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ORIGINAL RESEARCH

An implicit finite difference scheme for the numerical solutions of two-dimensional Burgers equations

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Abstract In this study, the system of two-dimensional Burgers equations is solved by a new approximation that approaches the solution at two time legs: approximation is explicit in x-direction and implicit in y-direction at the first leg while approximation is implicit in x-direction and explicit in y-direction at the second leg. Two test problems are used to illustrate the accuracy of the present approximation. Comparisons are made with the existing methods in the literature. The approximation is analyzed by von-Neumann stability analysis method and it is displayed that the approximation is unconditionally stable. The method is shown to be consistent and second order accurate in time and space. The obtained results show that the present approximation is successful to solve the system of two-dimensional Burgers equations.

Keywords Two-dimensional Burgers equation · Von-Neumann stability analysis

Mathematics Subject Classification Primary 65N06 · 65M06

1 Introduction

Burgers equation is an important partial differential equation in fluid dynamics. It has many applications in a various physical and engineering fields such as modeling of dynamics, turbulence [5], acoustic waves, heat conduction, in examining the chemical reaction, diffusion model of Brusselator [2] and approximate theory of flow through a shock wave travelling in viscous fluid [7].

In this study we consider the system of two-dimensional Burgers equations:

$$u_t + uu_x + vu_y = \frac{1}{Re}\left(u_{xx} + u_{yy}\right) \tag{1}$$

$$v_t + uv_x + vv_y = \frac{1}{Re}\left(v_{xx} + v_{yy}\right) \tag{2}$$

with initial conditions:

$$\begin{aligned} u(x,y,0) &= f(x,y),\ (x,y) \in D, \\ v(x,y,0) &= g(x,y),\ (x,y) \in D, \end{aligned} \tag{3}$$

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and boundary conditions:

$$\begin{aligned} u(x,y,t) &= \varphi(x,y,t),\ x,y \in \partial D,\ t > 0, \\ v(x,y,t) &= \psi(x,y,t),\ x,y \in \partial D,\ t > 0, \end{aligned} \tag{4}$$

where; $D = \{(x,y) : a \le x \le b,\ a \le y \le b\}$ and ∂D is its boundary, f, g, φ and ψ are known functions, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined and *Re* is the Reynolds number.

In 1983, Flecther [9] has obtained analytical solution of the system eqs. (1)–(2) by using the Hopf–Cole transformation. Many researchers have studied the numerical solutions of the system of equations. Jain and Holla [12] have proposed two algorithms based on cubic spline function technique for obtaining numerical solutions of one and two dimensional Burgers equations. Fletcher [10] studied numerical solutions of one and two dimensional Burgers equations by finite element and finite difference methods and compared the results obtained by the two methods. Wubs and Goede [20] used an explicit and implicit method. Goyon [11] used variety multilevel approach to solve the equation. Bahadır [4] has obtained the numerical solutions of the system of equations by using the fully implicit finite difference method. El-Sayed and Kaya [8] have obtained the numerical solutions of the system of two dimensional Burgers equations by using the decomposition method. Abdou and Soliman [1] used the variational iteration method to solve the two dimensional Burgers equation. Mittal and Jiwari [14] have obtained the numerical solutions of the system for rather large Reynolds numbers (Re = 1200) by using the differential quadrature method. Liu and Weiping [13] used the lattice Boltzmann method to obtain the numerical solutions of the system of two-dimensional equations. Zhao et al. [21] applied the Hopf–Cole transformation to the system of two-dimensional Burgers equations and then solved the resulting two-dimensional heat equation by local discontinuous Galerkin (LDG) finite element method. Srivastava et al. [16] used the Crank–Nicolsan finite difference method to solve the system of equations. Wani and Thakar [19] obtained the numerical solutions of the system by using a linear finite difference method that approximate to nonlinear terms by using central difference scheme and second order terms by using Crank–Nicolsan scheme. Srivastava et al. [17, 18] obtained the numerical solutions of equations by using implicit exponential finite difference and implicit logarithmic finite difference methods. Shukla et al. [15] used the modified cubic B-spline differential quadrature method to obtain numerical solutions of equation and compared the obtained results with other studies in the literature. Celikten and Aksan [6] used the Alternating Direction Implicit method to solve the system of equations. Ahmad et al. [3] investigated two new modified variational iteration algorithms for the numerical solution of coupled Burgers' equations.

