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ORIGINAL RESEARCH

Generalized hyper-Lucas numbers and applications

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Abstract In this paper, we give some combinatorial properties of a new generalization of hyper-Lucas numbers in order to extend the Cassini determinant. We also study norms of some circulant and *s*-circulant matrices.

Keywords Generalized hyper-Fibonacci numbers · Generalized hyper-Lucas numbers · Cassini determinant · Circulant matrix · *s*-circulant matrix · Spectral norm

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1 Introduction

Drawing on hyperharmonic numbers, Dil and Mező [7] introduced the concept of hyper-Fibonacci numbers $\left(F_n^{(r)}\right)_{n\geq 0}$ and hyper-Lucas numbers $\left(L_n^{(r)}\right)_{n\geq 0}$ which are defined respectively by

$$F_n^{(r+1)} = \sum_{k=0}^{n} F_k^{(r)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1,$$

and

$$L_n^{(r+1)} = \sum_{k=0}^{n} L_k^{(r)}, \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1, \tag{1.1}$$

where r is a nonnegative integer, $(F_n)_{n\geq 0}$ and $(L_n)_{n\geq 0}$ are the classical Fibonacci and Lucas numbers respectively. Hyper-Fibonacci numbers and hyper-Lucas numbers satisfy many interesting number-theoretical and combinatorial properties [5].

In [1] the authors generalized the concept of hyper-Fibonacci numbers to the Lucas sequence of the first kind $(u_n)_{n\geq 0}$ defined by

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$$\begin{cases} u_0 = 0, u_1 = 1, \\ u_{n+2} = \alpha u_{n+1} + \beta u_n, \qquad (n \geq 0), \end{cases}$$

where α and β are any integers, as follows

$$u_n^{(r+1)} = \sum_{k=0}^{n} \alpha^{n-k} u_k^{(r)}, \quad u_n^{(0)} = u_n, \quad u_0^{(r)} = 0, \quad u_1^{(r)} = 1, \tag{1.2}$$

and extend the work of Martinjak and Urbiha [10] to this new generalization.

In this paper we generalize the concept of hyper-Lucas numbers (1.1) to the Lucas sequence of the second kind $(v_n)_{n\geq 0}$ defined by

$$\begin{cases} v_0 = 2, v_1 = \alpha, \\ v_{n+2} = \alpha v_{n+1} + \beta v_n, \qquad (n \geq 0). \end{cases}$$

For $\alpha = \beta = 1$ we obtain the well known Lucas numbers $(L_n)_{n\geq 0}$. Moreover, if $\beta \neq 0$ and $\alpha^2 + 4\beta \neq 0$, then the Lucas sequences of the first kind and the second kind for negative subscripts are defined respectively by $u_{-n} = -(-\beta)^{-n} u_n$ and $v_{-n} = (-\beta)^{-n} v_n$ for $n \geq 0$. If $\beta \neq 0$ and $\alpha^2 + 4\beta = 0$ then $\alpha = 2t$ and $\beta = -t^2$, where t is the double root of $x^2 - \alpha x - \beta$. Thus, $u_n = nt^{n-1}$, $u_{-n} = -nt^{-n-1}$, $v_n = 2t^n$ and $v_{-n} = 2t^{-n}$ for $n \geq 0$. In Sections 2 and 3 of this paper we assume that $\beta \neq 0$.

We define the generalized hyper-Lucas numbers $\left(v_n^{(r)}\right)_{n\geq 0}$ associated to the sequence $(v_n)_{n\geq 0}$ by

$$v_n^{(r+1)} = \sum_{k=0}^{n} \alpha^{n-k} v_k^{(r)}, \quad v_n^{(0)} = v_n, \quad v_0^{(r)} = 2, \quad v_1^{(r)} = (2r+1)\alpha, \tag{1.3}$$

where r is a nonnegative integer.

In Section 2 we give some properties of the generalized hyper-Lucas numbers which will be used in Section 3 to extend the well known Cassini identities [14]

$$L_n L_{n+2} - L_{n+1}^2 = 5(-1)^n \tag{1.4}$$

and

$$v_n v_{n+2} - v_{n+1}^2 = (-\beta)^n (\alpha^2 + 4\beta), \tag{1.5}$$

to hyper-Lucas numbers and generalized hyper-Lucas numbers respectively. Obviously, for $\alpha = \beta = 1$ identity (1.5) is simply (1.4). Various authors [1, 9–11, 13] have generalized, in several manners, only the classical Cassini identity for the Fibonacci numbers. Recently, in [2], the authors generalized identity (1.4) to (a, b)-hyper-Fibonacci numbers. In Section 3, we generalize (1.5) to the generalized hyper-Lucas numbers. In Section 4 we investigate norms of circulant and s-circulant matrices. An $n \times n$ matrix C_s is called s-circulant matrix if it is of the form

$$C_s = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ sc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ sc_{n-2} & sc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ sc_1 & sc_2 & sc_3 & \cdots & sc_{n-1} & c_0 \end{pmatrix}.$$

The matrix C_s is determined by its first row elements and s, thus we denote $C_s = \mathrm{Circs}(c_0, c_1, \ldots, c_{n-1})$. It is clear that, for $s = 1$, C_s is the classical circulant matrix that will be denoted by $C = \mathrm{Circ}(c_0, c_1, \ldots, c_{n-1})$. Circulant matrices have many connections to problems in physics, image processing, Fourier analysis, probability and statistics, numerical analysis, number theory and geometry [6].

