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ORIGINAL RESEARCH

The stability of the Riemann solutions for the non-symmetry Keyfitz–Kranzer system with Chaplygin pressure

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Abstract Under the local small perturbation of the Riemann initial data, we study the stability of Riemann solutions for Chaplygin gas equation as a special case of non-symmetric Keyfitz–Kranzer systems. We construct the global solutions of the perturbed Riemann problem. As the perturbed parameter ε tends to zero, we show that there is no mass concentration even the initial perturbed density depends on the parameter ε.

Keywords Asymmetric Keyfitz–Kranzer system · Stability · Riemann problem · Delta shock wave

Mathematics Subject Classification 35L65 · 35L67 · 76L05 · 76N10

1 Introduction

Many fields in systems of conservation laws require the study of delta-shock waves. The original definition of delta-shock wave was first introduced into [12]. In nonlinear hyperbolic systems, the solution to these systems contain delta-shock waves. From the point view of physical science, a delta-shock wave represents the process of concentration of mass. For related results of delta-shock waves, we refer to [6, 7, 11–13, 15, 18, 24, 25] and the references therein.

In this paper, we use the local perturbation of the Riemann initial data to study the stability of solutions for Chaplygin gas equations as special cases of non-symmetric Keyfitz-Kranzer systems. The non-symmetric Keyfitz-Kranzer system was first introduced by Keyfitz and Kranzer [10] as a model for the elastic string, it can be written as

$$\begin{cases} \rho_t + (\rho\phi(\rho, u_1, u_2, \cdots, u_n))_x = 0, \\ (\rho u_i)_t + (\rho u_i \phi(\rho, u_1, u_2, \cdots, u_n))_x = 0, i = 1, 2, \cdots, n, \end{cases} \tag{1.1}$$

where ρ and u denote the density and the velocity, respectively, and

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$$\phi(\rho,u)=\phi(u)-P(\rho). \tag{1.2}$$

Using the equation (1. 2) and simplifying by $\phi(u)=u$, then (1. 1) can be reduced to

$$\begin{cases} \rho_t+(\rho(u-P))_x=0, \\ (\rho u)_t+(\rho u(u-P))_x=0. \end{cases} \tag{1.3}$$

For Chaplygin gas $P=-\frac{1}{\rho}$, there are many research results; see, for example, [1, 3, 4, 9, 17, 23]. As one of important discussion of the equation (1. 3), the Riemann problem associated with $P=-\frac{1}{\rho}$ has been developed (see [5]). In particular, Cheng [5] obtained the delta shock waves for a linearly degenerate hyperbolic system of conservation law with initial data

$$(u,\rho)(x,0)=\begin{cases} (u_-,\rho_-), & x<0, \\ (u_+,\rho_+), & x>0, \end{cases} \tag{1.4}$$

where $(u_\pm,\rho_\pm)$ denote the constants pairs. In the case of the restriction of the initial data $(u_0,m_0\delta(x))$ for $x=0$, Li and Shao [14] found (1. 4) corresponding to the following form

$$(u,\rho)(x,0)=\begin{cases} (u_-,\rho_-), & x<0, \\ (u_0,m_0\delta(x)), & x=0, \\ (u_+,\rho_+), & x>0. \end{cases} \tag{1.5}$$

Under the local small perturbation of the Riemann initial data (1. 4), the purpose of this paper is to investigate the stability of the Riemann solutions for system (1. 3) associated with $P=-\frac{1}{\rho}$ by the following initial data

$$(u,\rho)(x,0)=\begin{cases} (u_-,\rho_-), & x<-\varepsilon, \\ (u_m,\rho_m), & -\varepsilon<x<\varepsilon, \\ (u_+,\rho_+), & x>\varepsilon, \end{cases} \tag{1.6}$$

where $\varepsilon>0$ is called the perturbed parameter. Motivated by [2, 6, 16, 19, 21, 22], we prove that there is no mass concentration as ε vanishing, which means the solution is stable.

This paper is organised as follows. In Section 2, we give the basic background of the Riemann solutions for (1. 3) associated with $P=-\frac{1}{\rho}$. In Section 3, we research the perturbed Riemann problems (1. 3) and (1. 6) associated with $P=-\frac{1}{\rho}$ with initial data ρ_m,u_m. In section 4, we treat the perturbed Riemann problem of the case that ρ_m is a function of ε satisfying $\lim\limits_{\varepsilon\to0}\rho_m=\infty$ and $\lim\limits_{\varepsilon\to0}\rho_m\varepsilon=0$.

2 Riemann problem for (1. 3) and (1. 4) with $P=-\frac{1}{\rho}$

Before turning to our main subject, we insert a brief account of the Riemann solutions for (1. 3) and (1. 4) with $P=-\frac{1}{\rho}$. The detailed results of Riemann solutions for systems (1. 3) and (1. 4) with $P=-\frac{1}{\rho}$ can be found in [5].

The two eigenvalues of systems (1. 3) and (1. 4) are

$$\lambda_1=u,\quad \lambda_2=u+\frac{1}{\rho}. \tag{2.1}$$

The corresponding eigenvectors are

$$\overrightarrow{r_1}=(1,0)^T,\quad \overrightarrow{r_2}=(1,\frac{1}{\rho^2})^T. \tag{2.2}$$

From (2. 1) and (2. 2), it follows that the system (1. 3) is strictly hyperbolic and $\nabla\lambda_i\cdot r_i=0$ for $i\in\{1,2\}$. Therefore, λ_1 and λ_2 are linearly degenerate and the elementary waves are contact discontinuities.

The first family contact discontinuity is

$$J_1 : \sigma = u = u_-, \tag{2.3}$$

and the second family contact discontinuity is

$$J_2 : \sigma = u + \frac{1}{\rho} = u_- + \frac{1}{\rho_-}, \tag{2.4}$$

where σ is a velocity of contact discontinuity.

In the phase plane, the upper half plane $\rho > 0$ is divided by (2. 3), (2. 4) and the curve $S_\delta : u + \frac{1}{\rho} = u_-$ into three regions I, II, III, see Fig. 1.

If $(u_+, \rho_+) \in \mathrm{I} \cup \mathrm{II}$, the solution of the initial value problems for (1. 3) and (1. 4) can be expressed briefly as

$$(u_-, \rho_-) + J_1 + (u_*, \rho_*) + J_2 + (u_+, \rho_+),$$

where "+" means followed by, $(u_*, \frac{1}{\rho_*}) = (u_-, u_+ + \frac{1}{\rho_+} - u_-)$, see Fig. 2(a).

If $(u_+, \rho_+) \in \mathrm{III}$, the Riemann solution of systems (1. 3) and (1. 4) with $P = -\frac{1}{\rho}$ can be written as

$$(u, \rho) = \begin{cases} (u_-, \rho_-), & -\infty < x < x(t), \\ (u_\delta, w(t)\delta(x - x(t))), & x = x(t), \\ (u_+, \rho_+), & x(t) < x < +\infty, \end{cases}$$

where $x(t)$, $w(t)$ and $u_\delta(t)$ denote the location, weight and the velocity of delta shock wave, respectively. In particular, $x(t)$, $w(t)$ and $u_\delta(t)$ satisfy the generalized Rankine-Hugoniot conditions

$$\begin{cases} \dfrac{dx(t)}{dt} = u_\delta(t), \\ \dfrac{dw(t)}{dt} = u_\delta(t)[\rho] - [\rho(u + \dfrac{1}{\rho})], \\ \dfrac{d(w(t)u_\delta(t))}{dt} = u_\delta(t)[\rho u] - [\rho u(u + \dfrac{1}{\rho})], \end{cases} \tag{2.5}$$

with initial data $(x, w, u_\delta)(0) = (0, 0, 0)$, see Fig. 2(b).