In this study, a new approximation is used to solve the system of eqs. (1)–(2). In order to show accuracy of the present method two model problems are investigated. Obtained results are compared with exact solutions and numerical solutions that were obtained in other studies in the literature. The results obtained by present method show that the results are acceptable and consistent with other studies in the literature.

2 Numerical Method

In this study, the system of two-dimensional Burgers equations is solved by a new approximation that approaches the solution at two time legs: approximation is explicit in x-direction and implicit in y-direction at the first leg while approximation is implicit in x-direction and explicit in y-direction at the second leg. The system of two-dimensional Burgers equations can be solved by using explicit or implicit finite difference approximations for both x and y-direction. But disadvantage of implicit scheme is that a large system of equations must be solved at each time level and disadvantage of explicit scheme is that the explicit scheme is conditionally stable. The basic motivations of present scheme are that the present scheme is unconditionally stable and a tridiagonal system of equations must be solved at each time level.

We indicate the discrete approximation of $u(x,y,t)$ and $v(x,y,t)$ at the grid point $\left(ih_x, jh_y, nk\right)$ by $u_{i,j}^n$ and $v_{i,j}^n$, respectively $(i = 0,1,2,\ldots,N_x;\ j = 0,1,2,\ldots,N_y;\ n = 0,1,2,\ldots)$ where h_x and h_y the grid sizes in x-direction and y-direction, respectively and k represents the increment in time.

New finite difference approximation to eq. (1) is given by:

$$\begin{aligned}
&\frac{u_{i,j}^{n+\frac{1}{2}}-u_{i,j}^{n}}{k/2}+\left(\frac{u_{i,j}^{n}+u_{i+1,j}^{n}}{2}\right)\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2h_x}\right)\\
&\quad+\left(\frac{v_{i,j}^{n}+v_{i+1,j}^{n}}{2}\right)\left(\frac{u_{i,j+1}^{n+\frac{1}{2}}-u_{i,j-1}^{n+\frac{1}{2}}}{2h_y}\right)\\
&\quad=\frac{1}{Re}\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h_x^2}+\frac{u_{i,j+1}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i,j-1}^{n+\frac{1}{2}}}{h_y^2}\right)
\end{aligned}\tag{5}$$

$$\begin{aligned}
&\frac{u_{i,j}^{n+1}-u_{i,j}^{n+\frac{1}{2}}}{k/2}+\left(\frac{u_{i,j}^{n+\frac{1}{2}}+u_{i,j+1}^{n+\frac{1}{2}}}{2}\right)\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2h_x}\right)\\
&\quad+\left(\frac{v_{i,j}^{n+\frac{1}{2}}+v_{i,j+1}^{n+\frac{1}{2}}}{2}\right)\left(\frac{u_{i,j+1}^{n+\frac{1}{2}}-u_{i,j-1}^{n+\frac{1}{2}}}{2h_y}\right)\\
&\quad=\frac{1}{Re}\left(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h_x^2}+\frac{u_{i,j+1}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i,j-1}^{n+\frac{1}{2}}}{h_y^2}\right)
\end{aligned}\tag{6}$$

For simplicity, we let $h_x = h_y = h$ hence new finite difference approximation to eq. (1) is in the following form:

$$\begin{aligned}
&\left(-r_1\left(\frac{v_{i,j}^{n}+v_{i+1,j}^{n}}{2}\right)-r_2\right)u_{i,j-1}^{n+\frac{1}{2}}+(1+2r_2)u_{i,j}^{n+\frac{1}{2}}\\
&\quad+\left(r_1\left(\frac{v_{i,j}^{n}+v_{i+1,j}^{n}}{2}\right)-r_2\right)u_{i,j+1}^{n+\frac{1}{2}}\\
&\quad=\left(r_1\left(\frac{u_{i,j}^{n}+u_{i+1,j}^{n}}{2}\right)+r_2\right)u_{i-1,j}^{n}+(1-2r_2)u_{i,j}^{n}\\
&\quad+\left(-r_1\left(\frac{u_{i,j}^{n}+u_{i+1,j}^{n}}{2}\right)+r_2\right)u_{i+1,j}^{n}
\end{aligned}\tag{7}$$