In 2010, Shen and Cen [12] gave bounds (lower and upper) of the spectral norm of s-circulant matrices involving Fibonacci and Lucas numbers. In 2013, Yazlik and Taskara [15] presented bounds of the spectral norm of an s-circulant matrix with the generalized k-Horadam numbers. In 2014, Bahşī and Solak [3] computed the spectral norm of circulant matrices involving hyper-Fibonacci and hyper-Lucas numbers, from which they derive bounds of the spectral norm of the corresponding s-circulant matrices. Recently in 2017, Köme and Yazlik [8] present bounds for the spectral norm of the s-circulant matrices whose entries

are the biperiodic Fibonacci and biperiodic Lucas numbers. Our purpose is to generalize results of [3] to the sequences (1.2) and (1.3) using another method.

Throughout this paper we denote by C_n^k the binomial coefficient which is defined for a nonnegative integer n and an integer k by

$$C_n^k = \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } 0 \le k \le n \\ 0 & \text{othewise.} \end{cases} \tag{1.6}$$

For a negative integer n and an integer k we have

$$C_n^k = \begin{cases} (-1)^k C_{-n+k-1}^k & \text{if } k \ge 0 \\ (-1)^{n-k} C_{-k-1}^{n-k} & \text{if } k \le n \\ 0 & \text{othewise.} \end{cases} \tag{1.7}$$

2 Some properties of the generalized hyper-Lucas numbers

In this section we investigate properties satisfied by the generalized hyper-Lucas numbers $\left(v_n^{(r)}\right)_n$. Let us start with some technical results.

Lemma 2.1 *Let $n \ge 0$ be an integer, then*

$$v_n^{(1)} = \frac{1}{\beta} v_{n+2} - \frac{\alpha^{n+2}}{\beta}. \tag{2.1}$$

Proof By induction on n. For $n = 0$, identity (2.1) is trivially checked. Now assume that (2.1) is true for an integer $n \ge 0$, then

$$\begin{aligned} v_{n+1}^{(1)} &= \sum_{k=0}^{n+1} \alpha^{n+1-k} v_k \\ &= \alpha \sum_{k=0}^{n} \alpha^{n-k} v_k + v_{n+1} \\ &= \alpha v_n^{(1)} + v_{n+1} \\ &= \alpha \left(\frac{1}{\beta} v_{n+2} - \frac{\alpha^{n+2}}{\beta} \right) + v_{n+1} \\ &= \frac{1}{\beta} v_{n+3} - \frac{\alpha^{n+3}}{\beta}. \end{aligned}$$

We conclude that (2.1) is true for all $n \ge 0$. □

Lemma 2.2 *Let $P_r(n) = C_{n+r+2}^r + C_{n+r+1}^{r-1}$ where $r \ge 0$ and $n \ge 0$ are integers, then*

$$P_{r+1}(n) = 2r + 3 + \sum_{k=0}^{n} P_r(k).$$

Proof Since $\sum_{k=0}^{n} C_{k+r+1}^{r-1} = -(r+1) + C_{n+r+2}^r$, we get

$$\begin{aligned} 2r + 3 + \sum_{k=0}^{n} P_r(k) &= 2r + 3 + \sum_{k=0}^{n} C_{k+r+2}^r + \sum_{k=0}^{n} C_{k+r+1}^{r-1} \\ &= C_{n+r+3}^{r+1} + C_{n+r+2}^r \\ &= P_{r+1}(n). \end{aligned}$$

□

Proposition 2.3 *Let $r \ge 0$ be an integer, then*

$$v_n^{(r+1)} = \frac{1}{\beta} v_{n+2}^{(r)} - P_r(n)\frac{\alpha^{n+2}}{\beta}, \quad n \geq 0. \tag{2.2}$$

Proof By induction on $r \geq 0$. We deduce from Lemma 2.1 that (2.2) is true for $r = 0$. Now assume that (2.2) is true for an integer $r \geq 0$, then

$$\begin{aligned}
v_n^{(r+2)} &= \sum_{k=0}^{n} \alpha^{n-k} v_k^{(r+1)} \\
&= sum_{k=0}^{n} \alpha^{n-k}\left(\frac{1}{\beta} v_{k+2}^{(r)} - P_r(k)\frac{\alpha^{k+2}}{\beta}\right) \\
&= \frac{1}{\beta}\sum_{k=0}^{n} \alpha^{n-k} v_{k+2}^{(r)} - \sum_{k=0}^{n} P_r(k)\frac{\alpha^{n+2}}{\beta} \\
&= \frac{1}{\beta}\sum_{l=2}^{n+2} \alpha^{n+2-l} v_l^{(r)} - \sum_{k=0}^{n} P_r(k)\frac{\alpha^{n+2}}{\beta} \\
&= \frac{1}{\beta}\sum_{l=0}^{n+2} \alpha^{n+2-l} v_l^{(r)} - \frac{1}{\beta}\left[2\alpha^{n+2} + (2r+1)\alpha^{n+2}\right] - \sum_{k=0}^{n} P_r(k)\frac{\alpha^{n+2}}{\beta} \\
&= \frac{1}{\beta} v_{n+2}^{(r+1)} - \left((2r+3) + \sum_{k=0}^{n} P_r(k)\right)\frac{\alpha^{n+2}}{\beta} \\
&= \frac{1}{\beta} v_{n+2}^{(r+1)} - P_{r+1}(n)\frac{\alpha^{n+2}}{\beta}.
\end{aligned}$$