By (2. 5) and the initial data $(x, w, u_\delta)(0) = (0, 0, 0)$, we have

$$w(t) = \begin{cases} \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}t, & \rho_- \neq \rho_+, \\ (\rho_- u_- - \rho_+ u_+)t, & \rho_- = \rho_+, \end{cases} \tag{2.6}$$

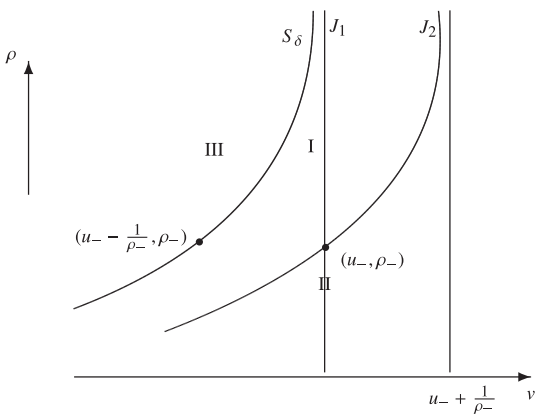

Fig. 1 The phase plane of Riemann solution for (1.3) and (1.4) with Chaplygin pressure

$$u_\delta(t) = \begin{cases} \frac{[\rho u]+\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]}, & \rho_- \neq \rho_+, \\ \frac{u_-+u_++\frac{1}{\rho_-}}{2}, & \rho_- = \rho_+, \end{cases} \tag{2.7}$$

$$x(t) = \begin{cases} \frac{[\rho u]+\sqrt{[\rho u]^2-[\rho][\rho u^2+u]}}{[\rho]}t, & \rho_- \neq \rho_+, \\ \frac{u_-+u_++\frac{1}{\rho_-}}{2}t, & \rho_- = \rho_+. \end{cases} \tag{2.8}$$

3 Solution of (1. 3) and (1. 6), and its limitation as $\varepsilon \to 0$

In this section, we construct the global solutions to (1. 3) and (1. 6) with ρ_m, u_m being constants independent of ε, and prove that the solutions of (1. 3) and (1. 4) are stable.

According to the relations of $u_-, u_m, u_m + \frac{1}{\rho_m}$ and $u_+ + \frac{1}{\rho_+}$, there are several cases as follows:

(1) $u_m + \frac{1}{\rho_m} \le u_- < u_+ + \frac{1}{\rho_+}$;
(2) $u_- < u_+ + \frac{1}{\rho_+} \le u_m$;
(3) $u_- < \min\{u_+ + \frac{1}{\rho_+}, u_m + \frac{1}{\rho_m}\}$ and $u_m < u_+ + \frac{1}{\rho_+}$;
(4) $u_m < u_+ + \frac{1}{\rho_+} \le u_-$ and $u_m + \frac{1}{\rho_m} < u_-$;
(5) $u_+ + \frac{1}{\rho_+} \le u_m < u_m + \frac{1}{\rho_m} \le u_-$;
(6) $u_+ + \frac{1}{\rho_+} \le u_- < u_m + \frac{1}{\rho_m}$ and $u_+ + \frac{1}{\rho_+} \le u_m$;
(7) $u_m < u_+ + \frac{1}{\rho_+} < u_- < u_m + \frac{1}{\rho_m}$.

Case 3. 1. $u_m + \frac{1}{\rho_m} \le u_- < u_+ + \frac{1}{\rho_+}$ *(if* $u_- < u_+ + \frac{1}{\rho_+} \le u_m$*, then structure of solution is similar).*

Since $u_m + \frac{1}{\rho_m} \le u_-$ and $u_m < u_m + \frac{1}{\rho_m} < u_+ + \frac{1}{\rho_+}$, when t is small enough, the solution of (1. 3) and (1. 6) can be expressed briefly as follows

$$(u_-, \rho_-) + \delta S + (u_m, \rho_m) + J_1 + (u_1, \rho_1) + J_2 + (u_+, \rho_+),$$

where $(u_1, \frac{1}{\rho_1}) = (u_m, u_+ + \frac{1}{\rho_+} - u_m)$, see Fig. 3. The weight and propagation speeds of the δS are given by (2. 6) and (2. 7) in which the subscript "+" replaced by "m". The speeds of J_1 and J_2 are $\lambda_1^m = u_m, \lambda_2^+ = u_+ + \frac{1}{\rho_+}$, respectively. By the entropy condition $u_m + \frac{1}{\rho_m} \le u_\delta \le u_-$, we have $u_\delta > \lambda_1^m$, which means that the delta shocks to wave δS will overtake the contacts discontinuity J_1 in finite time. The intersection point is

$$(x^*, t^*) = \left(\frac{(u_\delta + u_m)\varepsilon}{u_\delta - u_m}, \frac{2\varepsilon}{u_\delta - u_m}\right). \tag{3.1}$$

At the time $t = t^*$, the weight and the speed of delta wave are

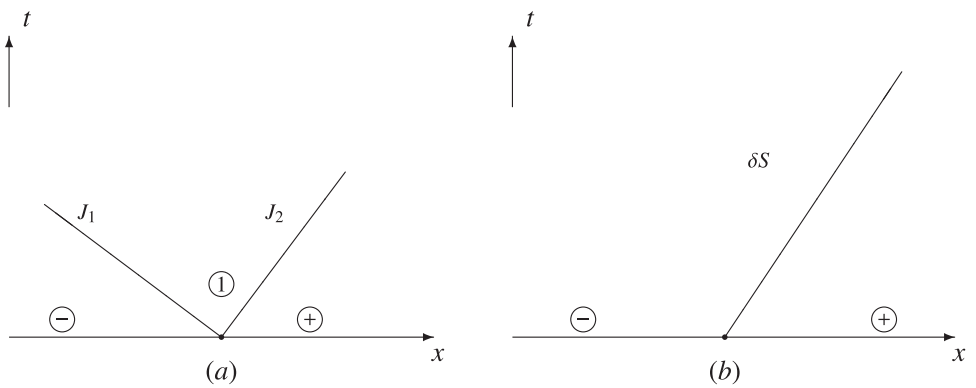

Fig. 2 The physical plane of Riemann solution for (1.3) and (1.4) with Chaplygin pressure

$$w(t^*) = \begin{cases} (\rho_- u_- - \rho_m u_m)t^*, & \rho_- = \rho_m, \\ \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}t^*, & \rho_- \neq \rho_m, \end{cases}$$
$$u_\delta(t^*) = \begin{cases} \dfrac{u_- + u_m + \frac{1}{\rho_-}}{2}, & \rho_- = \rho_m, \\ \dfrac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}, & \rho_- \neq \rho_m, \end{cases} \tag{3.2}$$

respectively. Then there is a new Riemann problem with delta initial data

$$(\rho, u)(x, t^*) = \begin{cases} (\rho_-, u_-), & x < x_1, \\ (w^*\delta(x - x^*), u^*_\delta), & x = x_1, \\ (\rho_1, u_1), & x > x_1, \end{cases} \tag{3.3}$$

where $u^*_\delta = u_\delta(t^*), w^* = w(t^*)$ and $(u_1, \frac{1}{\rho_1}) = (u_m, u_+ + \frac{1}{\rho_+} - u_m)$.

For $t > t^*$, similar to the proof of [14, case3. 2], the solutions of (1. 3) and (3. 3) are

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_-), & x < x_1(t), \\ (w_1(t)\delta(x - x(t)), u_{\delta_1}(t)), & x = x_1(t), \\ (\overline{\rho}, \overline{u}), & x_1(t) < x < (u_+ + \dfrac{1}{\rho_+})(t - t^*) + x^*, \\ (\rho_1, u_1), & x > (u_+ + \dfrac{1}{\rho_+})(t - t^*) + x^*, \end{cases}$$

where $(\overline{\rho}, \overline{u})(x, t) = (\rho_*, u_*)(\overline{t})$ along $(u_1 + \frac{1}{\rho_1})t + x = (u_1 + \frac{1}{\rho_1})\overline{t} - x(\overline{t})$ for $t \geq t^*$, and $(\rho_*, u_*)(t) = (\frac{1}{u_1 + \frac{1}{\rho_1} - u_{\delta_1}}, u_{\delta_1})$ is the right state of the delta shock wave δS_1.

