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ORIGINAL RESEARCH

Completion of skew completable unimodular rows

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Abstract In this paper, we prove that if R is a local ring of dimension $d \geq 3$, d odd and $\frac{1}{(d-1)!} \in R$ then any skew completable unimodular row $v \in Um_d(R[X])$ is completable. It is also proved that skew completable unimodular rows of size $d \geq 3$ over a regular local ring of dimension d are first row of a 2- stably elementary matrix.

Keywords Local ring · Regular local ring · Witt group

Mathematics Subject Classification 13C10 · 13H99 · 19B10 · 19B14

1 Introduction

In 1955, J.P. Serre asked whether there were non-free projective modules over a polynomial extension $k[X_1, \ldots, X_n]$, over a field k. D. Quillen ([6]) and A.A. Suslin ([9]) settled this problem independently in early 1976; and is now known as the Quillen–Suslin theorem. Since every finitely generated projective module over $k[X_1, \ldots, X_n]$ is stably free, to determine whether projective modules are free, it is enough to determine that unimodular rows over $k[X_1, \ldots, X_n]$ are completable. Therefore, problem of completion of unimodular rows is a central problem in classical K-Theory.

In [11], R.G. Swan and J. Towber showed that if $(a^2, b, c) \in Um_3(R)$ then it can be completed to an invertible matrix over R. This result of Swan and Towber was generalised by Suslin in [10] who showed that if $(a_0^{r!}, a_1, \ldots, a_r) \in Um_{r+1}(R)$ then it can be completed to an invertible matrix. In [7], Ravi Rao studied the problem of completion of unimodular rows over $R[X]$, where R is a local ring. Ravi Rao showed that if R is a local ring of dimension $d \geq 2, \frac{1}{d!} \in R$, then any unimodular row over $R[X]$ of length $d+1$ can be mapped to a factorial row by elementary transformations. In [8], Ravi Rao proved that if R is a local ring of dimension 3 with $2R = R$, then unimodular rows of length 3 are completable. In [2], Ravi Rao generalised his result with Anuradha Garge and proved that if R is a local ring of dimension 3 with $2R = R$ then any unimodular row of length 3 can be mapped to a factorial row via a two stably elementary matrix.

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Throughout this article we will assume R to be a commutative noetherian ring with $1 \neq 0$.

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In this article, we generalise the result of Garge–Rao for skew completable unimodular rows. We prove :

Theorem 1.1. *Let R be a local ring of Krull dimension $d \geq 3$ with d odd and $\frac{1}{(d-1)!} \in R$. Let $v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X])$ be skew-completable unimodular row over $R[X]$. Then there exists $\rho \in SL_d(R[X]) \cap E_{d+2}(R[X])$ and an invertible alternating matrix $W \in SL_{d+1}(R[X])$ such that*

$$v\rho = e_1 K(W).$$

In the last section, we study the completion of skew completable unimodular rows over regular local rings. Since $SK_1(R[X])$ is trivial for a regular local ring R, we get the following result :

Theorem 1.2. *Let R be a regular local ring of Krull dimension $d \geq 3$ with d odd and $\frac{1}{(d-1)!} \in R$. Let $v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X])$ be skew-completable unimodular row over $R[X]$. Then there exists $\rho \in SL_d(R[X]) \cap E_{d+2}(R[X])$ such that $v = e_1\rho$.*

2 Preliminary Remarks

A row $v = (a_0, a_1, \ldots, a_r) \in R^{r+1}$ is said to be unimodular if there is a $w = (b_0, b_1, \ldots, b_r) \in R^{r+1}$ with $\langle v, w\rangle = \Sigma_{i=0}^{r} a_i b_i = 1$ and $Um_{r+1}(R)$ will denote the set of unimodular rows (over R) of length $r+1$.

The group of elementary matrices is a subgroup of $GL_{r+1}(R)$, denoted by $E_{r+1}(R)$, and is generated by the matrices of the form $E_{ij}(\lambda) = I_{r+1} + \lambda e_{ij}$, where $\lambda \in R, i \neq j, 1 \leq i, j \leq r+1, e_{ij} \in M_{r+1}(R)$ whose ij^{th} entry is 1 and all other entries are zero. The elementary linear group $E_{r+1}(R)$ acts on the rows of length $r+1$ by right multiplication. Moreover, this action takes unimodular rows to unimodular rows : $\frac{Um_{r+1}(R)}{E_{r+1}(R)}$ will denote set of orbits of this action; and we shall denote by $[v]$ the equivalence class of a row v under this equivalence relation.