We deduce that (2.2) is true for all $r \geq 0$. □

Corollary 2.4 *Let $r \geq 0$ and $n \geq 0$ be integers, then*

$$v_{n+2}^{(r+1)} = \alpha v_{n+1}^{(r+1)} + \beta v_n^{(r+1)} + P_r(n)\alpha^{n+2}. \tag{2.3}$$

Proof According to definition (1.3), we have

$$v_{n+1}^{(r)} = v_{n+1}^{(r+1)} - \alpha v_n^{(r+1)}. \tag{2.4}$$

Replacing in (2.2) we obtain

$$\begin{aligned}
v_n^{(r+1)} &= \tfrac{1}{\beta}\left(v_{n+2}^{(r+1)} - \alpha v_{n+1}^{r+1}\right) - P_r(n)\tfrac{\alpha^{n+2}}{\beta} \\
&= \tfrac{1}{\beta} v_{n+2}^{(r+1)} - \tfrac{\alpha}{\beta} v_{n+1}^{(r+1)} - P_r(n)\tfrac{\alpha^{n+2}}{\beta}.
\end{aligned}$$

We deduce that

$$v_{n+2}^{(r+1)} = \alpha v_{n+1}^{(r+1)} + \beta v_n^{(r+1)} + P_r(n)\alpha^{n+2}.$$

□

Remark 2.5 For $r \geq 1$ and $\alpha \neq 0$ it follows from identity (2.3) that

$$v_{-n}^{(r)} = 0, \qquad 1 \leq n \leq r.$$

The following proposition is the key assertion behind the computation of Cassini determinant.

Proposition 2.6 *Assume that $\alpha \neq 0$ and let $r \geq 0$ be an integer, then*

$$v_{n+r+2}^{(r)} = \sum_{k=0}^{r+1}(-1)^{r+1-k}\alpha^{r-k}\left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) v_{n+k}^{(r)}, \qquad n \geq -r. \tag{2.5}$$

Proof Let us show identity (2.5) by induction on $r \geq 0$. For $r = 0$ we get $v_{n+2} = \alpha v_{n+1} + \beta v_n$ for $n \geq 0$

which is true by definition of the sequence $(v_n)_n$. Now assume that (2.5) is true for an integer $r \geq 0$, since $n+1 \geq n \geq -r$, we have

$$v_{n+r+3}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) v_{n+k+1}^{(r)}. \tag{2.6}$$

Since $n+r+3 \geq n+r+2 \geq 0$, we have $v_{n+r+3}^{(r)} = v_{n+r+3}^{(r+1)} - \alpha v_{n+r+2}^{(r+1)}$. For $k = 0, 1 \ldots, r+1$, we have $v_{n+k+1}^{(r)} = v_{n+k+1}^{(r+1)} - \alpha v_{n+k}^{(r+1)}$ because

- If $n+k+1 < 0$ then we obtain $0 = 0 - 0$.
- If $n+k+1 \geq 0$ and $n+k < 0$ then $n+k = -1$, we obtain $2 = 2 - 0$.
- If $n+k \geq 0$ then $n+k+1 > 0$ and we obtain $v_{n+k+1}^{(r)} = v_{n+k+1}^{(r+1)} - \alpha v_{n+k}^{(r+1)}$.

Thus, we get from (2.6) that

$$v_{n+r+3}^{(r+1)} - \alpha v_{n+r+2}^{(r+1)} = \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) \left(v_{n+k+1}^{(r+1)} - \alpha v_{n+k}^{(r+1)}\right).$$

We deduce that

$$\begin{aligned}
v_{n+r+3}^{(r+1)} &= \alpha v_{n+r+2}^{(r+1)} + \sum_{k=0}^{r+1} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) v_{n+k+1}^{(r+1)} \\
&\quad + \sum_{k=0}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) v_{n+k}^{(r+1)} \\
&= (\alpha r + 2\alpha) v_{n+r+2}^{(r+1)} + \sum_{k=0}^{r} (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) v_{n+k+1}^{(r+1)} \\
&\quad + (-1)^{r+1} \alpha^{r+1} \beta v_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) v_{n+k}^{(r+1)} \\
&= (\alpha r + 2\alpha) v_{n+r+2}^{(r+1)} + \sum_{l=1}^{r+1} (-1)^{r-l} \alpha^{r+1-l} \left(\alpha^2 C_{r+1}^{l-2} - \beta C_r^{l-1}\right) v_{n+l}^{(r+1)} \\
&\quad + (-1)^{r+1} \alpha^{r+1} \beta v_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right) v_{n+k}^{(r+1)} \\
&= \sum_{k=0}^{r+2} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+1}^{k-2} + \alpha^2 C_{r+1}^{k-1} - \beta C_r^{k-1} - \beta C_r^k\right) v_{n+k}^{(r+1)} \\
&= \sum_{k=0}^{r+2} (-1)^{r-k} \alpha^{r+1-k} \left(\alpha^2 C_{r+2}^{k-1} - \beta C_{r+1}^k\right) v_{n+k}^{(r+1)}.
\end{aligned}$$

We conclude that (2.5) is true for all $r \geq 0$. □

Corollary 2.7 *Let $r \geq 0$ be an integer and $\left(L_n^{(r)}\right)_n$ be the hyper-Lucas numbers defined by* (1.1)*, then*

$$L_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1}\right) L_{n+k}^{(r)}, \qquad n \geq -r.$$

Proof Follows from Identity (2.5) for $\alpha = \beta = 1$. □

Let us spread Lemma 2.1 to the *r*th generation.