For $t \geq t^*$, the weight $w_1(t)$ of δS_1 satisfy

$$\frac{dw_1(t)}{dt} = \rho_- u_- - \rho_- u_{\delta_1}, \tag{3.4}$$

$$w_1(u_- + \frac{1}{\rho_-} - u_{\delta_1}) = w^*(u_- + \frac{1}{\rho_-} - u^*_\delta) > 0. \tag{3.5}$$

Clearly, if $w^* = 0$, by (3. 5), we have $w_1(t) = 0$. Moreover, by using (3. 5), we also know, if $w^* \neq 0$, $w_1(t)$ satisfies

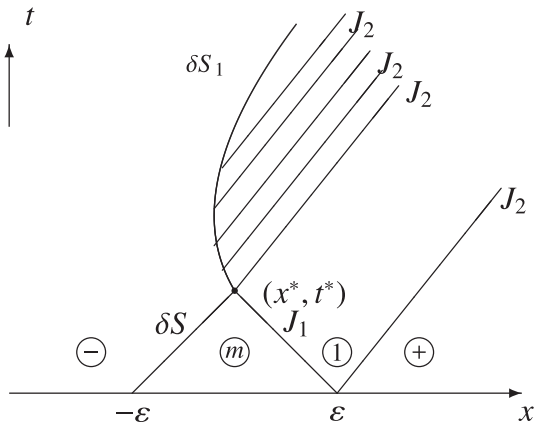

Fig. 3 The structure of solution in the physical plane for Case 3.1

$$w_1 + A_0 \ln(A_0 - w_1) - w^* - A_0 \ln(A_0 - w^*) = t^* - t, \tag{3.6}$$

where $A_0 = \rho_- w^*(u_- + \frac{1}{\rho_-} - u_\delta^*)$. Thus, the weight $w_1(t)$ of δS_1 is given by (3. 6). The propagation speed and the location of δS_1 is

$$\begin{aligned} u_{\delta_1}(t) &= u_- + \frac{1}{\rho_-} - \frac{A_0}{\rho_- w_1(t)}, \\ x(t) &= \int_{t^*}^{t} u_{\delta_1}(\tau) d\tau + x^*, \end{aligned} \tag{3.7}$$

respectively. Notice that w^* and A_0 depend on ε (see (3. 1) and (3. 6)).

By using (3. 7), we have

$$\lim_{t\to\infty} u_{\delta_1} = u_- < u_+ + \frac{1}{\rho_+},$$

which implies that δS_1 never overtakes J_2, see Fig. 3.

Now, we assume that $\Omega_m = \{(x,t) | u_\delta t - \varepsilon \le x \le u_m t + \varepsilon, 0 \le t \le t^*\}$. Then we have Ω_m shrinks to the origin point with no mass concentration due to

$$\lim_{\varepsilon\to 0}(x^*, t^*) = (0,0), \tag{3.8}$$

and ρ_m is constant. From (3. 2) and (3. 8), it follows that $\lim w^* = 0$, which, together with (3. 6), implies that $\lim_{\varepsilon\to 0} A_0(\varepsilon) = 0$. Inserting $w^* = 0$ into (3. 5) yields that $w_1^{\varepsilon\to 0}(t) = 0$ since $w_1(t) \ge 0$. It follows from (3. 4) that $u_{\delta_1}(t) = u_-$. Here $(u_*, \rho_*)(t) = (u_-, \frac{1}{u_+ + \frac{1}{\rho_+} - u_-})$ is constant state.

Thus, the limit solution to the systems (1. 3) and (1. 6) are consistent with the corresponding Riemann solutions of (1. 3) and (1. 4). So, the Riemann solution for (1. 3) and (1. 4) in this case is stable.

Case 3. 2. $u_+ + \frac{1}{\rho_+} \le u_m < u_m + \frac{1}{\rho_m} < u_-$.

In this case, when t is small enough, the solution of system (1. 3) and (1. 6) can be expressed as

$$(u_-, \rho_-) + \delta S_1 + (u_m, \rho_m) + \delta S_2 + (u_+, \rho_+).$$

The speeds and the weights of these two delta waves are

$$u_{\delta_1} = \begin{cases} \dfrac{u_- + u_m + \frac{1}{\rho_-}}{2}, & \rho_- = \rho_m, \\ \dfrac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]} t, & \rho_- \ne \rho_m, \end{cases}$$

$$u_{\delta_2} = \begin{cases} \dfrac{u_m + u_+ + \frac{1}{\rho_m}}{2}, & \rho_+ = \rho_m, \\ \dfrac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}, & \rho_+ \ne \rho_m, \end{cases}$$

$$w_1(t) = \begin{cases} \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}\, t, & \rho_- \ne \rho_m, \\ (\rho_- u_- - \rho_m u_m) t, & \rho_- = \rho_m, \end{cases}$$

$$w_2(t) = \begin{cases} \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}\, t, & \rho_+ \ne \rho_m, \\ (\rho_m u_m - \rho_+ u_+) t, & \rho_+ = \rho_m, \end{cases}$$

respectively.

From the generalized entropy conditions, it follows that

$$u_+ + \frac{1}{\rho_+} \le u_{\delta_2} \le u_m < u_m + \frac{1}{\rho_m} \le u_{\delta_1} \le u_-, \tag{3.9}$$

which implies that δS_1 meets δS_2 in finite time. And hence

$$(x^*, t^*) = (\frac{(u_{\delta_1} + u_{\delta_2})\varepsilon}{u_{\delta_1} - u_{\delta_2}}, \frac{2\varepsilon}{u_{\delta_1} - u_{\delta_2}}). \tag{3.10}$$

The weight of the delta shock wave of the point of (3. 10) is the sum of weights for the δS_1 and δS_2. The conservation law of momentum yields

$$w_1(t^*)u_{\delta_1}(t^*) + w_2(t^*)u_{\delta_2}(t^*) = (w_1(t^*) + w_2(t^*))u_{\delta_3}(t^*),$$

which implies that

$$u_{\delta_3}(t^*) = \frac{w_1(t^*)u_{\delta_1}(t^*) + w_2(t^*)u_{\delta_2}(t^*)}{(w_1(t^*) + w_2(t^*)}. \tag{3.11}$$

Let $u^*_{\delta_3} =: u_{\delta_3}(t^*)$ and $w^*_3 =: w_3(t^*) = w_1(t^*) + w_2(t^*)$. From (3. 9) and (3. 11), it follows that $u_{\delta_2}(t^*) < u_{\delta_3}(t^*) < u_{\delta_1}(t^*)$.

At the point (x^*, t^*) , the new Riemann problem with delta initial data

$$(\rho, u)(x, t^*) = \begin{cases} (\rho_-, u_-), & x < x_3(t), \\ (w^*_3\delta(x - x_3(t)), u^*_{\delta_3}), & x = x_3(t), \\ (\rho_+, u_+), & x > x_3(t), \end{cases} \tag{3.12}$$

is formed, see Fig. 4.

From (3. 9), it follows that

$$u_+ + \frac{1}{\rho_+} < u^*_{\delta_3} < u_-.$$

Similar to the proof of [14, case 3. 3], the location, the weight and the speed of the solution of (1. 3), (1. 6) and (3. 12) are

$$x_3(t) = \begin{cases} \dfrac{w^*_3u^*_{\delta_3}(t - t^*) - \frac{1}{2}[\rho u^2 + u](t - t^*)^2}{w^*_3 - [\rho u](t - t^*)} + x^*, & \rho_+ = \rho_-, \\ \dfrac{-w^*_3 + [\rho u](t - t^*) + w_3(t)}{[\rho]} + x^*, & \rho_+ \ne \rho_-, \end{cases}$$

$$w_3(t) = \begin{cases} w^*_3 - [\rho u](t - t^*), & \rho_+ = \rho_-, \\ \sqrt{(w^*_3 - [\rho u](t - t^*))^2 - [\rho]([\rho u^2 + u](t - t^*)^2 - 2w^*_3u^*_{\delta_3}(t - t^*))}, & \rho_+ \ne \rho_-, \end{cases}$$

$$u_{\delta_3}(t) = \frac{[\rho u](x_3 - x^*) - [\rho u^2 + u](t - t^*) + w^*_3u^*_{\delta_3}}{w_3}.$$

The structure of the solution is as in Fig. 4. Then the Riemann solution for systems (1. 3) and (1. 4) is stable as $\varepsilon \to 0$.