2.1 The elementary symplectic Witt group $W_E(R)$

If $\alpha \in M_r(R), \beta \in M_s(R)$ are matrices then $\alpha \perp \beta$ denotes the matrix $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in M_{r+s}(R)$. ψ_1 will denote $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in E_2(\mathbb{Z})$, and ψ_r is inductively defined by $\psi_r = \psi_{r-1} \perp \psi_1 \in E_{2r}(\mathbb{Z})$, for $r \geq 2$.

A skew-symmetric matrix whose diagonal elements are zero is called an alternating matrix. If $\phi \in M_{2r}(R)$ is alternating then $\det(\phi) = (\text{ pf } (\phi))^2$ where pf is a polynomial (called the Pfaffian) in the matrix elements with coefficients ± 1. Note that we need to fix a sign in the choice of pf ; so we insist pf $(\psi_r) = 1$ for all r. For any $\alpha \in M_{2r}(R)$ and any alternating matrix $\phi \in M_{2r}(R)$ we have pf $(\alpha^t \phi \alpha) =$ pf (ϕ) det (α). For alternating matrices ϕ, ψ it is easy to check that pf $(\phi \perp \psi) = ($ pf $(\phi))($ pf $(\psi))$.

Two matrices $\alpha \in M_{2r}(R), \beta \in M_{2s}(R)$ are said to be equivalent (w.r.t. $E(R)$) if there exists a matrix $\varepsilon \in SL_{2(r+s+l)}(R) \bigcap E(R)$, such that $\alpha \perp \psi_{s+l} = \varepsilon^t(\beta \perp \psi_{r+l})\varepsilon$, for some l. Denote this by $\alpha \overset{E}{\sim} \beta$. Thus $\overset{E}{\sim}$ is an equivalence relation; denote by $[\alpha]$ the orbit of α under this relation.

It is easy to see ([12, p. 945]) that $\perp$ induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with pfaffian 1; this group is called elementary symplectic Witt group and is denoted by $W_E(R)$.

2.2 W. Van der Kallen's group structure on $Um_{d+1}(R)/E_{d+1}(R)$

Definition 2.1 Essential dimension: Let R be a ring whose maximal spectrum Max (R) is a finite union of subsets V_i, where each V_i, when endowed with the (topology induced from the) Zariski topology is a space of Krull dimension d. We shall say R is essentially of dimension d in such a case.

For instance, a ring of Krull dimension d is obviously essentially of dimension $\leq d$; a local ring of dimension d is essentially of dimension 0; whereas a polynomial extension $R[X]$ of a local ring R of dimension $d \geq 1$ has dimension $d+1$ but is essentially of dimension d as Max $(R[X]) =$ Max $(R/(a)[X]) \cup$ Max $(R_a[X])$ for any non-zero divisor $a \in R$.

In ([4, Theorem 3.6]), W. van der Kallen derives an abelian group structure on $\frac{Um_{d+1}(R)}{E_{d+1}(R)}$ when R is essentially of dimension d, for all $d \geq 2$. Let $*$ denote the group multiplication henceforth. He also proved in ([4, Theorem 3.16(iv)]), that the first row map is a group homomorphism

$$SL_{d+1}(R) \longrightarrow \frac{Um_{d+1}(R)}{E_{d+1}(R)}$$

when R is essentially of dimension d, for all $d \geq 2$.

Lemma 2.2. *Let R be essentially of dimension $d \geq 2$, and let $C_{d+1}(R)$ denote the set of all completable $(d+1)$ -rows in $Um_{d+1}(R)$. Then,*

- *The map $\sigma \longrightarrow [e_1\sigma]$, where $e_1 = (1, 0, \ldots, 0) \in Um_{d+1}(R)$, is a group homomorphism $SL_{d+1}(R) \longrightarrow \frac{Um_{d+1}(R)}{E_{d+1}(R)}$.*
- *$\frac{C_{d+1}(R)}{E_{d+1}(R)}$ is a subgroup of $\frac{Um_{d+1}(R)}{E_{d+1}(R)}$.*

Proof. First follows from ([4, Theorem 3.16(iv)]). Since $v \in C_{d+1}(R)$ can be completed to a matrix of determinant one, $\frac{C_{d+1}(R)}{E_{d+1}(R)}$ is the image of $SL_{d+1}(R)$ under the above mentioned homomorphism; whence is a subgroup of $\frac{Um_{d+1}(R)}{E_{d+1}(R)}$. □