$$\begin{aligned}
&\left(-r_1\left(\frac{u_{i,j}^{n+\frac{1}{2}}+u_{i,j+1}^{n+\frac{1}{2}}}{2}\right)-r_2\right)u_{i-1,j}^{n+1}+(1+2r_2)u_{i,j}^{n+1}\\
&\quad+\left(r_1\left(\frac{u_{i,j}^{n+\frac{1}{2}}+u_{i,j+1}^{n+\frac{1}{2}}}{2}\right)-r_2\right)u_{i+1,j}^{n+1}\\
&\quad=\left(r_1\left(\frac{v_{i,j}^{n+\frac{1}{2}}+v_{i,j+1}^{n+\frac{1}{2}}}{2}\right)+r_2\right)u_{i,j-1}^{n+\frac{1}{2}}+(1-2r_2)u_{i,j}^{n+\frac{1}{2}}\\
&\quad+\left(-r_1\left(\frac{v_{i,j}^{n+\frac{1}{2}}+v_{i,j+1}^{n+\frac{1}{2}}}{2}\right)+r_2\right)u_{i,j+1}^{n+\frac{1}{2}}
\end{aligned}\tag{8}$$

where $r_1 = \frac{k}{4h}$ and $r_2 = \frac{k}{2Reh^2}$.

Similarly, new finite difference approximation to eq. (2) is given by:

$$\begin{aligned}
&\frac{v_{i,j}^{n+\frac{1}{2}}-v_{i,j}^{n}}{k/2}+\left(\frac{u_{i,j}^{n}+u_{i+1,j}^{n}}{2}\right)\left(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2h_x}\right)\\
&\quad+\left(\frac{v_{i,j}^{n}+v_{i+1,j}^{n}}{2}\right)\left(\frac{v_{i,j+1}^{n+\frac{1}{2}}-v_{i,j-1}^{n+\frac{1}{2}}}{2h_y}\right)\\
&\quad=\frac{1}{Re}\left(\frac{v_{i+1,j}^{n}-2v_{i,j}^{n}+v_{i-1,j}^{n}}{h_x^2}+\frac{v_{i,j+1}^{n+\frac{1}{2}}-2v_{i,j}^{n+\frac{1}{2}}+v_{i,j-1}^{n+\frac{1}{2}}}{h_y^2}\right)
\end{aligned}\tag{9}$$

$$\begin{aligned}
&\frac{v_{i,j}^{n+1}-v_{i,j}^{n+\frac{1}{2}}}{k/2}+\left(\frac{u_{i,j}^{n+\frac{1}{2}}+u_{i,j+1}^{n+\frac{1}{2}}}{2}\right)\left(\frac{v_{i+1,j}^{n+1}-v_{i-1,j}^{n+1}}{2h_x}\right)\\
&\quad+\left(\frac{v_{i,j}^{n+\frac{1}{2}}+v_{i,j+1}^{n+\frac{1}{2}}}{2}\right)\left(\frac{v_{i,j+1}^{n+\frac{1}{2}}-v_{i,j-1}^{n+\frac{1}{2}}}{2h_y}\right)\\
&\quad=\frac{1}{Re}\left(\frac{v_{i+1,j}^{n+1}-2v_{i,j}^{n+1}+v_{i-1,j}^{n+1}}{h_x^2}+\frac{v_{i,j+1}^{n+\frac{1}{2}}-2v_{i,j}^{n+\frac{1}{2}}+v_{i,j-1}^{n+\frac{1}{2}}}{h_y^2}\right)
\end{aligned}\tag{10}$$

For simplicity, we let $h_x = h_y = h$ hence new finite difference approximation to eq. (2) is in the following form:

$$\begin{aligned}
&\left(-r_1\left(\frac{v_{i,j}^{n}+v_{i+1,j}^{n}}{2}\right)-r_2\right)v_{i,j-1}^{n+\frac{1}{2}}+(1+2r_2)v_{i,j}^{n+\frac{1}{2}}\\
&\quad+\left(r_1\left(\frac{v_{i,j}^{n}+v_{i+1,j}^{n}}{2}\right)-r_2\right)v_{i,j+1}^{n+\frac{1}{2}}\\
&\quad=\left(r_1\left(\frac{u_{i,j}^{n}+u_{i+1,j}^{n}}{2}\right)+r_2\right)v_{i-1,j}^{n}\\
&\quad+(1-2r_2)v_{i,j}^{n}+\left(-r_1\left(\frac{u_{i,j}^{n}+u_{i+1,j}^{n}}{2}\right)+r_2\right)v_{i+1,j}^{n}
\end{aligned}\tag{11}$$

