Indian J Pure Appl Math (2022) 53:144–152
https://doi.org/10.1007/s13226-021-00028-5

ORIGINAL RESEARCH

Eigenvalues monotonicity of Witten-Laplacian along the mean curvature flow

Shahroud Azami

Received: 25 January 2020 / Accepted: 21 January 2021 / Published online: 14 June 2021
© The Indian National Science Academy 2021

Abstract In this paper, we derive the evolution equation for the first eigenvalue of the Witten-Laplace operator acting on the space of functions along the mean curvature flow on a closed oriented manifold. We show some interesting monotonic quantities under the mean curvature flow.

Keywords Laplace operator · Eigenvalue · Mean curvature flow

Mathematics Subject Classification 58C40 · 53C44

1 Introduction

In this paper, we study the monotonicity of the first eigenvalue of Witten-Laplace operator along the mean curvature flow. Over the last few years, mean curvature flow and other geometric flows have been a topic of active research interest in both mathematics and physics. In addition, many mathematicians consider this problem under various geometric flows. There are many results on the evolution and the monotonicity of eigenvalues of the geometric operators, for instance p-Lapalcian and Witten-Laplacian, on evolving manifolds with or without curvature assumptions.

Notice that the main study of the properties of eigenvalues of the geometric operator, in particular, p-Laplacian, on evolving closed manifolds is still very young. It began when Perelman [12] showed that the functional

$$F = \int_M (R + |\nabla f|^2) e^{-f} \, dv$$

is increasing under the Ricci flow coupled to a backward heat-type equation, where R is the scalar curvature with respect to the metric $g(t)$ and dv is the volume form of the metric $g = g(t)$. The nondecreasing of the functional F implies that the first eigenvalue of the geometric operator $-4\Delta + R$ is nondecreasing along the Ricci flow. Then, Li [10] and Cao [4] extended the geometric operator $-4\Delta + R$ to the operator $-\Delta + cR$. They both proved that the first eigenvalue of the geometric operator $-\Delta + cR$ for $c \geq \frac{1}{4}$ is nondecreasing along the Ricci flow. Zeng et al. [17] studied the monotonicity of the eigenvalues of the operator $-\Delta + cR$ along the Ricci-Bourguignon flow.

Communicated by K Sandeep.

S. Azami (✉)
Department of pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran
E-mail: azami@sci.ikiu.ac.ir; shahrood78@yahoo.com

Morover in [1, 16, 18] the evolution for the first eigenvalue of p-Laplacian along the Ricci-harmonic flow, Ricci flow and m-th mean curvature flow has been investigated, respectively. Mao [11], studied the monotonicity of the first eigenvalue of the Laplace and the p-Laplace operators under a forced mean curvature flow and given some monotonic quantities dependent to the first eigenvalue along the forced mean curvature flow. The author in [2] is shown that the first eigenvalue of the Witten-Laplace operator $-\Delta_\phi$ is monotonic along the Ricci-Bourguignon flow with the assumption that $\phi \in C^2(M)$. In [6] and [15] have been studied the evolution for the first eigenvalue of geometric operator $-\Delta_\phi + \frac{R}{2}$ under the Yamabe flow and Ricci flow, respectively, and constructed some monotonic quantities under this flow. Also, author in [3] has investigated the first eigenvalue of a class of (p, q)-Laplacian system along the mean curvature flow.

The mean curvature flow is a kind of heat flow equation for the immersion and occurs in the description of the interface evolution in certain physical models. This is related to the property that such flow is the gradient flow of the area functional and therefore appears naturally in problems where a surface energy is minimized [8, 9, 14]. Motivated by the above works, in this paper we will study the first eigenvalue of the Witten-Laplacian operator whose metric satisfies the mean curvature flow.

2 Preliminaries

In this section, we recall some basic knowledge about mean curvature flow. Let M be a closed oriented n-dimensional manifold with $n \geq 2$ and $F_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of M into the Euclidean space. The evolution of $M_0 = F_0(M)$ by unnormalized mean curvature flow is a one-parameter family of immersions $F : M \times [0, T) \to \mathbb{R}^{n+1}$ which satisfy the partial differential equation

$$\frac{\partial F}{\partial t}(x,t) = -H(x,t)v(x,t), \quad F(.,0) = F_0, \quad x \in M, \ \ t \geq 0, \tag{2.1}$$

where $H(x, t)$ and $v(x, t)$ are the mean curvature and the normal at the point $F(x, t)$ of the surface $M_t = F(., t)(M)$, respectively, and T is taken to be the maximum time of existence for the flow.