Proposition 2.3. *Let R be a local ring of dimension d, $d \geq 3$ and $\frac{1}{(d-1)!} \in R$. Let $v = (v_0, \ldots, v_d) \in Um_{d+1}(R[X])$. Then v is completable if and only if $v^{(d-1)} = (v_0^{(d-1)}, v_1, \ldots, v_d)$ is completable.*

Proof. In view of ([7, Remark 1.4.3]), we may assume that R is a reduced ring. By ([7, Lemma 1.3.1, Example 1.5.3]),

$$[v^{(d-1)}] = [v] * [v] * \cdots * [v], (d-1) \text{ times}$$

in $\frac{Um_{d+1}(R[X])}{E_{d+1}(R[X])}$. By Lemma 2.2, v is completable implies $v^{(d-1)}$ is also completable.

Conversely, let $v^{(d-1)}$ be completable. By ([7, Proposition 1.4.4]),

$$v \overset{E}{\sim} (w_0, w_1, \ldots, w_{d-1}, c)$$

with $c \in R$ a non-zero-divisor. Since $\dim (R/(c)) = d-1$ and $\frac{1}{(d-1)!} \in R$, by ([7, Corollary 2.3]),

$$(\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{d-1}) \in e_1 SL_d(R/(c)[X]).$$

By ([3, Proposition 1.2, Chapter 5]), $(w_0, w_1, \ldots, w_{d-1}, c^d)$ is completable. Thus,

- $(v_0, v_1, \ldots, v_{d-1}, v_d^d) \overset{E}{\sim} (w_0, w_1, \ldots, w_{d-1}, c^d)$ by ([14, Theorem 1]),
- $[v]^n = [(v_0, v_1, \ldots, v_{d-1}, v_d^n)]$ for all n by ([7, Lemma 1.3.1]).

Thus $[v]^d = [(w_0, w_1, \ldots, w_{d-1}, c^d)] \in \frac{C_{d+1}(R[X])}{E_{d+1}(R[X])}$ and by hypothesis $[v]^{d-1} = [v^{(d-1)}] \in \frac{C_{d+1}(R[X])}{E_{d+1}(R[X])}$. Therefore by Lemma 2.2, v is completable. □

3 Krusemeyer's completion of the square of a skew completable row

Definition 3.1. A row $v \in Um_{2r-1}(R)$ is said to be skew completable if there is an invertible alternating matrix $V \in GL_{2r}(R)$ with $e_1 V = (0, v)$.

First we note an example of skew completable unimodular row which is not completable.

Example 3.2. (Kaplansky) Let $A = \frac{\mathbb{R}[x_0, x_1, x_2]}{(x_0^2 + x_1^2 + x_2^2 - 1)}$ and $v = (\overline{x_0}, \overline{x_1}, \overline{x_2}) \in Um_3(A)$. In view of ([12, Section 5]), every unimodular row of length 3 is skew completable. Thus $v = (\overline{x_0}, \overline{x_1}, \overline{x_2})$ is skew completable. Next we will show that v is not completable.

Suppose to the contrary that $v = e_1\sigma$ for some $\sigma \in SL_3(A)$. Let $\sigma = (\sigma_{ij})$. We can think σ_{ij}'s as a function on S^2. Let us define tangent vector field

$$\phi : S^2 \longrightarrow \mathbb{R}^3$$
$$w \longmapsto ((\sigma_{21}^{-1})^t(w), \sigma_{22}^{-1})^t(w), \sigma_{23}^{-1})^t(w)).$$

As σ_{ij}'s are polynomials, ϕ is a differential function. Since $(\sigma^{-1})^t \in SL_3(A)$, ϕ is a nonvanishing continuous tangent vector field on S^2 which is a contradiction to Hairy ball theorem. Thus v is not completable.

Theorem 3.3. (M. Krusemeyer) ([5, Theorem 2.1]) *Let R be a commutative ring and $v = (v_1, \ldots, v_n)$ be skew completable. Let V be a skew completion of v , then $(v_1^2, v_2, \ldots, v_n)$ is completable.*

Notation 3.4. *In the above theorem we will denote $K(V) \in SL_n(R)$ to be a completion of $(v_1^2, v_2, \ldots, v_n)$ for a skew completable unimodular row $v = (v_1, \ldots, v_n)$ and its skew completion V.*

Remark 3.5. M. Krusemeyer's proof in ([5, Theorem 2.1]), shows that $V \in (1 \perp K(V))E_{n+1}(R)$.