$$\begin{aligned}
&\left(-r_1\left(\frac{u_{i,j}^{n+\frac{1}{2}}+u_{i,j+1}^{n+\frac{1}{2}}}{2}\right)-r_2\right)v_{i-1,j}^{n+1}+(1+2r_2)v_{i,j}^{n+1}\\
&\quad+\left(r_1\left(\frac{u_{i,j}^{n+\frac{1}{2}}+u_{i,j+1}^{n+\frac{1}{2}}}{2}\right)-r_2\right)v_{i+1,j}^{n+1}\\
&\quad=\left(r_1\left(\frac{v_{i,j}^{n+\frac{1}{2}}+v_{i,j+1}^{n+\frac{1}{2}}}{2}\right)+r_2\right)v_{i,j-1}^{n+\frac{1}{2}}\\
&\quad+(1-2r_2)v_{i,j}^{n+\frac{1}{2}}+\left(-r_1\left(\frac{v_{i,j}^{n+\frac{1}{2}}+v_{i,j+1}^{n+\frac{1}{2}}}{2}\right)+r_2\right)v_{i,j+1}^{n+\frac{1}{2}}
\end{aligned}\tag{12}$$

where $r_1 = \frac{k}{4h}$ and $r_2 = \frac{k}{2Reh^2}$.

In present scheme, we choose to use the average of the nodal and its forward point for the convective coefficients. We can also use the average of the nodal and its backward point or average of nodal and its forward and backward points. The calculations for these three selections are very close to each other. Therefore, an arbitrary choose was made. In addition, the Crout method is used to solve the tridiagonal systems occurring in each time level.

3 Order of Accuracy and Rate of Convergence

3.1 Order of Accuracy

In order to analyze the order of accuracy of the numerical schemes (5)–(6), the nonlinear terms $uu_x$and vu_y in the two-dimensional Burgers equation (1) have been linearized by replacing the quantities u and v by local constants. Thus the nonlinear terms uu_x and vu_y convert into $\hat{U}u_x$ and $\hat{V}u_y$ in equation (1) and in that case eq. (1) turns into eq. (25).

If we write the numerical schemes (5)–(6) for this linearized equation, newly approximations are taken as:

$$\frac{u_{i,j}^{n+\frac{1}{2}}-u_{i,j}^{n}}{k/2}+\hat{U}\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2h_x}\right)+\hat{V}\left(\frac{u_{i,j+1}^{n+\frac{1}{2}}-u_{i,j-1}^{n+\frac{1}{2}}}{2h_y}\right) =\frac{1}{Re}\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h_x^2}+\frac{u_{i,j+1}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i,j-1}^{n+\frac{1}{2}}}{h_y^2}\right) \tag{13}$$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n+\frac{1}{2}}}{k/2}+\hat{U}\left(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2h_x}\right)+\hat{V}\left(\frac{u_{i,j+1}^{n+\frac{1}{2}}-u_{i,j-1}^{n+\frac{1}{2}}}{2h_y}\right) =\frac{1}{Re}\left(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h_x^2}+\frac{u_{i,j+1}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i,j-1}^{n+\frac{1}{2}}}{h_y^2}\right) \tag{14}$$

In order to prove that the schemes given by eq. (13) and eq. (14) are consistent and are of order two in space and time, we will show the local truncation error of the schemes. If we sum the eq. (13) and eq. (14) and subtract eq. (14) from eq. (13), we get:

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{k/2}+\hat{U}\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2h_x}+\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2h_x}\right)+2\hat{V}\left(\frac{u_{i,j+1}^{n+\frac{1}{2}}-u_{i,j-1}^{n+\frac{1}{2}}}{2h_y}\right) =\frac{1}{Re}\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h_x^2}+\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h_x^2}+2\frac{u_{i,j+1}^{n+\frac{1}{2}}-2u_{i,j}^{n+\frac{1}{2}}+u_{i,j-1}^{n+\frac{1}{2}}}{h_y^2}\right) \tag{15}$$

$$\frac{2u_{i,j}^{n+\frac{1}{2}}-u_{i,j}^{n+1}-u_{i,j}^{n}}{k/2}+\hat{U}\left(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2h_x}-\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2h_x}\right) =\frac{1}{Re}\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h_x^2}-\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h_x^2}\right) \tag{16}$$