Case 3. 3. *$u_m < u_+ + \frac{1}{\rho_+} < u_-$ and $u_m + \frac{1}{\rho_m} \le u_-$ (if $u_+ + \frac{1}{\rho_+} \le u_- < u_m + \frac{1}{\rho_m}$, then structure of solution is similar).*

In this case, for sufficiently small t, the solution of (1. 3) and (1. 6) can be expressed as

$$(u_-, \rho_-) + \delta S + (u_m, \rho_m) + J_1 + (u_1, \rho_1) + J_2 + (u_+, \rho_+),$$

where $(u_1, \rho_1) = (u_m, \frac{1}{u_+ + \frac{1}{\rho_+} - u_m})$. The $w(t)$ and $u_\delta(t)$ of δS are given by (2. 6) and (2. 7) with the subscript "+" replaced by "m". The speed of J_1 and J_2 are $\lambda^m_1 = u_m, \lambda^+_2 = u_+ + \frac{1}{\rho_+}$ respectively. From $u_m < u_\delta(t) < u_-$, it follows that the δS will meet J_1 in finite time. And the intersection point is

$$(x^*, t^*) = (\frac{(u_\delta + u_m)\varepsilon}{u_\delta - u_m}, \frac{2\varepsilon}{u_\delta - u_m}).$$

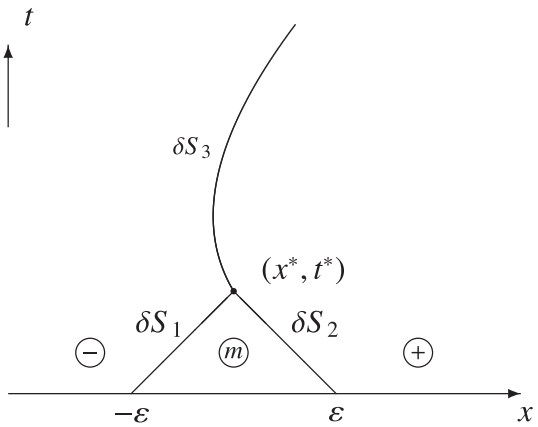

Fig. 4 The structure of solution in the physical plane for Case 3.2

When $t \geq t^*$, we need to solve the initial values problem of (1. 3), (1. 6) and (3. 3). By combining $u_m + \frac{1}{\rho_m} < u_-$ and the analysis in Case 3. 1, we have $u_m < u_\delta(t^*) < u_-$.

According to $u_m < u_\delta(t^*) < u_-$ and $u_+ + \frac{1}{\rho_+} < u_-$, we divide our discussion of two subcases as follows.

Subcase 3. 3. 1. $u_\delta(t^*) \geq u_+ + \frac{1}{\rho_+}$.

In this subcase, similar to analysis in Case 3. 2, after the interaction of δS and J_1, the location, the weight and the speed of the new delta shock wave δS_1 are

$$x_1(t) = \begin{cases} \dfrac{w^* u_\delta^*(t-t^*) - \frac{1}{2}[\rho u^2 + u](t-t^*)^2}{w^* - [\rho u](t-t^*)} + x^*, & \rho_1 = \rho_-, \\ \dfrac{-w^* + [\rho u](t-t^*) + w_1}{[\rho]} + x^*, & \rho_1 \neq \rho_-, \end{cases} \tag{3.13}$$

$$\begin{aligned} w_1(t) &= \begin{cases} \{(w^* - [\rho u](t-t^*))^2 - [\rho]([\rho u^2 + u](t-t^*)^2 - 2w^* u_\delta^*(t-t^*))\}^{\frac{1}{2}}, \rho_1 \neq \rho_-, \\ w^* - (\rho_1 u_1 - \rho_- u_-)(t-t^*), \qquad \rho_1 = \rho_-, \end{cases} \\ u_{\delta_1}(t) &= \frac{[\rho u](x_1 - x^*) - [\rho u^2 + u](t-t^*) + w^* u_\delta^*}{w_{\delta_1}(t)}, \end{aligned} \tag{3.14}$$

respectively. Due to $u_+ + \frac{1}{\rho_+} = u_1 + \frac{1}{\rho_1} < \lim\limits_{t\to\infty} u_{\delta_1}(t) < u_-$, δS_1 will meet J_2 at the point (x^+, t^+). (x^+, t^+) is determined by $x^+ = (u_+ + \frac{1}{\rho_+})t^+ + \varepsilon$ and $x^+ = \int_{t^*}^{t^+} u_{\delta_1}(\tau)d\tau + x^*$. At the time $t = t^*$, a new Riemann problem is formed with delta initial data. From $u_+ + \frac{1}{\rho_+} < u_{\delta_1}(t^*) < u_-$ and similar to analysis in case 3. 2, for $t > t^+$, we know that a new delta shock wave δS_2 is generated. The structure of the solution of (1. 3) and (1. 6) in this case is shown as in Fig. 5(a).

Subcase 3. 3. 2. $u_\delta(t^*) < u_+ + \frac{1}{\rho_+}$.

In this subcase, by the conclusion of [14, case3. 4], the interaction result of δS and J_1 is a delta shock wave δS_1 adjacent to a family of J_2 and there exists a unique $t^{**} > t^*$ such that $u_{\delta_1}(t^{**}) = u_1 + \frac{1}{\rho_1}$. When $t > t^{**}$, it will overtake and finally penetrate all of the J_2 at a point (x^+, t^+). Similar to analysis in Case 3. 2, for $t > t^+$, a new delta wave δS_2 is generated. Since $u_+ + \frac{1}{\rho_+} < u_{\delta_1}(t^+) < u_-$, δS_2 will meet the J_2 wave in finite time. By $u_+ + \frac{1}{\rho_+} < u_{\delta_2}(t^{++}) < u_-$ and similar to analysis in case 3. 2, for $t > t^{++}$, a new delta waves δS_3 arises from (x^{++}, t^{++}). The structure of the solution is shown as in Fig. 5(b).

Next, we go on to study the limit solution of Case 3. 3.

When $u_\delta(t^*) \geq u_+ + \frac{1}{\rho_+}$, $\lim\limits_{\varepsilon\to 0}(x^+, t^+) = (0, 0)$. In fact, if $\lim\limits_{\varepsilon\to 0} t^+ = k \neq 0$, by (3. 13) and $x(t^+) = (u_1 + \frac{1}{\rho_1})t^+ + \varepsilon$, we have

$$(u_1+\frac{1}{\rho_1})t^+ + \varepsilon = \frac{-w^* + [\rho u](t^+ - t^*) + w_1(t^+)}{[\rho]} + x^*. \tag{3.15}$$

Combining (3. 14), (3. 15) and letting $\varepsilon \to 0$, we get

$$[\rho](u_1+\frac{1}{\rho_1}) = [\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2+u]},$$

namely,

$$\rho_1(u_- - u_1) = 1,$$

then $u_- = u_+ + \frac{1}{\rho_+}$, a contradiction.

Noting that $\lim_{\varepsilon\to 0}\frac{1}{\rho_1} = u_+ + \frac{1}{\rho_+} - u_m > 0$, there is no mass concentration formed due to the shrink of the region with state (u_1, ρ_1) (see Fig. 5) and $w(0) = 0$ as $\varepsilon \to 0$. The limit solution consists of a delta shock wave.

When $u_\delta(t^*) < u_+ + \frac{1}{\rho_+}$, we have $\lim_{\varepsilon\to 0}(x^{**}, t^{**}) = (0, 0)$ and $w(0) = 0$. By (3. 8), we have $t^+ = 0$ for $w(0) = 0$, $u_\delta = u_+ + \frac{1}{\rho_+}$. Thus $t^{++} = 0$. Then, as $\varepsilon \to 0$, the limit solution of systems (1. 3) and (1. 6) in this subcase is corresponding Riemann solution of (1. 3) and (1. 4), see Fig. 2(b). Thus, the Riemann solution for systems (1. 3) and (1. 4) is stable in this subcase.