Proposition 2.8 *Let $r \geq 1$ be an integer, then*

$$v_n^{(r)} = \frac{1}{\beta^r} v_{n+2r} - \sum_{k=1}^{r} P_{k-1}(n+2r-2k) \frac{\alpha^{n+2r-2k+2}}{\beta^{r-k+1}}, \quad n \geq 0 \tag{2.7}$$

Proof By induction on $r \geq 1$. We deduce from (2.1) that (2.7) is true for $r = 1$. Now assume that (2.7) is true for an integer $r \geq 1$, then from (2.2) we get

$$\begin{aligned} v_n^{(r+1)} &= \frac{1}{\beta} v_{n+2}^{(r)} - P_r(n) \frac{\alpha^{n+2}}{\beta} \\ &= -\sum_{k=1}^{r} P_{k-1}(n+2r-2k+2) \frac{\alpha^{n+2r-2k+4}}{\beta^{r-k+2}} + \\ &\frac{1}{\beta^{r+1}} v_{n+2r+2} - P_r(n) \frac{\alpha^{n+2}}{\beta} \\ &= -\sum_{k=1}^{r+1} P_{k-1}(n+2r-2k+2) \frac{\alpha^{n+2r-2k+4}}{\beta^{r-k+2}} + \\ &\frac{1}{\beta^{r+1}} v_{n+2r+2}. \end{aligned}$$

We conclude that (2.7) is true for all $r \geq 1$. □

Corollary 2.9 *Let $r \geq 1$ be an integer, then we have*

$$L_n^{(r+1)} = L_{n+2}^{(r)} - P_r(n), \quad n \geq 0. \tag{2.8}$$

$$L_n^{(r)} = L_{n+2r} - \sum_{k=1}^{r} P_{k-1}(n+2r-2k), \quad n \geq 0. \tag{2.9}$$

Proof Identities (2.8) and (2.9) follow respectively from (2.2) and (2.7) for $\alpha = \beta = 1$. □

It is clear that for any integer n we have

$$v_n = u_{n+1} + \beta u_{n-1}, \tag{2.10}$$

and since we have [4]

$$u_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_{n-k-1}^{k} \alpha^{n-2k-1} \beta^k, \quad n \geq 1, \tag{2.11}$$

we deduce that

$$v_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P_k(n-2k-2) \alpha^{n-2k} \beta^k, \quad n \geq 0, \tag{2.12}$$

where $P_k(n-2k-2) = C_{n-k}^{k} + C_{n-k-1}^{k-1}$. We are able to generalize (2.12) to any generation r.

Corollary 2.10 *Let $r \geq 0$ be an integer, then*

$$v_n^{(r)} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P_{r+k}(n-2k-2) \alpha^{n-2k} \beta^k, \quad n \geq 0, \tag{2.13}$$

where $P_{r+k}(n-2k-2) = C_{n+r-k}^{r+k} + C_{n+r-k-1}^{r+k-1}$.

Proof If $r = 0$, then (2.13) is simply (2.12). Thus, assume that $r \geq 1$. From identities (2.7) and (2.12) we get

$$\begin{aligned} v_n^{(r)} &= \frac{1}{\beta^r} v_{n+2r} - \sum_{k=1}^{r} P_{k-1}(n+2r-2k) \frac{\alpha^{n+2r-2k+2}}{\beta^{r-k+1}} \\ &= \sum_{k=0}^{\lfloor \frac{n+2r}{2} \rfloor} P_k(n+2r-2k-2) \frac{\alpha^{n+2r-2k}}{\beta^{r-k}} \\ &- \sum_{k=0}^{r-1} P_k(n+2r-2k-2) \frac{\alpha^{n+2r-2k}}{\beta^{r-k}} \\ &= \sum_{k=r}^{\lfloor \frac{n+2r}{2} \rfloor} P_k(n+2r-2k-2) \frac{\alpha^{n+2r-2k}}{\beta^{r-k}} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P_{r+k}(n-2k-2) \alpha^{n-2k} \beta^k. \end{aligned}$$

□

3 Cassini determinant for generalized hyper-Lucas numbers

Cassini identities (1.4) and (1.5) can be expressed as determinants in the following way

$$\begin{vmatrix} L_n & L_{n+1} \\ L_{n+1} & L_{n+2} \end{vmatrix} = 5(-1)^n$$

and

$$\begin{vmatrix} v_n & v_{n+1} \\ v_{n+1} & v_{n+2} \end{vmatrix} = (-\beta)^n \left(\alpha^2 + 4\beta\right).$$

For $n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$, let's define the $(r+2) \times (r+2)$ matrix

$$C_{r,n} = \begin{pmatrix} v_n^{(r)} & v_{n+1}^{(r)} & \cdots & v_{n+r+1}^{(r)} \\ v_{n+1}^{(r)} & v_{n+2}^{(r)} & \cdots & v_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n+r+1}^{(r)} & v_{n+r+2}^{(r)} & \cdots & v_{n+2r+2}^{(r)} \end{pmatrix}.$$

Note that

$$C_{0,n} = \begin{pmatrix} v_n & v_{n+1} \\ v_{n+1} & v_{n+2} \end{pmatrix}.$$

Our aim is to evaluate the determinant of the matrix $C_{r,n}$.