We solve eq. (16) for $u_{i,j}^{n+\frac{1}{2}}$ and substitute into eq. (15) and then use the Taylor series about $\left(x_i,y_j,t_n\right)$ we get the local truncation error as follows:

$$\begin{aligned}T_{i,j}^{n}=&2\left[u_t+\hat{U}u_x+\hat{V}u_y-\frac{1}{Re}\left(u_{xx}+u_{yy}\right)\right]_{i,j}^{n}\\&+k\frac{\partial}{\partial t}\left[u_t+\hat{U}u_x+\hat{V}u_y-\frac{1}{Re}\left(u_{xx}+u_{yy}\right)\right]_{i,j}^{n}\\&+\frac{k^2}{3}\frac{\partial^3 u}{\partial t^3}+\frac{\hat{U}h_x^2}{6}\frac{\partial^3 u}{\partial x^3}+\frac{\hat{V}h_y^2}{6}\frac{\partial^3 u}{\partial y^3}+\cdots\end{aligned} \tag{17}$$

Thus the principal part of the local truncation error is as follows:

$$\left[\frac{k^2}{3}\frac{\partial^3 u}{\partial t^3}+\frac{\hat{U}h_x^2}{6}\frac{\partial^3 u}{\partial x^3}+\frac{\hat{V}h_y^2}{6}\frac{\partial^3 u}{\partial y^3}\right]_{i,j}^{n} \tag{18}$$

So the order of local truncation error is $O\left(h_x^2+h_y^2+k^2\right)$ and the schemes (5) and (6) are second order in time and space.

Similarly we can show the order of the numerical schemes (9) and (10). They are also second order in time and space.

3.2 Rate of Convergence

The accuracy of the method is measured in terms of the error norm defined by

$$E=\sqrt{\frac{\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\left|U_{i,j}-u_{i,j}\right|^2}{\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\left|U_{i,j}\right|^2}} \tag{19}$$

in which U and u indicate exact and computed numerical solutions, respectively. The rates of convergence of the method, computed by using

$$rate=\frac{log(E^h/E^{\frac{h}{2}})}{log(2)} \tag{20}$$

where $h_x=h_y=h$ and E^h and $E^{\frac{h}{2}}$ are the errors defined in eq. (19) with the grid size h and $\frac{h}{2}$, respectively.

4 Model Problems and Numerical Results

In order to illustrate the accuracy and efficiency of the present method two model problems are investigated. To demonstrate the correctness of results L_2, L_∞ error norms and absolute error:

$$\begin{aligned}
L_2 &=\|U-u\|_2=\sqrt{\sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\left|U_{i,j}-u_{i,j}\right|^2}\\
L_\infty &=\|U-u\|_\infty=\max_{i,j}\left|U_{i,j}-u_{i,j}\right|\\
A.E. &=|U-u|=\left|U_{i,j}-u_{i,j}\right|
\end{aligned}$$

are used, in which U and u indicate exact and computed numerical solutions, respectively.

4.1 Problem 1

We consider the system of two-dimensional Burgers eqs. (1)–(2) with exact solutions:

$$\begin{aligned}
u(x,y,t) &=\frac{3}{4}-\frac{1}{4(1+\exp((-4x+4y-t)Re/32))},\\
v(x,y,t) &=\frac{3}{4}+\frac{1}{4(1+\exp((-4x+4y-t)Re/32))}
\end{aligned} \tag{21}$$

that can be generated by using the Hopf–Cole transformation [9].

The initial and boundary conditions are taken from the exact solutions and the computational domain is $D=\{(x,y):0\le x\le 1,\ 0\le y\le 1\}$.

In numerical computations present by Tables 1, 2, 7, and 9 and Figs. 3 and 4, we used $h_x=h_y=h=0.05$ and $k=0.0001$.