Similar to analysis in case 3. 1, the limit solution of systems (1. 3) and (1. 6) is a delta shock wave, which is exactly the corresponding Riemann solution of systems (1. 3) and (1. 4).

Case 3. 4. $u_- < \min\{u_m + \frac{1}{\rho_m}, u_+ + \frac{1}{\rho_+}\}$ *and* $u_m < u_+ + \frac{1}{\rho_+}$.

When t is small enough, the solution of systems (1. 3) and (1. 6) can be expressed briefly as follows (see Fig. 6)

$$(u_-, \rho_-) + J_1 + (u_1, \rho_1) + J_2 + (u_m, \rho_m) + \widehat{J_1} + (u_2, \rho_2) + \widehat{J_2} + (u_+, \rho_+),$$

where $(u_1, \rho_1) = (u_-, \frac{1}{u_m + \frac{1}{\rho_m} - u_-}), (u_2, \rho_2) = (u_m, \frac{1}{u_+ + \frac{1}{\rho_+} - u_m})$. The speed of $J_1, J_2, \widehat{J_1}, \widehat{J_2}$ are $\lambda_1^- = u_-, \lambda_2^m = u_m + \frac{1}{\rho_m}, \widehat{\lambda_1^m} = u_m$ and $\lambda_2^+ = u_+ + \frac{1}{\rho_+}$, respectively. Since $\widehat{\lambda_1^m} < \lambda_2^m$, J_2 will overtake $\widehat{J_1}$ in finite time. The intersection point is

$$(x^*, t^*) = ((2\rho_m u_m + 1)\varepsilon, 2\rho_m \varepsilon).$$

According to $u_- = u_1 < u_+ + \frac{1}{\rho_+} = u_2 + \frac{1}{\rho_2}$, for $t \geq t^*$, the solution of (1. 3) and (1. 6) is

$$(u_-, \rho_-) + J_1 + (u_1, \rho_1) + \widetilde{J_1} + (u_3, \rho_3) + \widetilde{J_2} + (u_2, \rho_2) + \widehat{J_2} + (u_+, \rho_+),$$

where the speed of $\widetilde{J_1}$ and $\widetilde{J_2}$ are $\widetilde{\lambda_1} = u_-, \widetilde{\lambda_2} = u_+ + \frac{1}{\rho_+}$, and the intermediate state is $(u_3, \rho_3) = (u_-, \frac{1}{u_+ + \frac{1}{\rho_+} - u_-})$. The structure of the solution is shown as in Fig. 6. Letting $\varepsilon \to 0$, we get

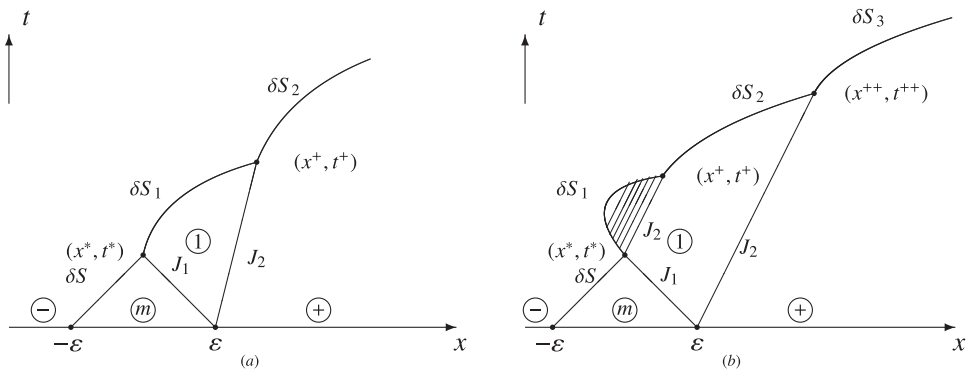

Fig. 5 The structure of solution in the physical plane for Case 3.3

$(x^*, t^*), (-\varepsilon, 0), (0, \varepsilon)$ all goes to (0, 0) and the region with state u_1, ρ_1 shrinks to the origin point with no mass concentration. In this case, the Riemann solution of (1. 3) and (1. 4) is stable.

Case 3. 5. $u_m < u_+ + \frac{1}{\rho_+} < u_- < u_m + \frac{1}{\rho_m}$.

In this case, for sufficiently small t, the solution of (1. 3) and (1. 6) can be expressed briefly as

$$(u_-, \rho_-) + \widehat{J_1} + (u_1, \rho_1) + \widehat{J_2} + (u_m., \rho_m) + \widetilde{J_1} + (u_2, \rho_2) + \widetilde{J_2} + (u_+, \rho_+),$$

where $(u_1, \frac{1}{\rho_1}) = (u_-, u_m + \frac{1}{\rho_m} - u_-), (u_2, \frac{1}{\rho_2}) = (u_m, u_+ + \frac{1}{\rho_+} - u_m)$, the speed of $\widehat{J_1}, \widehat{J_2}, \widetilde{J_1}, \widetilde{J_2}$ are $\widehat{\lambda_1^-} = u_-, \widehat{\lambda_2^m} = u_m + \frac{1}{\rho_m}, \widetilde{\lambda_1^m} = u_m$ and $\widetilde{\lambda_2^+} = u_+ + \frac{1}{\rho_+}$, respectively.

Since $\widehat{\lambda_2^m} = u_m + \frac{1}{\rho_m} > u_m = \widetilde{\lambda_1^m}$, $\widehat{J_2}$ meet $\widetilde{J_1}$ in finite time and intersection point is

$$(x^*, t^*) = ((2\rho_m u_m + 1)\varepsilon, 2\rho_m \varepsilon).$$

In view of $u_- = u_1 > u_+ + \frac{1}{\rho_+} = u_2 + \frac{1}{\rho_2}$, the interaction results in a delta wave δS. The weight and the propagation speed of δS are

$$u_\delta(t) = \begin{cases} \dfrac{1}{2}(u_1 + \dfrac{1}{\rho_1} + u_2), & \rho_1 = \rho_2, \\ \dfrac{[\rho u] + \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}}{[\rho]}, & \rho_1 \neq \rho_2, \end{cases}$$

$$w_\delta(t) = \begin{cases} (\rho_1 u_1 - \rho_2 u_2)(t - t^*), & \rho_1 = \rho_2, \\ \sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}(t - t^*), & \rho_1 \neq \rho_2. \end{cases}$$

By the generalized entropy conditions, we have

$$u_+ + \frac{1}{\rho_+} = u_2 + \frac{1}{\rho_2} < u_\delta(t) < u_1 = u_m.$$

There are three subcases as follows.

Subcase 3. 5. 1 In this subcase, δS meets $\widehat{J_1}$ and $\widetilde{J_2}$ at the same time, see Fig. 7(a), the intersection point is

$$(x^{**}, t^{**}) = \left(\frac{(u_- + u_+ + \frac{1}{\rho_+})\varepsilon}{u_- - u_+ - \frac{1}{\rho_+}}, \frac{2\varepsilon}{u_- - u_+ - \frac{1}{\rho_+}}\right).$$

The weight of the delta wave at that point is $w(t^{**})$ and the speed of delta wave is $u_\delta(t^{**})$.