From (2.1), we conclude that $-H(x,t)v(x,t) = \Delta_{M_t} F(x,t)$, where Δ_{M_t} is the Laplace-Beltrami operator on M_t. Thus, the unnormalized mean curvature flow is a kind of heat equation for the immersion, a parabolic problem and it has unique solution for small times. Moreover, we can apply the maximum principle for the mean curvature flow. The volume-preserving mean curvature flow is defined as

$$\frac{\partial F}{\partial t}(x,t) = (r(t) - H(x,t))v(x,t), \quad F(.,0) = F_0, \quad x \in M, \ \ t \geq 0, \tag{2.2}$$

where $r(t) = \frac{\int_M H d\mu_t}{\int_M d\mu_t}$ is a function of t. Short time existence and uniqueness for solution to the mean curvature flow on [0, T) have been shown in [13]. An extended mean curvature flow is defined as

$$\frac{\partial F}{\partial t}(x,t) = -H^k(x,t)v(x,t), \quad F(.,0) = F_0, \quad x \in M, \ \ t \geq 0, \tag{2.3}$$

where $k > 0$ and H^k is the k-th power of the mean curvature. This evolution equation is called unnormalized H^k- flow. We consider an interesting generalization of the mean curvature flow as follows:

$$\frac{\partial F}{\partial t}(x,t) = -\mathcal{S}(x,t)v(x,t), \quad F(.,0) = F_0, \quad x \in M, \ \ t \geq 0, \tag{2.4}$$

where $\mathcal{S}(x, t)$ is a given symmetric function of the principal curvatures of M_t at (x, t). For instance, if S is equal to H, $H - r(t)$ or H^k, then this flow is an unnormalized mean curvature flow, the volume-preserving mean curvature flow, or the H^k-flow, respectively. Ritoré and Sinestrari [13], have shown that if $k_1, ..., k_n$ be the principal curvatures of M and $F_0 : M \to \mathbb{R}^{n+1}$ be a smooth immersion of a closed n-dimensional manifold M such that at any point $x \in M$,

$$\frac{\partial \mathcal{S}}{\partial k_i}(k_1(x), ..., k_n(x)) > 0, \quad i = 1, 2, ...n,$$

then the equation (2.4) has a unique smooth solution on some time interval [0, T). Also, from [13] we have the following lemma:

Lemma 2.1 *If M_t evolves by* (2.4), *with $\mathcal{S}$ being a symmetric function of the curvatures homogeneous of degree $\gamma > 0$, then the geometric quantities associated to M_t satisfy the following equations*:

i: $\frac{\partial}{\partial t} g_{ij} = -2\mathcal{S}h_{ij}, \qquad \frac{\partial}{\partial t} g^{ij} = 2\mathcal{S}h^{ij},$

ii: $\frac{\partial}{\partial t} d\mu = -\mathcal{S}H d\mu,$

iii: $\frac{\partial}{\partial t} h_{ij} = \frac{\partial \mathcal{S}}{\partial h^m_l} \nabla^l \nabla_m h_{ij} + \frac{\partial^2 \mathcal{S}}{\partial h^s_r \partial h^k_l} \nabla_i h^s_r \nabla_j h^k_l + \frac{\partial \mathcal{S}}{\partial h^k_l} h^k_m h^m_l h_{ij} - (\gamma + 1) h_{ik} h^k_j \mathcal{S},$

iv: $\frac{\partial}{\partial t} H = \frac{\partial \mathcal{S}}{\partial h^j_i} \nabla^i \nabla_j H + \frac{\partial^2 \mathcal{S}}{\partial h^s_r \partial h^k_l} \nabla^i h^s_r \nabla_i h^k_l + \frac{\partial \mathcal{S}}{\partial h^k_l} h^k_m h^m_l H - (\gamma - 1)|A|^2 \mathcal{S},$

where $A = (h_{ij})$ with $h_{ij} = <\partial_i F, \partial_j \nu>_e$ is the second fundamental form associated with $F : M \to \mathbb{R}^{n+1}$ and $<.,.>_e$ denotes the Euclidean scalar product in $\mathbb{R}^{n+1}$.

2.1 Eigenvalues of the Witten-Laplace operator

In this section, we first will give the definitions for the first eigenvalue of the Witten-Laplace operator Δ_ϕ and then we will find the formula for the evolution of the first eigenvalue of the Witten-Laplace operator under the evolution equation (2.4) on a closed oriented manifold. Let M be a closed oriented Riemannian manifold, and $(M, g(t))$ be a smooth solution of the evolution equation (2.4) for $t \in [0, T)$. Let ∇ be the Levi-Civita connection on $(M, g(t))$ and $f : M \to \mathbb{R}$ be a smooth function on M or $f \in W^{1,2}(M)$ where $W^{1,2}(M)$ is the Sobolev space. The Laplacian of f is defined as

$$\Delta f = \operatorname{div}(\nabla f) = g^{ij}(\partial_i \partial_j f - \Gamma^k_{ij} \partial_k f). \tag{2.5}$$