Lemma 3.6. *Let R be a commutative ring and $v = (v_1, \ldots, v_n) \in Um_n(R)$ be skew completable to V. Then $[e_1K(V)] = [e_1K(V)^t]$.*

Proof. By Remark 3.5, $V \in (1 \perp K(V))E_{n+1}(R)$. Since $-I_{2k} \in E_{2k}(R)$, we have $V \in V^tE_{n+1}(R)$. Therefore $(1 \perp K(V))^t \in (1 \perp K(V))E_{n+1}(R)$. Since stably $K(V)$ and $K(V)^t$ are in same elementary class, therefore in view of ([13, Lemma 10]), we have $[e_1K(V)] = [e_1K(V)^t]$. □

Lemma 3.7. *Let R be a local ring with $1/2 \in R$ and let V be an invertible alternating matrix of Pfaffian 1. Let $e_1V = (0, v_1, \ldots, v_{2r-1})$. Then $[V^{2^n}] = [W]$, with $e_1W = (0, v_1^{2^n}, \ldots, v_{2r-1})$.*

Proof. We will prove it by induction on n. For $n = 1$, by ([2, Corollary 4.3]), $W_E(R[X]) \hookrightarrow SK_1(R[X])$ is injective, we have

$$V \perp V \overset{SK_1}{\equiv} V^2 \overset{SK_1}{\equiv} V^t\psi_r V \overset{SK_1}{\equiv} (1 \perp K(V)^t)\psi_r(1 \perp K(V)).$$

Therefore $[V^2] = [U]$ with $e_1U = (0, v_1^2, \ldots, v_{2r-1})$. Now assume that result is true for all $k \leq n-1$ and Let $[W_1] = [V^{2^{n-1}}]$ with $e_1W_1 = (0, v_1^{2^{n-1}}, \ldots, v_{2r-1})$. Since by lemma ([2, Corollary 4.3]), $W_E(R[X]) \hookrightarrow SK_1(R[X])$ is injective, we have

$$W_1 \perp W_1 \overset{SK_1}{\equiv} W_1^2 \overset{SK_1}{\equiv} W_1^t\psi_r W_1 \overset{SK_1}{\equiv} (1 \perp K(W_1)^t)\psi_r(1 \perp K(W_1)).$$

Therefore $[V^{2^n}] = [W]$ with $e_1W = (0, v_1^{2^n}, \ldots, v_{2r-1})$. □

4 Completion of skew-completable unimodular rows of length d

In this section, we prove that if R is a local ring of dimension $d \geq 3$, d odd and $\frac{1}{(d-1)!} \in R$ then any skew completable unimodular row $v \in Um_d(R[X])$ is completable.

Proposition 4.1. *Let R be a local ring of dimension $d \geq 3$ with d odd and $\frac{1}{(d-1)!} \in R$. Let $V \in SL_{d+1}(R[X])$ be an alternating matrix with Pfaffian 1. Then $[V] = [(1 \perp K(W)^t)\psi_{\frac{d+1}{2}}(1 \perp K(W))]$ for some $[W] \in W_E(R[X])$. Consequently, there is a 1-stably elementary matrix $\gamma \in SL_{d+1}(R[X])$ such that*

$$V = \gamma^t(1 \perp K(W)^t)\psi_{\frac{d+1}{2}}(1 \perp K(W))\gamma.$$

Proof. By ([8, Proposition 2.4.1]), $[V] = [W_1]^2$ for some $W_1 \in W_E(R[X])$. By ([9, Theorem 2.6]), $Um_r(R[X]) = e_1E_r(R[X])$ for $r \geq d+2$, so on applying ([12, Lemma 5.3 and Lemma 5.5]), a few times, if necessary, we can find an alternating matrix $W \in SL_{d+1}(R[X])$ such that $[W_1] = [W]$. Therefore $[V] = [W]^2$. Now, we have

$$W \perp W \overset{SK_1}{\equiv} W^2 \overset{SK_1}{\equiv} W^t\psi_{\frac{d+1}{2}}W \overset{SK_1}{\equiv} (1 \perp K(W)^t)\psi_{\frac{d+1}{2}}(1 \perp K(W)).$$

Since in view of ([2, Corollary 4.3]), $W_E(R[X]) \hookrightarrow SK_1(R[X])$ is injective. Thus $[W]^2 = [(1 \perp K(W)^t)\psi_{\frac{d+1}{2}}(1 \perp K(W))]$. Therefore, $[V] = [(1 \perp K(W)^t)\psi_{\frac{d+1}{2}}(1 \perp K(W))]$. The last statement follows by applying ([12, Lemma 5.5 and Lemma 5.6]). □