Let $u_\delta^{**} = u_\delta(t^{**})$ and $w(t^{**}) = w^{**}$, we have $u_+ + \frac{1}{\rho_+} < u_\delta^{**} < u_-$. Now at the point (x^{**}, t^{**}), a new Riemann problem with delta initial data

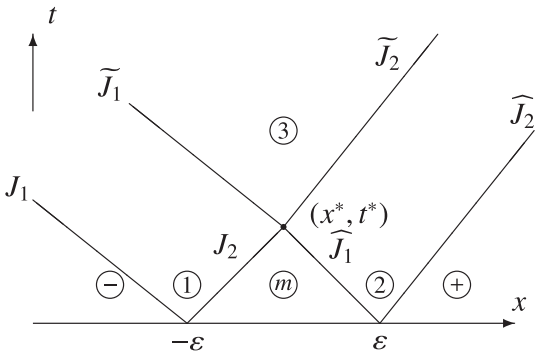

Fig. 6 The structure of solution in the physical plane for Case 3.4

$$(\rho,u)(x,t^{**})=\begin{cases}(\rho_-,u_-), & x<x_1(t),\\ (w^{**}\delta(x-x_1(t)),u_\delta^{**}), & x=x_1(t),\\ (\rho_+,u_+), & x>x_1(t),\end{cases}$$

is formed. Similar to analysis in Case 3. 2, for $t>t^{**}$, the location, the weight and the speed of a new delta wave δS_1 are

$$\begin{aligned}x_1(t)&=\begin{cases}\dfrac{-w^{**}+[\rho u](t-t^{**})+w_1(t)}{[\rho]}+x^{**}, & \rho_+\neq\rho_-,\\ \dfrac{w^{**}u_\delta^{**}(t-t^{**})-\frac{1}{2}[\rho u^2+u](t-t^{**})^2}{w_\delta^{**}-[\rho u](t-t^{**})}+x^{**}, & \rho_+=\rho_-,\end{cases}\\ w_1(t)&=\begin{cases}\sqrt{(w^{**}-[\rho u](t-t^{**}))^2-[\rho]([\rho u^2+u](t-t^{**})^2-w^{**}u_\delta^{**}(t-t^{**}))}, & \rho_+\neq\rho_-,\\ w^{**}-[\rho u](t-t^{**}), & \rho_+=\rho_-,\end{cases}\\ u_{\delta_1}(t)&=\frac{w^{**}u_\delta^{**}+[\rho u](x_1(t)-x^{**})-[\rho u^2+u](t-t^{**})}{w_1(t)}.\end{aligned}\tag{3.16}$$

The structure of solution for systems (1. 3) and (1. 6) is showed as in Fig. 7(a). Letting $\varepsilon\to 0$, we get $(x^*,t^*)\to(0,0),(x^{**},t^{**})\to(0,0)$ and $\lim\limits_{\varepsilon\to0}\frac{1}{\rho_1}=u_m+\frac{1}{\rho_m}-u_-\neq0,\lim\limits_{\varepsilon\to0}\frac{1}{\rho_2}=u_++\frac{1}{\rho_+}-u_m\neq0$. Then, the region with states (u_1,ρ_1) and (u_2,ρ_2) are no mass concentration. Since $\lim\limits_{\varepsilon\to0}w^*=0$ and $\lim\limits_{\varepsilon\to0}w_1^{**}=0$, the limit $u_\delta(t),w(t),x(t)$ of solution for systems (1. 3) and (1. 6) are (2. 6)-(2. 8). Thus, Riemann solution of systems (1. 3) and (1. 4) is stable in this case .

Subcase 3. 5. 2 In this subcase, δS meets $\widetilde{J}_2$ in finite time, see Fig. 7(b), the point of interaction is

$$\begin{aligned}(x^+,t^+)=(&\frac{(u_++\frac{1}{\rho_+})(u_\delta t^*-x^*+\varepsilon)}{u_\delta-(u_++\frac{1}{\rho_+})}\\ &+\varepsilon,\frac{u_\delta t^*-x^*+\varepsilon}{u_\delta-(u_++\frac{1}{\rho_+})}).\end{aligned}\tag{3.17}$$

The weight w_1^+ and propagation speed $u_{\delta_1}^+$ of δS at that point are given by (3. 16) and (3. 17) in which the subscript "t" replaced by "$t^{+''}$, i. e. , $w_1^+=w(t^+)$ and $u_{\delta_1}^+=u_\delta(t^+)$. Now at that point (x^+,t^+), a new Riemann problem with delta initial data

$$(\rho,u)(x,t^+)=\begin{cases}(\rho_1,u_1), & x<x_1(t),\\ (w_1^+\delta(x-x_1(t)),u_{\delta_1}^+), & x=x_1(t),\\ (\rho_+,u_+), & x>x_1(t),\end{cases}$$

is formed. Similar to analysis in Case 3. 2, for $t>t^*$, the location, the weight and the speed of δS_1 are

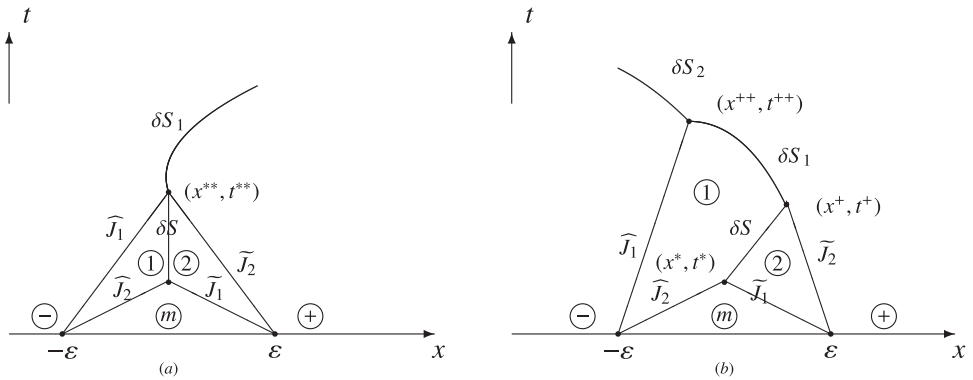

Fig. 7 The structure of solution in the physical plane for Case 3.5

$$x_1(t)=\begin{cases}\dfrac{-w_1^+ + [\rho u](t-t^+)+w_1(t)}{[\rho]}+x^+, & \rho_+\neq\rho_2,\\ \dfrac{w_1^+u_{\delta_1}^+(t-t^+)-\frac{1}{2}[\rho u^2+u](t-t^+)^2}{w_1^+-[\rho u](t-t^+)}, & \rho_+=\rho_2,\end{cases}\tag{3.18}$$

$$w_1(t)=\begin{cases}\sqrt{(w_1^+-[\rho u](t-t^+))^2-[\rho]([\rho u^2+u](t-t^+)^2-w_1^+u_{\delta_1}^+)(t-t^+)}, & \rho_+\neq\rho_2,\\ w_1^+-[\rho u](t-t^+), & \rho_+=\rho_2,\end{cases}$$

and

$$u_{\delta_1}(t)=\frac{w_1^+u_{\delta_1}^+ + [\rho u](x_1(t)-x^+)-[\rho u^2+u](t-t^+)}{w_1(t)}.\tag{3.19}$$

Since $u_{\delta_1}(t)<u_1=u_-=\widehat{\lambda_1^-}$, δS_1 will meet $\widehat{J}_1$ in finite time, the interaction point is

$$(x^{++},t^{++})=(\frac{u_-x^+ + u_-u_\delta t^+ - u_\delta\varepsilon}{u_\delta-u_-},\frac{x^+ + u_\delta t^+-\varepsilon}{u_\delta-u_-}).$$

The weight w_2^+ and propagation speed $u_{\delta_2}^+$ of δS_1 at that point are given by (3. 18) and (3. 19) in which the subscript "t" replaced t^{++}, i. e. , $w_2^+=w_1(t^{++})$ and $u_{\delta_2}^+=u_{\delta_1}(t^{++})$. From $u_+ + \frac{1}{\rho_+}<u_{\delta_2}^+<u_-$, at the point (x^{++},t^{++}), a new Riemann problem with delta initial data

$$(\rho,u)(x,t^{++})=\begin{cases}(\rho_-,u_-), & x<x_2(t),\\ (w_2^+\delta(x-x_2(t)),u_{\delta_2}^+), & x=x_2(t),\\ (\rho_+u_+), & x>x_2(t),\end{cases}$$

is formed. Similar to analysis in Case 3. 2, for $t>t^{++}$, the location, the weight and the speed of δS_2 are

$$x_2(t)=\begin{cases}\dfrac{-w_2^+ + [\rho u](t-t^{++})+w_2(t)}{[\rho]}+x^{++}, & \rho_+\neq\rho_-,\\ \dfrac{w_2^+u_{\delta_2}^+(t-t^{++})-\frac{1}{2}[\rho u^2+u](t-t^{++})^2}{w_2^+-[\rho u](t-t^{++})}+x^{++}, & \rho_+=\rho_-,\end{cases}$$

$$w_2(t)=\begin{cases}\sqrt{(w_2^+-[\rho u](t-t^{++}))^2-[\rho]([\rho u^2+u](t-t^{++})^2-w_2^+u_{\delta_2}^+(t-t^{++}))}, & \rho_+=\rho_-,\\ w_2^+-[\rho u](t-t^{++}), & \rho_+=\rho_-,\end{cases}$$

$$u_{\delta_2}(t)=\frac{w_2^+u_{\delta_2}^+ + [\rho u](x_1(t)-x^{++})-[\rho u^2+u](t-t^+)}{w_2(t)}.$$