Let dv be the Riemannian volume measure, and $d\mu$ be the weight volume measure on $(M, g(t))$ related to the function ϕ, i.e.,

$$d\mu = e^{-\phi(x)} dv \tag{2.6}$$

where $\phi \in C^2(M)$. The Witten-Laplacian is defined by

$$\Delta_\phi = \Delta - \nabla\phi.\nabla \tag{2.7}$$

which is a symmetric operator on $L^2(M, \mu)$ and satisfies the following integration by part formula:

$$\int_M <\nabla u, \nabla v> d\mu = -\int_M v \Delta_\phi u \, d\mu = -\int_M u \Delta_\phi v \, d\mu \qquad \forall u, v \in C^\infty(M).$$

The Witten-Laplacian is a generalization of Laplace operator. For instance, when ϕ is a constant function, then the Witten-Laplacian operator is just the Laplace-Belterami operator.

We say that $\lambda_1(t)$ is an eigenvalue of the Witten-Laplace operator Δ_ϕ at time $t \in [0, T)$ whenever for some $f \in W^{1,2}(M)$, we have

$$-\Delta_\phi f = \lambda_1(t) f, \tag{2.8}$$

or equivalently

$$\int_M <\nabla f, \nabla h> d\mu = \lambda_1 \int_M f \, h \, d\mu, \qquad \forall h \in C^\infty(M). \tag{2.9}$$

If in above formula we set $h = f$ then

$$\lambda_1 = \frac{\int_M |\nabla f|^2 d\mu}{\int_M f^2 \, d\mu}.$$

Therefore, the first eigenvalue of the Witten-Laplace operator is defined as

$$\lambda = \min_{f \neq 0} \left\{ \int_M |\nabla f|^2 d\mu : \; f \in C^\infty(M), \; \int_M f^2 d\mu = 1, \; \int_M f d\mu = 0 \right\}.$$

The author in [2] shown that the following Lemma:

Lemma 2.2 *If g_1 and g_2 are two metrics on the Riemannian manifold M which satisfy*

$$\frac{1}{1+\epsilon}g_1 \leq g_2 \leq (1+\epsilon)g_1,$$

then

$$\lambda(g_2) - \lambda(g_1) \leq \left((1+\epsilon)^{\frac{n}{2}+1} - (1+\epsilon)^{-\frac{n}{2}}\right)(1+\epsilon)^{\frac{n}{2}}\lambda(g_1).$$

In particular, λ is a continuous function with respect to the C^2-topology.

We say f is normalized eigenfunction corresponding to λ whenever $\lambda = \int_M |\nabla f|^2 d\mu$, $\int_M f d\mu = 0$, and $\int_M f^2 d\mu = 1$. The nonzero first eigenvalue of the Witten-Lapalcian and its corresponding eigenfunction are not known to be C^1-differentiable with respect to t along the flow. For this reason, we apply techniques of Cao [5] and Wu [16] to study the evolution and monotonicity of $\lambda(t) = \lambda(f(t), t)$, where $\lambda(f(t), t)$ and $f(t)$ are assumed to be smooth. For this end, we assume that at time $t_0 \in [0, T)$, $f_0 = f(t_0)$ is the eigenfunction for the eigenvalue $\lambda(t_0)$ of the Witten-Laplacian. We consider the following smooth function

$$l(t) = f_0 \left[\frac{\det(g_{ij}(t_0))}{\det(g_{ij}(t))}\right]^{\frac{1}{2}},$$

along the evolution equation (2.4). We assume that

$$f(t) = \frac{l(t)}{\left(\int_M (l(t))^2 d\mu\right)^{\frac{1}{2}}},$$

where $f(t)$ is a smooth function under the evolution equation (2.4) that satisfies $\int_M f^2 d\mu = 1$, $\int_M f d\mu = 0$, and f is the eigenfunction for λ of the Witten-Laplacian at time t_0. Now we define a smooth eigenvalue function by

$$\lambda(f, t) := \int_M |\nabla f|^2 d\mu \tag{2.10}$$

where $\lambda(f(t_0), t_0) = \lambda(t_0)$, f is a smooth function with , $\int_M f d\mu = 0$, and it satisfies

$$\int_M f^2 \, d\mu = 1. \tag{2.11}$$

3 Variation of $\lambda(t)$

In this section, we will give some useful evolution formulas for $\lambda(t)$ under the evolution equation (2.4). Now, we give a useful proposition about the variation of eigenvalues of the Witten-Laplacian under the evolution equation (2.4).