From Proposition 2.6, we can write for $\alpha \neq 0$

$$v_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} q_k v_{n+k}^{(r)}, \qquad n \geq -r,$$

where

$$q_k = (-1)^{r+1-k} \alpha^{r-k} \left(\alpha^2 C_{r+1}^{k-1} - \beta C_r^k\right), \quad 0 \leq k \leq r+1.$$

Let

$$V_{r+2} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ q_0 & q_1 & q_2 & \cdots & q_r & q_{r+1} \end{pmatrix}.$$

Thus, we deduce from Proposition 2.6 that the generalized hyper-Lucas numbers $\left(v_n^{(r)}\right)_n$ can be defined by the vector recurrence relation

$$\begin{pmatrix} v_{n+1}^{(r)} \\ v_{n+2}^{(r)} \\ \vdots \\ v_{n+r+2}^{(r)} \end{pmatrix} = V_{r+2} \begin{pmatrix} v_n^{(r)} \\ v_{n+1}^{(r)} \\ \vdots \\ v_{n+r+1}^{(r)} \end{pmatrix}, \tag{3.1}$$

where $n + r \geq 0$.

Lemma 3.1 *Let $\alpha, \beta, n, r \in \mathbb{Z}$ such that $\alpha \neq 0$, $n \geq 0$ and $r \geq 0$, then*

$$C_{r,n} = V_{r+2}^n C_{r,0}.$$

Proof From relation (3.1) we can write $C_{r,n} = V_{r+2} C_{r,n-1}$. It follows that

$$C_{r,n} = V_{r+2} C_{r,n-1} = V_{r+2}^2 C_{r,n-2} = \cdots = V_{r+2}^n C_{r,0}.$$

□

Lemma 3.2 *Assume that $\alpha \neq 0$ and let $r \geq 0$ be an integer, then*

$$\det(V_{r+2}) = -\alpha^r \beta.$$

Proof It is clear that

$$\det(V_{r+2}) = (-1)^{r+3} q_0 = (-1)^{r+3}(-1)^{r+2}\alpha^r \beta = -\alpha^r \beta.$$

□

Theorem 3.3 *Let $\alpha, \beta, n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$, then*

$$\det\left(C_{r,n}\right) = (-1)^{n+\lfloor (r+1)/2 \rfloor}(\alpha^2 + 4\beta)\alpha^{rn+r^2+r}\beta^{n+r}. \tag{3.2}$$

Proof For $r = 0$ the result follows from identity (1.5). Thus, assume that $r \geq 1$. For $\alpha \neq 0$, we deduce from (3.1) that multiplication by V_{r+2}^{-1} decreases by 1 the subscript of each component, i.e.,

$$V_{r+2}^{-1} C_{r,0} = \begin{pmatrix} v_{-1}^{(r)} & v_0^{(r)} & \cdots & v_r^{(r)} \\ v_0^{(r)} & v_1^{(r)} & \cdots & v_{r+1}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ v_r^{(r)} & v_{r+1}^{(r)} & \cdots & v_{2r+1}^{(r)} \end{pmatrix}.$$

Thus,

$$V_{r+2}^{-r} C_{r,0} = \begin{pmatrix} v_{-r}^{(r)} & v_{1-r}^{(r)} & \cdots & v_1^{(r)} \\ v_{1-r}^{(r)} & v_{2-r}^{(r)} & \cdots & v_2^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{(r)} & v_2^{(r)} & \cdots & v_{r+2}^{(r)} \end{pmatrix}.$$

Since $v_{-n}^{(r)} = 0$ for $1 \leq n \leq r$, then

$$V_{r+2}^{-r} C_{r,0} = \begin{pmatrix} 0 & \cdots & v_0^{(r)} & v_1^{(r)} \\ \vdots & \vdots & \iddots & \vdots \\ v_0^{(r)} & v_1^{(r)} & \cdots & v_{r+1}^{(r)} \\ v_1^{(r)} & v_2^{(r)} & \cdots & v_{r+2}^{(r)} \end{pmatrix}.$$

Thus,

$$\det(C_{r,0}) = \det(V_{r+2})^r \cdot \Delta, \tag{3.3}$$

where

$$\Delta = \begin{vmatrix} 0 & \cdots & v_0^{(r)} & v_1^{(r)} \\ \vdots & \vdots & \iddots & \vdots \\ v_0^{(r)} & v_1^{(r)} & \cdots & v_{r+1}^{(r)} \\ v_1^{(r)} & v_2^{(r)} & \cdots & v_{r+2}^{(r)} \end{vmatrix}.$$

