_2,\ldots,t_n)-h\varepsilon_i-h\varepsilon_j}=0. \tag{4.14}$$

Therefore, if we subtract from (S_1') the $\frac{1}{2}s'(s'-1)$ equations corresponding to $(t_1-1,t_2,\ldots,t_n)-(h+1)\varepsilon_i-h\varepsilon_j$, where $t_i,\ t_j\geq(2h+1)$ and $i<j$, we get a maximal rank. Thus, the rank of (3.4) is

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}s(s-1) = \binom{n+k-2}{k} - \frac{1}{2}s(s-1).$$

Case 2: $m = 2h$. In (S_1'), for $i \neq j$, $t_i \geq m$ and $t_j \geq 1$, according to (4.13), the equation corresponding to $(t_1 - 1, t_2, \ldots, t_n) - (2h-1)\varepsilon_i - \varepsilon_j$ gives

$$B_{(t_1-1,t_2,\ldots,t_n)-(2h-2)\varepsilon_i-\varepsilon_j} = 0.$$

Consider a fixed $t_{i_0} > 1$. The equation corresponding to $(t_1 - 1, t_2, \ldots, t_n) - (2h-2)\varepsilon_i - \varepsilon_{i_0} - \varepsilon_j$, for $t_i \geq m$, gives

$$B_{(t_1-1,t_2,\ldots,t_n)-(2h-3)\varepsilon_i-\varepsilon_{i_0}-\varepsilon_j} = 0.$$

Step by step, for t_i, $t_j \geq m$, $i \neq j$, the equations corresponding respectively to $(t_1 - 1, t_2, \ldots, t_n) - \varepsilon_{i_0} - (h-1)\varepsilon_i - h\varepsilon_j$ and $(t_1 - 1, t_2, \ldots, t_n) - \varepsilon_{i_0} - h\varepsilon_i - (h-1)\varepsilon_j$ are equivalent to the same equation which is

$$B_{(t_1-1,t_2,\ldots,t_n)-\varepsilon_{i_0}-(h-1)\varepsilon_i-(h-1)\varepsilon_j} = 0. \tag{4.15}$$

Thus, we have the same result as in the previous case.

4.2 If $t_1 > m$

In this case the system (S_2) is of rank

$$\Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}(s-2)(s-1).$$

The system (S_1') is of rank

$$\Gamma_{n-1}^{k-t_1} - \frac{1}{2}s'(s'-1)$$

where s' is the number of $t_i \geq m$ for $i \geq 2$. We proceed as in the previous case, but here, since $t_1 > m$, we prove that in (S_1') the equations (4.14) and (4.15) become trivial for $i = j$ and $t_i > m$. The corresponding $(s-1)$ equations in (S_1) are respectively

$$B_{(t_1,t_2,\ldots,t_n)-(2h+1)\varepsilon_i} = 0 \quad \text{and} \quad B_{(t_1,t_2,\ldots,t_n)-\varepsilon_{i_0}-(2h-1)\varepsilon_i} = 0.$$

But, these equations appear also in (S_2) as equations corresponding respectively to $(t_1, t_2, \ldots, t_n) - (2h+2)\varepsilon_i$ and $(t_1, t_2, \ldots, t_n) - \varepsilon_{i_0} - 2h\varepsilon_i$ for any $i \geq 2$ such that $t_i > m$. Thus, the rank of (3.4) is

$$\Gamma_n^{k-1} - \Gamma_{n-1}^{k-t_1-1} + \Gamma_{n-1}^{k-t_1-1} - \frac{1}{2}(s-2)(s-1) - (s-1) = \binom{n+k-2}{k} - \frac{1}{2}s(s-1).$$

Theorem 4.9 is proved □

Note that for $n = 2$ and $\sigma_2 \geq k-1$, we have always

$$s(s-1) = 2(s-1) = s(s-1) - 2r = 2$$

and we know that, in this case, we have $\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = 3 = \binom{2+k-2}{k} + 2.$

Now, we summarize our results in the following theorem

Theorem 4.10

$$\dim\mathrm{H}^1_{\mathrm{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\bar{\lambda},\mu}) = \begin{cases} \Gamma_{n-1}^k & \text{if} \quad \sigma_n < k-1, \\ \Gamma_{n-1}^k + 2 & \text{if} \quad \sigma_n = k-1, \\ \Gamma_{n-1}^k + 2(s-1) & \text{if} \quad \sigma_n = k, \\ \Gamma_{n-1}^k + (s+r)(s+r-1) - 2r & \text{if} \quad \sigma_n = k+1, \ \text{and} \ \max t_i \geq 2 \\ \Gamma_{n-1}^k & \text{if} \quad \sigma_n = k+1, \ \text{and} \ \max t_i = 1 \\ \Gamma_{n-1}^k + s(s-1) & \text{if} \quad \sigma_n = k+m, \quad m \geq 2, \end{cases}.$$

where s is the number of $t_i > \sigma_n - k$, *r is the number of* $t_i = 1$ *and* $\Gamma_{n-1}^k = \binom{n+k-2}{k}$.

References

1. Basdouri Okba and Nasri Elamine, $\mathfrak{sl}(2)$ *-invariant 3-ary differential operators cohomology of* $\mathfrak{sl}(2)$*acting on 3-ary differential operators*, Int. J. Geom. Methods Mod. Phys. 14, 1750015 (2017) [17 pages], https://doi.org/10.1142/S0219887817500153
2. Bouarroudj S, *Cohomology of the vector fields Lie algebras on* $\mathbb{RP}^1$*acting on bilinear differential operators*, International Journal of Geometric Methods in Modern Physics (2005), **2**; N 1, 23-40.
3. Bouarroudj S, Ovsienko V, Three cocycles on Diff(S^1) generalizing the Schwarzian derivative, *Internat. Math. Res. Notices* **1** (1998), 25–39.
4. Feigin B L, Fuchs D B, Homology of the Lie algebras of vector fields on the line, *Func. Anal. Appl.*, 14 (1980) 201–212.
5. Fuchs D B, *Cohomology of infinite-dimensional Lie algebras*, Plenum Publ. New York, 1986.
6. Gargoubi H, *Sur la géométrie de l'espace des opérateurs différentiels linéaires sur* $\mathbb{R}$, Bull. Soc. Roy. Sci. Liège. Vol. 69, 1, 2000, 21–47.
7. Lecomte B P A, *On the cohomology of* $\mathfrak{sl}(n+1;\mathbb{R})$*acting on differential operators and* $\mathfrak{sl}(n+1;\mathbb{R})$*-equivariant symbols,* Indag. Math. NS. 11 (1), (2000), 95 114.
8. A. Nijenhuis, R.W. Richardson, *Cohomology and deformations in graded Lie algebras*. Bull. Amer. Math. Soc., 72 (1966), 1-29.