Proposition 3.1 *Let $(M^n, g(t))$ be a solution of the evolution equation (2.4) on the smooth closed manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution of the first eigenvalue under the evolution equation (2.4), then*

$$\frac{d}{dt}\lambda(f, t)|_{t=t_0} = \lambda(t_0)\int_M \mathcal{S}Hf^2 d\mu + 2\int_M \mathcal{S}h^{ij}\nabla_i f \nabla_j f d\mu - \int_M |\nabla f|^2 \mathcal{S}H d\mu. \tag{3.1}$$

Proof According to the above description, $\lambda(f, t)$ is a smooth function and by derivation (2.10) we have

$$\frac{d}{dt}\lambda(f, t) = \int_M \frac{d}{dt}(|\nabla f|^2) d\mu + \int_M |\nabla f|^2 \frac{d}{dt}(d\mu). \tag{3.2}$$

Moreover, we have

$$\frac{d}{dt}(d\mu_t) = -\mathcal{S}H d\mu \tag{3.3}$$

and

$$\begin{aligned}\frac{d}{dt}\left(|\nabla f|^2\right) =& \frac{d}{dt}\left(g^{ij}\nabla_i f\nabla_j f\right) = \frac{\partial}{\partial t}(g^{ij})\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f_t\nabla_j f \\ =& -g^{il}g^{jk}\frac{\partial}{\partial t}(g_{lk})\nabla_i f\nabla_j f + 2<\nabla f_t, \nabla f>.\end{aligned} \tag{3.4}$$

Plugging (3.3), (2.4) and (3.4) into (3.2) yields

$$\frac{d}{dt}\lambda(f,t) = 2\int_M \left\{ \mathcal{S}h^{ij}\nabla_i f\nabla_j f + <\nabla f_t, \nabla f> \right\} d\mu - \int_M |\nabla f|^2 \mathcal{S}H d\mu. \tag{3.5}$$

Now, using (2.11), from the condition

$$\int_M f^2 d\mu = 1,$$

and the time derivative, we get

$$2\int_M f_t f d\mu = \int_M f^2 \mathcal{S}H d\mu, \tag{3.6}$$

equations (2.9) and (3.6) imply that at time $t = t_0$, we have

$$\int_M <\nabla f_t, \nabla f> d\mu = \lambda(t_0)\int_M f_t f d\mu = \frac{\lambda(t_0)}{2}\int_M f^2 \mathcal{S}H d\mu. \tag{3.7}$$

Replacing (3.7) in (3.5), we obtain (3.1). □

If in the pervious proposition we assume that $\mathcal{S} = H$, $\mathcal{S} = H - r$ or $\mathcal{S} = H^k$ then we can obtain variation of $\lambda(t)$ under the volume-preserving, unnormalized mean curvature flow and H^k-flow indicated in the following corollary.

Corollary 3.2 *Let $(M^n, g(t))$ be a solution of the unnormalized mean curvature flow on the smooth closed oriented manifold (M^n, g_0). If $\lambda(t)$ denotes the evolution of a first eigenvalue under the unnormalized mean curvature flow, then*

$$\frac{d\lambda}{dt}(f,t)|_{t=t_0} = \lambda(t_0)\int_M H^2 f^2 d\mu + 2\int_M H h^{ij}\nabla_i f\nabla_j f d\mu - \int_M |\nabla f|^p H^2 d\mu. \tag{3.8}$$

Also, for the volume-preserving mean curvature flow, we have

$$\begin{aligned}\frac{d\lambda}{dt}(f,t)|_{t=t_0} =& \lambda(t_0)\int_M (H-r)Hf^2 d\mu + 2\int_M (H-r)h^{ij}\nabla_i f\nabla_j f d\mu \\ & - \int_M |\nabla f|^2 (H-r)H d\mu.\end{aligned} \tag{3.9}$$

and for unnormalized H^k-flow, we obtain

$$\frac{d\lambda}{dt}(f,t)|_{t=t_0} = \lambda(t_0)\int_M H^{k+1} f^2 d\mu + 2\int_M H^k h^{ij}\nabla_i f\nabla_j f d\mu - \int_M |\nabla f|^p H^{k+1} d\mu. \tag{3.10}$$

In the following theorem, we show that the evolution of the first eigenvalue (2.10) under the unnormalized mean curvature flow with some conditions at the initial time, is nondecreasing.

Theorem 3.3 *Let $(M^n, g(t))$ be a solution of the unnormalized mean curvature flow on the smooth closed manifold (M^n, g_0) and $\lambda(t)$ denotes the evolution of the first eigenvalue under the unnormalized mean curvature flow. At the initial time $t = 0$, if $H > 0$, and there exists a non-negative constant ϵ such that*

$$h_{ij} \geq \epsilon H g_{ij}, \quad \frac{1}{2} \leq \epsilon \tag{3.11}$$

then $\lambda(t)$ is nondecreasing under the unnormalized mean curvature flow.