The structure of the solution to system (1. 3) and (1. 6) is showed in Fig. 7(b). Letting $\varepsilon\to0,(x^*,t^*)\to(0,0),(x^+,t^+)\to(0,0),(x^{++},t^{++})\to(0,0),\quad \lim_{\varepsilon\to0}\frac{1}{\rho_1}=u_m+\frac{1}{\rho_m}-u_-\neq0$ and $\lim_{\varepsilon\to0}\frac{1}{\rho_2}=u_+ + \frac{1}{\rho_+}-u_-\neq0$. Then the region with state (u_1,ρ_1) and (u_2,ρ_2) is no mass concentration. Since $\lim_{\varepsilon\to0}w_1^*=0,\lim_{\varepsilon\to0}w_2^+=0$, the limit solution of system (1. 3) and (1. 6) is (2. 6)-(2. 8), see Fig. 2(b). Thus, Riemann solution of system (1. 3) and (1. 4) is stable in this case .

Subcase 3. 5. 3. δS meets $\widehat{J}_1$ in finite time, this case is similar to subcase 3. 7. 2.

4 Perturbation depending on parameter ε

In this section, we study the solution of systems (1. 3) and (1. 6) under the assumption that u_m is constant, and ρ_m satisfies

$$\begin{aligned}\lim_{\varepsilon\to 0}\rho_m &= \lim_{\varepsilon\to 0}\rho_m(\varepsilon)\\ &= +\infty, \lim_{\varepsilon\to 0}\rho_m\varepsilon = 0.\end{aligned} \tag{4.1}$$

According to the relations among u_-, $u_+ + \frac{1}{\rho_+}$ and u_m, the Riemann problem is divided into 9 cases:

(1) $u_m < u_- < u_+ + \frac{1}{\rho_+}$;
(2) $u_- < u_+ + \frac{1}{\rho_+} < u_m$;
(3) $u_- = u_m < u_+ + \frac{1}{\rho_+}$ or $u_- < u_m = u_+ + \frac{1}{\rho_+}$;
(4) $u_- < u_m < u_+ + \frac{1}{\rho_+}$;
(5) $u_m < u_+ + \frac{1}{\rho_+} < u_-$;
(6) $u_+ + \frac{1}{\rho_+} = u_m < u_-$;
(7) $u_+ + \frac{1}{\rho_+} < u_m < u_-$;
(8) $u_+ + \frac{1}{\rho_+} < u_- < u_m$;
(9) $u_+ + \frac{1}{\rho_+} < u_m = u_-$.

In the sequel, we construct the solution of each case for sufficiently small positive ε with emphasis on the limit structure of the perturbed solution by passing $\varepsilon \to 0$.

Case 4. 1. $u_m < u_- < u_+ + \frac{1}{\rho_+}$.

This case corresponds to Case 3. 1 in Section 3. Then, the solution of system (1. 3) and (1. 6) in this case is given, see Fig. 3.

Next, we investigate the limit of the perturbed solution as $\varepsilon \to 0$. By (3. 1) and (3. 2), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}(x^*, t^*) &= \lim_{\varepsilon\to 0}\left(\frac{(u_\delta + u_m)\varepsilon}{u_\delta - u_m}, \frac{2\varepsilon}{u_\delta - u_m}\right) = (0,0), \ \lim_{\varepsilon\to 0} w(t^*)\\ &= \lim_{\varepsilon\to 0}\sqrt{[\rho u]^2 - [\rho][\rho u^2 + u]}t^* = 0\end{aligned}$$

and $\lim_{\varepsilon\to 0}(u_1, \frac{1}{\rho_1}) = (u_m, u_+ + \frac{1}{\rho_+} - u_m)$. Since

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^{t^*}\int_{u_\delta t-\varepsilon}^{u_m t+\varepsilon}\rho_m dx dt\\ &\quad = \lim_{\varepsilon\to 0}(2\rho_m\varepsilon t^*\\ &\qquad + \frac{1}{2}\rho_m(t^*)^2(u_m - u_\delta)) = 0\end{aligned} \tag{4.2}$$

and $\frac{1}{\rho_1} = u_+ + \frac{1}{\rho_+} - u_m > 0$, there is no mass concentration formed as the region with state (u_m, ρ_m) and (u_1, ρ_1) shrinks.

Case 4. 2. $u_- < u_+ + \frac{1}{\rho_+} < u_m$.

This case is similar to Case 4. 1 and there is no mass concentration formed as the region with state (u_1, ρ_1) and (u_m, ρ_m) shrinks.

Case 4. 3. $u_- = u_m < u_+ + \frac{1}{\rho_+}$ *or* $u_- < u_m = u_+ + \frac{1}{\rho_+}$.

If $u_- = u_m < u_+ + \frac{1}{\rho_+}$, for fixed small positive ε, we have $u_- < \min\{u_+ + \frac{1}{\rho_+}, u_m + \frac{1}{\rho_m}\}$ and $u_m < u_+ + \frac{1}{\rho_+}$, this case corresponds to Case 3. 4 in Section 3 (see Fig. 6).

Note that $\lim_{\varepsilon\to 0}(u_1, \frac{1}{\rho_1}) = \lim_{\varepsilon\to 0}(u_-, u_m + \frac{1}{\rho_m} - u_-) = (u_-, u_m - u_-) = (u_-, 0)$, $\lim_{\varepsilon\to 0}(u_2, \frac{1}{\rho_2}) = \lim_{\varepsilon\to 0}(u_m, u_+ + \frac{1}{\rho_+} - u_m) = (u_m, u_+ + \frac{1}{\rho_+} - u_m)$ and $u_+ + \frac{1}{\rho_+} - u_m > 0$. Since

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^{t^*}\int_{u_- t-\varepsilon}^{(u_m+\frac{1}{\rho_m})t-\varepsilon}\rho_1 dxdt\\
&\quad=\lim_{\varepsilon\to 0}\int_0^{t^*}\int_{u_- t-\varepsilon}^{(u_m+\frac{1}{\rho_m})t-\varepsilon}\frac{1}{u_m+\frac{1}{\rho_m}-u_-}dxdt\\
&\quad=\lim_{\varepsilon\to 0}\frac{1}{2}(t^*)^2=0,\\
&\lim_{\varepsilon\to 0}\int_0^{t^*}\int_{(u_m+\frac{1}{\rho_m})t-\varepsilon}^{u_m t+\varepsilon}\rho_m dxdt\\
&\quad=\lim_{\varepsilon\to 0}(-\frac{1}{2}(t^*)^2+2\rho_m\varepsilon t^*)=0,
\end{aligned}\tag{4.3}$$

there is no mass concentration formed as the region with state (u_1,ρ_1) and (u_m,ρ_m) shrinks. Similarly, for $u_-<u_m=u_+ +\frac{1}{\rho_+}$, there is no mass concentration formed as the region with state (u_2,ρ_2) and (u_m,ρ_m) shrinks.

Case 4. 4. $u_-<u_m<u_+ +\frac{1}{\rho_+}$.

For fixed small positive ε, we have $u_-<u_m<u_m+\frac{1}{\rho_m}<u_+ +\frac{1}{\rho_+}$. It corresponds to case 3. 3 in section 3 (see Fig. 5). Since $(u_1,\frac{1}{\rho_1})\to(u_-,u_m-u_-)$, $(u_2,\frac{1}{\rho_2})\to(u_m,u_+ +\frac{1}{\rho_+}-u_m)$, $u_m-u_->0$, $u_+ +\frac{1}{\rho_+}-u_m>0$ and (4. 3), there is no mass concentration formed as the region with state (u_m,ρ_m) shrinks.