Theorem 4.2. *Let R be a local ring of Krull dimension $d \geq 3$ with d odd and $\frac{1}{(d-1)!} \in R$. Let $v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X])$ be skew-completable unimodular row over $R[X]$. Then there exists $\rho \in SL_d(R[X]) \cap E_{d+2}(R[X])$ and an invertible alternating matrix $W \in SL_{d+1}(R[X])$ such that*

$$v\rho = e_1 K(W).$$

Proof. Let $V \in SL_{d+1}(R[X])$ be an invertible alternating matrix of Pfaffian 1 which is a skew completion of $v = (v_0, v_1, \ldots, v_{d-1})$. By Proposition 4.1, there exists an alternating matrix $W \in SL_{d+1}(R[X])$ of Pfaffian 1 such that

$$[V] = [(1 \perp K(W))^t \psi_{\frac{d+1}{2}} (1 \perp K(W))].$$

Therefore there exists $\gamma \in SL_{d+1}(R[X]) \cap E_{d+2}(R[X])$ such that

$$\gamma^t V \gamma = (1 \perp K(W))^t \psi_{\frac{d+1}{2}} (1 \perp K(W)).$$

In view of ([2, Corollary 5.17]), $e_1\gamma$ can be completed to an elementary matrix. Thus there exists $\varepsilon \in E_{d+1}(R[X])$, and $\rho_1 \in SL_d(R[X]) \cap E_{d+2}(R[X])$ such that

$$\varepsilon^t (1 \perp \rho_1)^t V (1 \perp \rho_1)\varepsilon = (1 \perp K(W))^t \psi_{\frac{d+1}{2}} (1 \perp K(W)).$$

By ([1, Corollary 4.5]), there exists $\varepsilon_1 \in E_d(R[X])$ such that

$$(1 \perp \varepsilon_1)^t (1 \perp \rho_1)^t V (1 \perp \rho_1)(1 \perp \varepsilon_1) = (1 \perp K(W))^t \psi_{\frac{d+1}{2}} (1 \perp K(W)).$$

Now we set $\rho = \rho_1\varepsilon_1$. Thus $v\rho = e_1 K(W)$. Hence v is completable. □

5 Completion of unimodular 3-vectors

([8, Theorem 3.1]), Ravi A. Rao proved that for a local ring of dimension 3, every $v \in Um_3(R[X])$ is completable. We get stronger results than Theorem 4.2, when we work with a local ring R of dimension 3. We reprove Anuradha Garge and Ravi Rao's result in ([2, Corollary 5.18]).

Proposition 5.1. *Let R be a local ring of dimension 3 with $\frac{1}{2k} \in R$ and let $V \in SL_4(R[X])$ be an alternating matrix of Pfaffian 1. Then $[V] = [V^*]$ in $W_E(R[X])$ with $e_1 V^* = (0, a^{2k}, b, c)$, and $V^* \in SL_4(R[X])$. Consequently, there is a stably elementary $\gamma \in SL_4(R[X])$ such that $V = \gamma^t V^* \gamma$.*

Proof. By ([8, Proposition 2.4.1]), $[V] = [W_1]^{2k}$ for some $W_1 \in W_E(R[X])$. By ([9, Theorem 2.6]), $Um_r(R[X]) = e_1 E_r(R[X])$ for $r \geq 5$, so on applying ([12, Lemma 5.3 and Lemma 5.5]), a few times, if necessary, we can find an alternating matrix $V^* \in SL_4(R[X])$ such that $[W_1] = [V_1^*]$. Therefore $[V] = [V_1^*]^{2k}$. Let $[V_1^*]^{2k} = [V^*]$, thus $[V] = [V^*]$. By ([2, Lemma 4.8]), $e_1 V^* = (0, a^{2k}, b, c)$. The last statement follows by applying ([12, Lemma 5.3 and Lemma 5.5]). □

Theorem 5.2. *Let R be a local ring of Krull dimension 3 with $\frac{1}{2k} \in R$. Let $v = (v_0, v_1, v_2) \in Um_3(R[X])$. Then there exists $\rho \in SL_3(R[X]) \cap E_5(R[X])$ such that*

$$v\rho = (a^{2k}, b, c) \text{ for some } (a, b, c) \in Um_3(R[X]).$$

Proof. Choose $w = (w_0, w_1, w_2)$ such that $\Sigma_{i=0}^2 v_i w_i = 1$, and consider the alternating matrix V with Pfaffian 1 given by

$$V = \begin{bmatrix} 0 & v_0 & v_1 & v_2 \\ -v_0 & 0 & w_2 & -w_1 \\ -v_1 & -w_2 & 0 & w_0 \\ -v_2 & w_1 & -w_0 & 0 \end{bmatrix} \in SL_4(R[X]).$$