Proof G. Huisken in [8], by the weak maximum principle for symmetric tensors, has shown that

$$h_{ij} \geq \epsilon H g_{ij} \quad \text{in } M^n \times [0,T),$$

thus $h_{ij} \geq \epsilon H g_{ij}$ is preserved by the unnormalized mean curvature flow. This implies that

$$h^{ij}\nabla_i f \nabla_j f \geq \epsilon H |\nabla f|^2 \quad \text{in } M^n \times [0,T).$$

Replacing this into (3.8) we infer

$$\frac{d\lambda}{dt}(f,t)|_{t=t_0} \geq \lambda(t_0)\int_M H^2 f^2 d\mu + (2\epsilon - 1)\int_M |\nabla f|^2 H^2 d\mu,$$

and we arrive at $\frac{d}{dt}\lambda(f(t),t) > 0$ in any sufficiently small neighborhood of t_0, then

$$\lambda(f(t_1),t_1) < \lambda(f(t_0),t_0) \quad \text{on} \quad [t_1,t_0].$$

Since $\lambda(f(t_0),t_0) = \lambda(t_0)$ and $\lambda(f(t_1),t_1) \geq \lambda(t_1)$ we conclude that $\lambda(t_1) < \lambda(t_0)$ which show that $\lambda(t)$ is strictly monotone increasing in any sufficiently small neighborhood of t_0. Then $\lambda(t)$ is strictly increasing along the evolution equation (2.4) on [0, T) since t_0 is arbitrary. □

Now consider the evolution equation (2.4), with $\mathcal{S} = (H^2 - \tilde{r}(t))$ where $\tilde{r}(t) = \frac{\int_M H^2 d\mu}{\int_M d\mu}$, i.e.,

$$\frac{\partial F}{\partial t}(x,t) = (\tilde{r}(t) - H^2(x,t))v(x,t), \quad F(.,0) = F_0, \quad x \in M, \ \ t \geq 0. \tag{3.12}$$

In the next theorem we study some monotonic quantities dependent to the first eigenvalue (2.10) under the equation (3.12).

Theorem 3.4 *Let $(M^n, g(t))$ be a solution of the evolution equation* (3.12) *on the smooth closed oriented manifold (M^n, g_0) and $\lambda(t)$ denotes the evolution of the first eigenvalue* (2.10) *under the evolution equation* (3.12). *If at the initial time $t = 0$, $H > 0$ and there exist non-negative constant ϵ such that $h_{ij} \geq \epsilon H g_{ij}$ then there exist $\varphi(t)$ and $\psi(t)$ such that the quantities*

$$e^{-\int_0^t (\psi(\tau) - \varphi(\tau) + 2\epsilon\psi(\tau))d\tau}\lambda(t)$$

is nondecreasing along the evolution equation (3.12). *Similarly*,

$$e^{-\int_0^t (\varphi(\tau) - \psi(\tau) + 2\varphi(\tau))d\tau}\lambda(t)$$

is nonincreasing along the evolution equation (3.12).

Proof Rivas and Sinestrari in [14] have shown that the inequality $\epsilon H g_{ij} \leq h_{ij} \leq H g_{ij}$ preserves along the evolution equation (3.12), therefore

$$\epsilon H |\nabla f|^2 \leq h^{ij}\nabla_i f \nabla_j f \leq H |\nabla f|^2 \quad \text{in } M^n \times [0,T).$$

Moreover, since $H > 0$ in $t = 0$ by the maximum principle they show that the following inequalities also preserve.

$$0 \leq C_1 e^{-C_2 t} \leq H \leq C_3, \tag{3.13}$$

for some constants C_1, C_2 and C_3. Therefore,

$$C_1^2 e^{-2C_2 t} - \tilde{r}(t) \leq \mathcal{S} = (H^2 - \tilde{r}(t)) \leq C_3^2 - \tilde{r}(t)$$

and

$$\psi(t) = (C_1^2 e^{-2C_2 t} - \tilde{r}(t))C_1 e^{-C_2 t} \leq \mathcal{S}H = (H^2 - \tilde{r}(t))H \leq (C_3^2 - \tilde{r}(t))C_3 = \varphi(t).$$