Let L_j denotes the jth line of Δ where $j = 1, 2, \ldots, r+2$. First, we replace L_{i+1} by $L_{i+1} - \alpha L_i$ for $i = r+1, r, \ldots, 1$. Since $v_{i+1}^{(r-1)} = v_{i+1}^{(r)} - \alpha v_i^{(r)}$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & \cdots & v_0^{(r-1)} & v_1^{(r-1)} + 2\alpha \\ 0 & 0 & \cdots & v_1^{(r-1)} & v_2^{(r-1)} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ v_0^{(r-1)} & v_1^{(r-1)} & \cdots & v_r^{(r-1)} & v_{r+1}^{(r-1)} \\ v_1^{(r-1)} & v_2^{(r-1)} & \cdots & v_{r+1}^{(r-1)} & v_{r+2}^{(r-1)} \end{vmatrix}.$$

Using the same method $(r-2)$ times again, we obtain

$$\Delta = \begin{vmatrix} 0 & 0 & \cdots & 0 & v_0 & v_1 + d_1 \\ 0 & 0 & \cdots & v_0 & v_1 & v_2 + d_2 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & v_0 & \cdots & v_{r-2} & v_{r-1} & v_r + d_r \\ v_0 & v_1 & \cdots & v_{r-1} & v_r & v_{r+1} \\ v_1 & v_2 & \cdots & v_r & v_{r+1} & v_{r+2} \end{vmatrix},$$

where $d_i = 2(-1)^{i-1} C_r^i \alpha^i$ for $1 \le i \le r$. Now let C_j denotes the jth column of this last determinant, where $j = 1, \ldots, r+2$. Replacing the column C_i by $C_i - (\alpha C_{i-1} + \beta C_{i-2})$ for $i = r+2, r+1, \ldots, 3$ and using the fact that $v_i = \alpha v_{i-1} + \beta v_{i-2}$ gives

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & v_0 & v_1 - \alpha v_0 + d_1 \\ 0 & 0 & 0 & \cdots & v_0 & v_1 - \alpha v_0 & d_2 \\ 0 & 0 & 0 & \cdots & v_1 - \alpha v_0 & 0 & d_3 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & v_0 & v_1 - \alpha v_0 & \cdots & 0 & 0 & d_r \\ v_0 & v_1 & 0 & \cdots & 0 & 0 & 0 \\ v_1 & v_2 & 0 & \cdots & 0 & 0 & 0 \end{vmatrix}.$$

Now we permute the column C_i with column C_{r+3-i} for $1 \le i \le \lfloor (r+2)/2 \rfloor$, we obtain

$$\Delta = (-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor} \begin{vmatrix} v_1 - \alpha v_0 + d_1 & v_0 & 0 & \cdots & 0 & 0 & 0 \\ d_2 & v_1 - \alpha v_0 & v_0 & \cdots & 0 & 0 & 0 \\ d_3 & 0 & v_1 - \alpha v_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_r & 0 & 0 & \cdots & v_1 - \alpha v_0 & v_0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & v_1 & v_0 \\ 0 & 0 & 0 & \cdots & 0 & v_2 & v_1 \end{vmatrix}.$$

We deduce that

$$\Delta = (-1)^{\left\lfloor \frac{r+2}{2} \right\rfloor} \Delta' \begin{vmatrix} v_1 & v_0 \\ v_2 & v_1 \end{vmatrix}, \tag{3.4}$$

where

$$\Delta' = \begin{vmatrix} d_1 - \alpha & 2 & 0 & \cdots & 0 & 0 \\ d_2 & -\alpha & 2 & \cdots & 0 & 0 \\ d_3 & 0 & -\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{r-1} & 0 & 0 & \cdots & -\alpha & 2 \\ d_r & 0 & 0 & \cdots & 0 & -\alpha \end{vmatrix}.$$

Let L_j denotes the jth line of Δ' where $j = 1, \ldots, r$, replacing the line L_i by $L_i + 2\alpha^{-1}L_{i+1}$ for $i = r-1, \ldots, 1$ gives

$$\Delta' = \begin{vmatrix} \sum_{i=1}^{r} 2^{i-1}\alpha^{1-i}d_i - \alpha & 0 & \cdots & 0 & 0 \\ & -\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{r-1} + 2\alpha^{-1}d_r & 0 & \cdots & -\alpha & 0 \\ d_r & 0 & \cdots & 0 & -\alpha \end{vmatrix}.$$

We deduce that

$$\begin{aligned} \Delta' &= (-\alpha)^{r-1}\left(\sum_{i=1}^{r} 2^{i-1}\alpha^{1-i}d_i - \alpha\right) \\ &= (-\alpha)^{r-1}\left(\sum_{i=1}^{r} 2^{i-1}\alpha^{1-i}\left[2(-1)^{i-1}C_r^i\alpha^i\right] - \alpha\right) \\ &= (-\alpha)^{r-1}\left(\sum_{i=0}^{r} 2^i\alpha(-1)^{i-1}C_r^i\right) \\ &= (-\alpha)^{r}\left(\sum_{i=0}^{r}(-2)^iC_r^i\right) \\ &= \alpha^r. \end{aligned}$$

Since $\begin{vmatrix} v_1 & v_0 \\ v_2 & v_1 \end{vmatrix} = -(\alpha^2 + 4\beta)$, we deduce from identities (3.3), (3.4) and Lemmas 3.1, 3.2 that

$$\det\left(C_{r,n}\right) = (-1)^{n+r+1+\lfloor (r+2)/2 \rfloor}(\alpha^2 + 4\beta)\alpha^{rn+r^2+r}\beta^{n+r}.$$