By Proposition 5.1, there exists an alternating matrix $V^* \in SL_4(R[X])$, with $e_1 V^* = (0, a^{2k}, b, c)$, of Pfaffian 1 such that

$$[V] = [V^*].$$

Therefore there exists $\gamma \in SL_4(R[X]) \cap E_5(R[X])$ such that

$$\gamma^t V \gamma = V^*.$$

In view of ([2, Corollary 5.17]), $e_1\gamma$ can be completed to an elementary matrix. Thus there exists $\varepsilon \in E_4(R[X])$, and $\rho_1 \in SL_3(R[X]) \cap E_5(R[X])$ such that

$$\varepsilon^t(1 \perp \rho_1)^t V(1 \perp \rho_1)\varepsilon = V^*.$$

By ([1, Corollary 4.5]), there exists $\varepsilon_1 \in E_3(R[X])$ such that

$$(1 \perp \varepsilon_1)^t(1 \perp \rho_1)^t V(1 \perp \rho_1)(1 \perp \varepsilon_1) = V^*.$$

Now we set $\rho = \rho_1\varepsilon_1$. Thus $v\rho = (a^{2k}, b, c)$ for some $(a, b, c) \in Um_3(R[X])$. □

6 Completion over regular local rings

In this section, we prove that skew completable unimodular rows of size $d \geq 3$ over a regular local ring of dimension d are first row of a 2- stably elementary matrix.

We note a result of Rao and Garge in ([2, Corollary 4.3]).

Lemma 6.1. *Let R be a local ring with $2R = R$. Then the natural map*

$$W_E(R[X]) \hookrightarrow SK_1(R[X])$$

is injective.

Corollary 6.2. *Let R be a regular local ring with $2R = R$. Then the Witt group $W_E(R[X])$ is trivial.*

Proof. Since R is a regular local ring, $SK_1(R[X]) = 0$. Thus the result follows in view of Lemma 6.1. □

Lemma 6.3. *Let R be a regular local ring of Krull dimension $d \geq 3$ with d odd and $\frac{1}{(d-1)!} \in R$. Let $v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X])$ be skew-completable unimodular row over $R[X]$. Then there exists $\rho \in SL_d(R[X]) \cap E_{d+2}(R[X])$ such that $v = e_1\rho$.*

Proof. Let V be a skew completion of v. In view of Corollary 6.2, $W_E(R[X]) = 0$, we have $[V] = [\psi_{\frac{d+1}{2}}]$. Thus upon applying ([12, Lemma 5.5 and Lemma 5.6]), there exists $\gamma \in SL_{d+1}(R[X]) \cap E_{d+2}(R[X])$ such that $\gamma^t V \gamma = \psi_{\frac{d+1}{2}}$.

By ([2, Corollary 5.17]), $e_1\gamma$ can be completed to an elementary matrix. Thus there exists $\varepsilon \in E_{d+2}(R[X])$, and $\rho_1 \in SL_d(R[X]) \cap E_{d+2}(R[X])$ such that

$$\varepsilon^t(1 \perp \rho_1)^t V(1 \perp \rho_1)\varepsilon = \psi_{\frac{d+1}{2}}.$$

By ([1, Corollary 4.5]), there exists $\varepsilon_1 \in E_d(R[X])$ such that

$$(1 \perp \varepsilon_1)^t(1 \perp \rho_1)^t V(1 \perp \rho_1)(1 \perp \varepsilon_1) = \psi_{\frac{d+1}{2}}.$$

Now we set $\rho = (\rho_1\varepsilon_1)^{-1}$. Thus we have $v = e_1\rho$. □

Corollary 6.4. *Let R be a regular local ring of Krull dimension 3 with $\frac{1}{2} \in R$. Let $v = (v_0, v_1, v_2) \in Um_3(R[X])$. Then there exists $\rho \in SL_3(R[X]) \cap E_5(R[X])$ such that $v = e_1\rho$.*

Proof. Since every $v \in Um_3(R[X])$ is skew completable, thus the result follows in view of Lemma 6.3. □

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