Now (3.1) implies that

$$\begin{aligned}\frac{d\lambda}{dt}(f,t)|_{t=t_0} &\geq \lambda(t_0)\int_M \mathcal{S}Hf^2 d\mu + 2\epsilon\int_M \mathcal{S}H|\nabla f|^2 d\mu - \int_M |\nabla f|^2 \mathcal{S}H d\mu \\ &\geq \lambda(t_0)\psi(t_0) + 2\epsilon\psi(t_0)\int_M |\nabla f|^2 d\mu - \varphi(t)\int_M |\nabla f|^2 d\mu \\ &\geq \lambda(t_0)(\psi(t_0) - \varphi(t_0) + 2\epsilon\psi(t_0)),\end{aligned}$$

this results that in any enough small neighborhood of t_0 as I_0, we have

$$\frac{d}{dt}\lambda(f,t) \geq \lambda(t)(\psi(t) - \varphi(t) + 2\epsilon\psi(t)).$$

Integrating the last inequality with respect to variable time t on $[t_1, t_0] \subset I_0$, we get

$$\ln\frac{\lambda(f(t_0),t_0)}{\lambda(f(t_1),t_1)} > \int_{t_1}^{t_0}(\psi(\tau) - \varphi(\tau) + 2\epsilon\psi(\tau))d\tau.$$

Since $\lambda(f(t_0),t_0) = \lambda(t_0)$ and $\lambda(f(t_1),t_1) \geq \lambda(t_1)$ we conclude that

$$\ln\frac{\lambda(t_0)}{\lambda(t_1)} > \int_{t_1}^{t_0}(\psi(\tau) - \varphi(\tau) + 2\epsilon\psi(\tau))d\tau,$$

that is the quantity $e^{-\int_0^t(\psi(\tau)-\varphi(\tau)+2\epsilon\psi(\tau))d\tau}\lambda(t)$ is strictly increasing in any sufficiently small neighborhood of t_0. Since t_0 is arbitrary, then

$$e^{-\int_0^t(\psi(\tau)-\varphi(\tau)+2\epsilon\psi(\tau))d\tau}\lambda(t)$$

is nondecreasing along the mean curvature flow (3.12) on [0, T). Similarly,

$$e^{-\int_0^t(\varphi(\tau)-\psi(\tau)+2\varphi(\tau))d\tau}\lambda(t)$$

is nonincreasing along the mean curvature flow (3.12). □

In the following, we need to use Hamilton's maximum principle for tensors on manifolds (see [7]).

Theorem 3.5 ([7]). *Suppose that on* [0, T) *the evolution equation*

$$\frac{\partial}{\partial t}M_{ij} = \Delta M_{ij} + u^k\nabla_k M_{ij} + N_{ij}$$

holds, where $N_{ij} = P(M_{ij}, g_{ij})$, *a polynomial in* M_{ij} *formed by contracting products of* M_{ij} *with itself using the metric, satisfies* $N_{ij}X^iX^j \geq 0$ *for any null eigenvector* $X = \{X^i\}$ *of* M_{ij}. *If* $M_{ij} \geq 0$ *at* $t = 0$ *then it remains so on* [0, T).

Lemma 3.6 *Let* M_t^n *be a solution of unnormalized* H^k*-flow on the smooth closed oriented manifold* M_0^n. *If there exist positive constant* $a_1, a_2, ..., a_n$ *such that the initial hypersurface* M_0^n *satisfies*

$$h_{ij} = a_i H g_{ij}, \quad \text{where } \sum_{i=1}^n a_i = 1, \text{ and } \quad 0 \leq \frac{1}{n} - a_i \leq \epsilon \tag{3.14}$$

for small enough ϵ *only depending on* n, *then* (3.14) *remains along the unnormalized* H^k *-flow for any* $t \in [0, T)$.

Proof By (3.14), we have

$$a_i H g_{ij} \leq h_{ij} \leq a_i H g_{ij}, \quad \text{on } M_0.$$

Moreover, by Lemma (2.1) for $\mathcal{S} = H^k$ we get

$$\frac{\partial H}{\partial t} = kH^{k-1}\Delta H + k(k-1)H^{k-2}|\nabla H|^2 + |A|^2 H^k \tag{3.15}$$

and

$$\frac{\partial h_{ij}}{\partial t} = -(k+1)H^k h_{il}h^l_j + kH^{k-1}\Delta h_{ij} + k(k-1)H^{k-2}\nabla_i H\nabla_j H + kH^{k-1}|A|^2 h_{ij}. \tag{3.16}$$

From the classical maximum principle, we conclude that $H^k \geq 0$ for all $k \in \mathbb{N}$ and $t \in [0, T)$. If $M_{ij} = h_{ij} - \frac{1}{2}a_i H g_{ij}$ then

$$\begin{aligned}\frac{\partial M_{ij}}{\partial t} =& kH^{k-1}\Delta M_{ij} + k(k-1)H^{k-2}\nabla_i H\nabla_j H - \frac{1}{2}a_i k(k-1)H^{k-1}|\nabla H|^2 g_{ij}\\ &- (k+1)H^k h_{il}h^l_j - \frac{1}{2}a_i|A|^2 H^k g_{ij} + a_i H^{k+1}h_{ij} + kH^{k-1}|A|^2 h_{ij}.\end{aligned} \tag{3.17}$$