It is easy to see that $(-1)^{r+1+\lfloor (r+2)/2 \rfloor} = (-1)^{\lfloor (r+1)/2 \rfloor}$, thus

$$\det\left(C_{r,n}\right) = (-1)^{n+\lfloor (r+1)/2 \rfloor}(\alpha^2 + 4\beta)\alpha^{rn+r^2+r}\beta^{n+r}.$$

For $\alpha = 0$, we have

$$\begin{cases} v_{2n} = 2\beta^n \\ v_{2n+1} = 0 \end{cases}, \quad n \geq 0$$

and

$$v_n^{(r)} = v_n, \quad n \geq 0.$$

Thus, for $r \geq 1$, it is clear that $\det\left(C_{r,n}\right) = 0$. □

Corollary 3.4 *Let $n \geq 0$ and $r \geq 0$ be integers, then*

$$\begin{vmatrix} L_n^{(r)} & L_{n+1}^{(r)} & \cdots & L_{n+r+1}^{(r)} \\ L_{n+1}^{(r)} & L_{n+2}^{(r)} & \cdots & L_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+r+1}^{(r)} & L_{n+r+2}^{(r)} & \cdots & L_{n+2r+2}^{(r)} \end{vmatrix} = 5(-1)^{n+d(r+1)/2t}.$$

Proof Follows from Identity (3.2) for $\alpha = \beta = 1$. □

4 Spectral norm of circulant matrices

The spectral norm of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is

$$|| A ||_2 = \sqrt{\rho(A^H A)},$$

where $\rho(A^H A)$ is the spectral radius of $A^H A$ and A^H is conjugate transpose of A. Recall that

$$|| A ||_\infty = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} | a_{i,j} | \right).$$

The Euclidian norm of A is

$$|| A ||_E = \sqrt{\left(\sum_{i,j=1}^{n} | a_{ij} |^2 \right)}.$$

The following inequalities are well known

$$|| A ||_2 \le || A ||_E \le \sqrt{n} || A ||_2 . \tag{4.1}$$

In this section, we compute the spectral norm of the circulant matrix $M = \mathrm{Circ}\left(\alpha^{n-1} x_0^{(r)}, \alpha^{n-2} x_1^{(r)}, \ldots, x_{n-1}^{(r)}\right)$, where $\left(x_n^{(r)}\right)_{r,n \ge 0}$ denotes one of the sequences $\left(u_n^{(r)}\right)_{r,n \ge 0}$ or $\left(v_n^{(r)}\right)_{r,n \ge 0}$. Thereafter, we give upper and lower bounds for the spectral norm of the s-circulant matrix M_s defined by $M_s = \mathrm{Circs}\left(\alpha^{n-1} x_0^{(r)}, \alpha^{n-2} x_1^{(r)}, \ldots, x_{n-1}^{(r)}\right)$.

For our purpose we need to have all the $x_n^{(r)}$ nonnegative, but this is not the case for all $\alpha, \beta \in \mathbb{Z}$. Thus, we have make sufficient conditions on α and β.

It is clear that $x_n^{(r)} \ge 0$ for $\alpha \ge 0$ and $\beta \ge 0$. When $\alpha \ge 0$ and $\beta < 0$ we apply the well known Binet's formulas, which are given by

$$u_n = \frac{a^n - b^n}{a - b} \quad \text{and} \quad v_n = a^n + b^n,$$

where

$$a = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2} \quad \text{and} \quad b = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2}.$$

We remark that if $\alpha^2 + 4\beta > 0$ then $a > b > 0$. It results that $u_n > 0$ and $v_n > 0$, i.e., $x_n > 0$ and so $x_n^{(r)} > 0$. When $\alpha^2 + 4\beta = 0$, then $u_n = n\left(\frac{\alpha}{2}\right)^{n-1}$ and $v_n = 2\left(\frac{\alpha}{2}\right)^n$ which are clearly nonnegative.

We therefore set the conditions $\alpha \ge 0$ and $\alpha^2 + 4\beta \ge 0$ in this section.

Theorem 4.1 *The spectral norm of the matrix M is*

$$|| M ||_2 = x_{n-1}^{(r+1)}.$$

Proof Since the circulant matrix M is normal, its spectral norm $|| M ||_2$ is equal to its spectral radius $\rho(M)$. Let us consider the vector $v = (1, 1, \ldots, 1)^t$, then

$$M \cdot v = \left(\sum_{k=0}^{n-1} \alpha^{n-1-k} x_k^{(r)} \right) \cdot v = x_{n-1}^{(r+1)} \cdot v.$$

Since $v \neq 0$, then $x_{n-1}^{(r+1)}$ is an eigenvalue of M. But, since we have $\alpha^{n-1-k} x_k^{(r)} \geq 0$ for $0 \leq k \leq n-1$, then

$$|| M ||_\infty = x_{n-1}^{(r+1)}.$$

It is well known that $\rho(M) \leq || M ||_\infty$, then we get

$$\rho(M) \leq x_{n-1}^{(r+1)}.$$

Since $x_{n-1}^{(r+1)}$ is an eigenvalue of M then

$$x_{n-1}^{(r+1)} \leq \rho(M).$$

We conclude that $\rho(M) = x_{n-1}^{(r+1)}$, i.e., $|| M ||_2 = x_{n-1}^{(r+1)}$. □

Corollary 4.2 *We have the following spectral norms*

Let $A = Circ\left(F_0^{(r)}, F_1^{(r)}, \ldots, F_{n-1}^{(r)}\right)$, then

$$|| A ||_2 = F_{n-1}^{(r+1)}.$$

Let $B = Circ\left(L_0^{(r)}, L_1^{(r)}, \ldots, L_{n-1}^{(r)}\right)$, then

$$|| B ||_2 = L_{n-1}^{(r+1)}.$$

Proof Follows from Theorem 4.1 for $\alpha = \beta = 1$ in $\left(u_n^{(r)}\right)_{r,n \geq 0}$ and $\left(v_n^{(r)}\right)_{r,n \geq 0}$ respectively. □

Remark 4.3 Results in Corollary 4.2 are the same as those given in Theorems 1 and 2 of [3].