Case 4. 5. $u_m<u_+ +\frac{1}{\rho_+}<u_-$.

For fixed small positive ε, this case corresponds to Case 3. 3 in section 3 (see Fig. 5). Note that

$$\lim_{\varepsilon\to 0}(u_1,\frac{1}{\rho_1})=(u_m,u_+ +\frac{1}{\rho_+}-u_m).$$

Since $u_+ +\frac{1}{\rho_+}-u_m>0$ and (4. 2), there is no mass concentration formed as the region with state (u_m,ρ_m) shrinks.

Case 4. 6. $u_+ +\frac{1}{\rho_+}=u_m<u_-$.

For fixed small positive ε, this case corresponds to Case 3. 3 in Section 3 (see Fig. 5). Similar to the analysis in Case 4. 5, we have $\lim_{\varepsilon\to 0}(x^*,t^*)=(0,0)$, $\lim_{\varepsilon\to 0}(x^{**},t^{**})=(0,0)$ in the case shown as in Fig. 5 and $\lim_{\varepsilon\to 0}(x^*,t^*)=\lim_{\varepsilon\to 0}(x^{++},t^{++})=(0,0)$ in the case shown as in Fig. 5. For $u_\delta(t^*)<u_+\frac{1}{\rho_+}$, noting that

$$\lim_{\varepsilon\to 0}(u_1,\frac{1}{\rho_1})=(u_m,u_+ +\frac{1}{\rho_+}-u_m)=0.$$

Since

$$\lim_{\varepsilon\to 0}\int_0^{t^*}\int_{u_m t+\varepsilon}^{(u_+ +\frac{1}{\rho_+})t+\varepsilon}\rho_1 dxdt+\lim_{\varepsilon\to 0}\int_{t^*}^{t^+}\int_{x_1}^{(u_+ +\frac{1}{\rho_+})t+\varepsilon}\rho_1 dxdt=0,$$

and (4. 2), there is no mass concentration formed as the region with state (u_1,ρ_1) and (u_m,ρ_m). Similarly, for $u_\delta(t^*)<u_+\frac{1}{\rho_+}$, there is no mass concentration formed as the region with state (u_1,ρ_1) and (u_m,ρ_m) shrinks.

Case 4. 7. $u_+ +\frac{1}{\rho_+}<u_m<u_-$.

For fixed small positive ε, this case corresponds to Case 3. 5 in Section 3 (see Fig. 7). Since

$$\lim_{\varepsilon\to 0}\int_0^{t^*}\int_{u_{\delta_1}t-\varepsilon}^{u_{\delta_2}t+\varepsilon}\rho_m dxdt=\lim_{\varepsilon\to 0}(\frac{1}{2}\rho_m(t^*)^2(u_{\delta_2}-u_{\delta_1})+2\varepsilon\rho_m t^*)=0,$$

there is no mass concentration formed as the region with state (u_m,ρ_m).

Similar to analysis in Case 4. 1, there is no mass concentration formed as the region with state (u_1,ρ_1) and (u_m,ρ_m) shrinks for Cases 4. 8 and 4. 9.

5 Discussion

From the above analysis, under the perturbation (1. 6), we can see that the solution of systems (1. 3) and (1. 4) is stable if $\lim_{\varepsilon \to 0} \rho_m \varepsilon = 0$.

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References

1. M. C. Bento, O. Bertolami and A. A. Sen, Generalized Chaplygin Gas, Accelerated Expansion, and Dark-Matter Unification, Phy. Rev. , No. D66, 2002.
2. F. Bouchut, On zero pressure gas dynamics, in: Advances in Kinetic Theory and Computing, Ser. Adv. Math. Appl. Aci. , World Sci. Publishing, River Edge, NJ, No. 22, (1994), 171-190.
3. Y. Brenier, Solutions with concentration to the Riemann problem for the one-dimensional Chaplygin gas equation, J. Math. Fluid Mech. , No. 7, (2005), 326-331.
4. S. A. Chaplygin, On gas jets, in: Math. Phys. Sec. , Moscow Univ. 1902 Eng. trans. pub. by Brown Univ, No. 21, (1944), 1-121.
5. H. Cheng, Shock waves for a linearly degenerate hyperbolic system of conservation laws of Keyfitz-Kranzer type, Advances in Mathematical Physics, No. 2006, (2013).
6. V. G. Danilov and V. M. Shelkovich, Dynamics of propagation and interaction of delta shock waves in conservation law systems, J. Differential Equations, No. 221, (2005), 333-381.
7. L. H. Guo and G. Yin, Limit of Riemann solutions to the non-symmetric system of Keyfitz-Kranzer type, The scientific world Journal, No. 2014. (2014), Article I287256, 11pages.
8. F. Huang and Z. Wang, Well-posedness for pressureless flow, Comm. Math. Phys. , No. 222, (2001), 117-146.
9. T. Karman, Compressibility effects in aerodynamics, J. Aeron. Sci. , No. 8, (1941), 337-365.
10. B. L. Keyfitz and H. C. Kranzer, A system of non-strictly hyper-bolic conservation laws arising in elasticity theory, Archive for Rational Mechanics and Analysis, No. 3, (1980), 219-241, .
11. B. L. Keyfitz, Conservation laws, delta shocks and singular shocks, M. Grosses, G. Hormann, M. Kunzinger, M. Obergguggenberger(Eds.), Nonlinear Theory of Generalized Functions, in: Res. Note Math. ,Chapman and Hall/CRC, No. 401, (1999), 99-111.
12. D. J. Korchinski, Solution of a Riemann problem for a system of conservation laws possessing no classical weak solution. Thesis, Adelphi University, (1977).
13. J. Li, T. Zhang and S. Yang, The Two-Dimensional Rirmann Problem in Gas Dynamics, in: Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific and Technical, No. 98, (1998).
14. H. Li and Z. Shao, The Riemann problem with delta initial data for the non-symmetric Keyfitz-Kranzer system with Chaplygin pressure, Applied Analysis and Computation, No. 7, (2017), 681-701.
15. Y. Lu, Existence of global bounded weak solutions to nonsymmetric systems of Keyfitz-Kranzer type, Journal of Functional Analysis, No. 261, (2011), 2797-2815.
16. A. Qu, Z. Wang, Stability of the Riemann solutions for a Chaplygin gas, J. Math. Appl. , No. 409, (2014), 347-361.
17. M. R. Setare, Holographic Chaplygin DCG cosmologies. Internet. J. Modern Phys. , No. D18, (2009), 419-427.
18. Z. Shao, Riemann problem with delta initial data for the isentropic relativistic Chaplygin Euler equations. Zeitschrift fr angewandte Mathematik und Physik, No. 67, (2016), 376-395.
19. C. Shen and M. Sun, Stability of the Riemann solutions for a nonstrictly hyperbolic system of conservation laws, Nonlinear Analysis, No. 73, (2010), 3284-3294.
20. C. Shen, Delta shock wave solution for a synnetric a keyfitz-Kranzer system, Applied Mathematics Letters, No. 77, (2018), 35-43.
21. M. Sun and W. Sheng, The ignition problem for a scalar nonconvex combustion model, J. Differential Equations, No. 231, (2006), 401-427.
22. M. Sun, Interactions of elementary waves for Aw-Rascle model, SIAM J. Appl. Math. , No. 69, (2009), 1542-1558.
23. H. Tsien, Two dimensional subsonic flow of cimpressible fluids, J. Aeron. Sci. , No. 6, (1939), 399-407.
24. Z. Wang and Q. Zhang, The Riemann problem with delta initial data for the one-deimentional Chaplygin gas equation, Acta Mathematica Scientia, No. 32B, (2012), 825-841.
25. H. Yang and Y. Zhang, New developments of delta shock waves and its applications in systems of conservation laws, J. Differential Equations, No. 11, (2012), 5951-5993.