Suppose that

$$N_{ij} = -(k+1)H^k h_{il}h^l_j - \frac{1}{2}a_i|A|^2 H^k g_{ij} + a_i H^{k+1}h_{ij} + kH^{k-1}|A|^2 h_{ij}. \tag{3.18}$$

By the assumption of the lemma $M_{ij} \geq 0$ at $t = 0$. Now we show that N_{ij} is nonnegative on the null-eigenvectors of M_{ij}. For some vector $X = \{X^i\}$, assume we have $h_{ij}X^j = \frac{1}{2}a_i H X_i$. Now we can write

$$\begin{aligned}N_{ij}X^iX^j =& -\frac{1}{4}(k+1)H^{k+2}a_i a_j g^{ij}X_iX_j - \frac{1}{2}a_i|A|^2 H^k g_{ij}X^iX^j + \frac{1}{2}a_i^2 H^{k+2}X^iX_i\\ &+ \frac{1}{2}ka_i H^k|A|^2 X^iX_i\\ =& \frac{1}{4}H^{k+2}(-(k+1)a_i a_j g^{ij}X_iX_j + 2a_i^2 X^iX_i)\\ &+ \frac{1}{2}H^k|A|^2(-a_i g_{ij}X^iX^j + ka_i X^iX_i)\\ =& \frac{1}{4}(k-1)H^k a_i(2|A|^2 - a_i H^2)X_iX^i.\end{aligned}$$

Since $|A|^2 \geq \frac{H^2}{n}$, we have $N_{ij}X^iX^j \geq 0$. Note

$$\nabla H = \nabla(h_{ij}g^{ij}) = \nabla[(M_{ij} + \frac{1}{2}a_i H g_{ij})g^{ij}] = g^{ij}\nabla M_{ij} + \frac{1}{2}\nabla H \tag{3.19}$$

which implies $\nabla H = 2g^{ij}\nabla M_{ij}$. Now, Hamilton's maximum principle to tensor implies that $M_{ij} \geq 0$ on M_t for any $t \in [0, T)$, that is $a_i H g_{ij} \leq h_{ij}$. Similarly, on M_t we have $a_i H g_{ij} \geq h_{ij}$ for any $t \in [0, T)$. So we get $h_{ij} = a_i H g_{ij}$ on $M \times [0, T)$. □

Theorem 3.7 *Let $(M^n, g(t))$ be a solution of the unnormalized H^k-flow on the smooth closed oriented manifold (M^n, g_0) and $\lambda(t)$ denotes the evolution of a first eigenvalue under the unnormalized mean curvature flow. If initial hypersurface M_0 satisfies* (3.14) *then under the unnormalized m H^k-flow $\lambda(t)$ is nondecreasing.*

Proof From Lemma 3.6 we have

$$h_{ij} = a_i H g_{ij}, \quad \text{on } M_t. \tag{3.20}$$

So, from (3.8) we conclude that

$$\begin{aligned}\frac{d\lambda}{dt}(f,t)|_{t=t_0} =& \lambda(t_0)\int_M H^2 f^2 d\mu + 2\int_M H^2 a_i g^{ij}\nabla_i f\nabla_j f d\mu - \int_M |\nabla f|^2 H^2 d\mu\\ \geq& \lambda(t_0)\int_M H^2 f^2 d\mu + 2(\frac{1}{n} - \epsilon - \frac{1}{2})\int_M H^2|\nabla f|^2 d\mu\end{aligned}$$

so for $\epsilon \leq \frac{1}{n} - \frac{1}{2}$, we have $\frac{d}{dt}\lambda(f(t), t) > 0$ in any sufficiently small neighborhood of t_0, then

$$\lambda(f(t_1), t_1) < \lambda(f(t_0), t_0) \quad \text{on} \quad [t_1, t_0].$$

Since $\lambda(f(t_0), t_0) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$ we conclude that $\lambda(t_1) < \lambda(t_0)$, hence $\lambda(t)$ is nondecreasing in any sufficiently small neighborhood of t_0. Then $\lambda(t)$ is nondecreasing along H^k-flow on [0, T) because of t_0 is arbitrary. □