Corollary 4.4 *For the Euclidian norm of the matrix M we have*

$$x_{n-1}^{(r+1)} \leq || M ||_E \leq \sqrt{n} x_{n-1}^{(r+1)}. \tag{4.2}$$

Proof Follows from Theorem 4.1 and relation (4.1). □

Corollary 4.5 *We have*

$$\frac{1}{\sqrt{n}} x_{n-1}^{(r+1)} \leq \sqrt{\sum_{i=0}^{n-1} \left(\alpha^{n-1-i} x_i^{(r)} \right)^2} \leq x_{n-1}^{(r+1)}.$$

Proof Follows from the definition of Euclidian norm and relation (4.2). □

Theorem 4.6 *Let* $M_s = \text{Circs}\left(\alpha^{n-1} x_0^{(r)}, \alpha^{n-2} x_1^{(r)}, \ldots, x_{n-1}^{(r)}\right)$ *be an s-circulant matrix.*

(i) If $| s | \geq 1$, then

$$\frac{1}{\sqrt{n}} x_{n-1}^{(r+1)} \leq || M_s ||_2 \leq | s | \sqrt{n} x_{n-1}^{(r+1)}.$$

(ii) If $| s | < 1$, then

$$\frac{| s |}{\sqrt{n}} x_{n-1}^{(r+1)} \leq || M_s ||_2 \leq \sqrt{n} x_{n-1}^{(r+1)}.$$

Proof Note that relation (4.1) can be written as

$$\frac{1}{\sqrt{n}} || A ||_E \leq || A ||_2 \leq || A ||_E . \tag{4.3}$$

(i) Since $| s | \geq 1$, we have

$$|| M ||_E \leq || M_s ||_E \leq || sM ||_E . \tag{4.4}$$

Replacing A by M_s in (4.3) and combining with (4.2) and (4.4), we get

$$\frac{1}{\sqrt{n}} x_{n-1}^{(r+1)} \leq || M_s ||_2 \leq | s | \sqrt{n} x_{n-1}^{(r+1)}.$$

(ii) Since $| s | < 1$, we have

$$|| sM ||_E \leq || M_s ||_E \leq || M ||_E . \tag{4.5}$$

Replacing A by M_s in (4.3) and combining with (4.2) and (4.5), we get

$$\frac{| s |}{\sqrt{n}} x_{n-1}^{(r+1)} \leq || M_s ||_2 \leq \sqrt{n} x_{n-1}^{(r+1)}.$$

□

Corollary 4.7 *Let* $F_s = \text{Circs}\left(F_0^{(r)}, F_1^{(r)}, \ldots, F_{n-1}^{(r)}\right)$ *be an s-circulant matrix with the hyper-Fibonacci numbers.*

(i) If $| s | \geq 1$, then

$$\frac{1}{\sqrt{n}} F_{n-1}^{(r+1)} \leq || F_s ||_2 \leq | s | \sqrt{n} F_{n-1}^{(r+1)}.$$

(ii) If $| s | < 1$, then

$$\frac{| s |}{\sqrt{n}} F_{n-1}^{(r+1)} \leq || F_s ||_2 \leq \sqrt{n} F_{n-1}^{(r+1)}.$$

Proof Follows from Theorem 4.6 for $\left(x_n^{(r)}\right)_{r,n\geq 0} = \left(u_n^{(r)}\right)_{r,n\geq 0}$ and $\alpha = \beta = 1$. □

Corollary 4.8 *Let* $L_s = \text{Circs}\left(L_0^{(r)}, L_1^{(r)}, \ldots, L_{n-1}^{(r)}\right)$ *be an s-circulant matrix with the hyper-Lucas numbers.*

(i) If $| s | \geq 1$, then

$$\frac{1}{\sqrt{n}} L_{n-1}^{(r+1)} \leq || L_s ||_2 \leq | s | \sqrt{n} L_{n-1}^{(r+1)}.$$

(ii) If $| s | < 1$, then

$$\frac{| s |}{\sqrt{n}} L_{n-1}^{(r+1)} \leq || L_s ||_2 \leq \sqrt{n} L_{n-1}^{(r+1)}.$$

Proof Follows from Theorem 4.6 for $\left(x_n^{(r)}\right)_{r,n\geq 0} = \left(v_n^{(r)}\right)_{r,n\geq 0}$ and $\alpha = \beta = 1$. □

Remark 4.9 By comparing our results with those given in Theorems 3 and 4 of [3], we see that for an arbitrary s we get the same lower bound, for $| s | \geq 1$ we obtain a better upper bound and this by using straightforward methods.

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