Example 3.8 Let $M_0 = \mathbb{S}^n_R(0)$ be the sphere of radius R around the origin. The mean curvature $\mathbb{S}^n_R(0)$ is equal to $\frac{n}{R}$. If we set $F(x,t) = r(t)F_0(x)$, where $F_0(x)$ is the standard immersion of M_0 in $\mathbb{R}^{n+1}$, then we have

$$R'(t)F_0(x) = \frac{\partial}{\partial t}F(x,t) = -H(x,t)v(x,t) = -\frac{n}{r(t)}F_0(x),$$

therefore the evolution of M_0 is given by sphere $M_t = \mathbb{S}^n_{r(t)}(0)$, where the radius $r(t)$ evolves according to the ordinary differential equation

$$r'(t) = -\frac{n}{r(t)}, \quad r(0) = R.$$

The solution is given by $r(t) = \sqrt{R^2 - 2nt}$, thus

$$H(x,t) = \frac{n}{\sqrt{R^2 - 2nt}}, \quad h^{ij}(x,t) = \frac{1}{\sqrt{R^2 - 2nt}}g^{ij}(t) = \frac{H(x,t)}{n}g^{ij}(t).$$

These equations together with (3.8) imply that

$$\frac{d\lambda}{dt}(f,t)|_{t=t_0} = 2\frac{H^2(x,t_0)}{n}\int_M |\nabla f|^2 d\mu = 2\frac{H^2(x,t_0)}{n}\lambda(t_0). \tag{3.21}$$

we arrive at $\frac{d}{dt}\lambda(f(t),t) > 0$ in any sufficiently small neighborhood of t_0, therefore

$$\lambda(f(t_1),t_1) < \lambda(f(t_0),t_0) \quad \text{on} \quad [t_1,t_0].$$

Since $\lambda(f(t_0),t_0) = \lambda(t_0)$ and $\lambda(f(t_1),t_1) \geq \lambda(t_1)$ we get $\lambda(t_1) < \lambda(t_0)$ which show that $\lambda(t)$ is strictly monotone increasing in any sufficiently small neighborhood of t_0. Then $\lambda(t)$ is strictly increasing along the mean curvature flow.

References

1. A. Abolarinwa, *Evolution and monotonicity of the first eigenvalue of p-Laploacian under the Ricci-harmonic flow*, J. Appl. Anal. **21** (2015), 147-160.
2. S. Azami, *Monotonicity of eigenvalues of Witten-Laplace operator along the Ricci-Bourguignon flow*, AIMS Mathematics, **2**(2)(2017), 230-243.
3. S. Azami, *Eigenvalues of a (p,q)-Laplacian system under the mean curvature flow*, Iran J. Sci. Technol. Trans. Sci., **43** (2019), 1881-1888.
4. X. D. Cao, *First eigenvalues of geometric operators under the Ricci flow*, Proc. Amer. Math. Soc. **136** (2008), 4075-4078.
5. X. Cao, *Eigenvalues of $(\Delta + \frac{R}{2})$on manifolds with nonnegative curvature operator*, Math. Ann., **337**(2) (2007),435-442.
6. S. Fang and F. Yang, *First eigenvalues of geometric operators under the Yamabe flow*, Bull. Korean Math. Soc. **53** (2016), 1113-1122.
7. R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry, **17** (1982), 255-306.
8. G. Huisken, *Flow by mean curvature of convex surfaces into sphere*, J. Differ. Geom. 20, 237-266, 1984.
9. G. Huisken and A. Polden, *Geometric evolution equations for hypersurfaces, calculuse of variations and geometric evolution problems*, Springer-Verlag, Berlin, 1999, 45-84.
10. J. F. Li, *Eigenvalues and energy functionals with monotonicity formula under Ricci flow*, Math. Ann.**338** (2007), 927-946.
11. J. Mao, *Monotonicity of the first eigenvalue of the Laplace and the p-Laplace operators under a forced mean curvature flow*, Journal of the Korean Mathematical Society, **55**(6)(2013), 1435-1458.
12. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, (2002), ArXiv: 0211159.
13. M. Ritoré and C. Sinestrari, *Mean curvature flow and isoperimetric inequalities*, Birkhäuser, 2010.
14. E. C. Rivas, C. Sinestrai, *Volume-preserving flow by powers of the mth mean curvature*, Calc. Var. **38** (2010) 441-469.
15. F. S. Wen, X. H. Feng, Z. Peng, *Evolution and monotonicity of eigenvalues under the Ricci flow*, Sci. China Math. **58** (2015), no.8, 1737-1744.
16. J. Y. Wu, *First eigenvalue monotonicity for the p-Laplace operator under the Ricci flow*, Acta mathematica Sinica, English senes, Vol. 27, **8**(2011), 1591-1598.
17. F. Zeng, Q. He, B. Chen, *Monotonicity of eigenvalues of geometric operators along the Ricci-Bourguignon flow*, Pacific journal of mathematics, **296**(1)(2018), 1-20.
18. L. Zhao, *The first eigenvalue of the p-laplacin operator under powers of the mth mean curvature flow*, Results in Math.**63**(2012) 937-948.