Tables 1 and 2 compare the numerical solutions obtained by present method with exact solutions and numerical solutions of existing methods [4, 6, 15–18] for $Re=100$ at $t=0.5$. Table 3 shows L_2 and L_∞ error norms for $Re=200$ at $t=0.01$, Table 4 shows L_2 and L_∞ error norms for $Re=100$ at $t=1$ while Table 5 shows L_2 and L_∞ error norms for $Re=100$ at $t=2$. It is observed from Tables 3, 4 and 5 that the

Table 1 Comparison of numerical solutions and exact solutions of $u(x, y, t)$ for $r_2 = 0.0002$ at $t = 0.5$

(x, y)	Present	Exact	[4]	[6]	[15]	[16]	[17]	[18]
(0.1,0.1)	0.54320	0.54332	0.54235	0.54299	0.54412	0.54300	0.54300	0.54300
(0.5,0.1)	0.50034	0.50035	0.49964	0.50034	0.50037	0.50034	0.50034	0.50034
(0.9,0.1)	0.50000	0.50000	0.49931	0.50000	0.50000	0.50000	0.50000	0.50000
(0.3,0.3)	0.54326	0.54332	0.54207	0.54268	0.54388	0.54269	0.54270	0.54269
(0.7,0.3)	0.50032	0.50035	0.49961	0.50032	0.50037	0.50032	0.50032	0.50032
(0.1,0.5)	0.74211	0.74221	0.74130	0.74215	0.74196	0.74215	0.74215	0.74215
(0.5,0.5)	0.54315	0.54332	0.54222	0.54249	0.54347	0.54250	0.54252	0.54251
(0.9,0.5)	0.50030	0.50035	0.49997	0.50030	0.50035	0.50030	0.50030	0.50030
(0.3,0.7)	0.74202	0.74221	0.74146	0.74211	0.74211	0.74212	0.74212	0.74211
(0.7,0.7)	0.54311	0.54332	0.54243	0.54245	0.54327	0.54246	0.54247	0.54246
(0.1,0.9)	0.74994	0.74995	0.74913	0.74994	0.74994	0.74995	0.74994	0.74994
(0.5,0.9)	0.74200	0.74221	0.74201	0.74210	0.74219	0.74213	0.74210	0.74210
(0.9,0.9)	0.54300	0.54332	0.54232	0.54227	0.54333	0.54640	0.54229	0.54228

Table 2 Comparison of numerical solutions and exact solutions of $v(x, y, t)$ for $r_2 = 0.0002$ at $t = 0.5$

(x, y)	Present	Exact	[4]	[6]	[15]	[16]	[17]	[18]
(0.1,0.1)	0.95680	0.95668	0.95577	0.95701	0.95589	0.95700	0.95700	0.95700
(0.5,0.1)	0.99966	0.99965	0.99827	0.99966	0.99963	0.99966	0.99966	0.99966
(0.9,0.1)	1.00000	1.00000	0.99861	1.00000	1.00000	1.00000	1.00000	1.00000
(0.3,0.3)	0.95674	0.95668	0.95596	0.95732	0.95612	0.95731	0.95731	0.95731
(0.7,0.3)	0.99968	0.99965	0.99827	0.99968	0.99964	0.99968	0.99968	0.99968
(0.1,0.5)	0.75789	0.75779	0.75699	0.75785	0.75804	0.75785	0.75785	0.75785
(0.5,0.5)	0.95685	0.95668	0.95685	0.95751	0.95654	0.95750	0.95749	0.95749
(0.9,0.5)	0.99970	0.99965	0.99903	0.99970	0.99965	0.99970	0.99970	0.99970
(0.3,0.7)	0.75798	0.75779	0.75723	0.75789	0.75789	0.75789	0.75789	0.75789
(0.7,0.7)	0.95689	0.95668	0.95746	0.95755	0.95673	0.95754	0.95754	0.95754
(0.1,0.9)	0.75006	0.75005	0.74924	0.75006	0.75006	0.75006	0.75006	0.75006
(0.5,0.9)	0.75800	0.75779	0.75781	0.75790	0.75781	0.75787	0.75790	0.75790
(0.9,0.9)	0.95700	0.95668	0.95777	0.95773	0.95667	0.95360	0.95772	0.95772

Table 3 L_2 and L_∞ error norms for $Re = 200$ at $t = 0.01$

h	$u(x, y, t)$		$v(x, y, t)$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.1	4.16927600	1.23491700	4.16927600	1.23491700
0.05	2.07551400	0.39738000	2.07551400	0.39738000
0.025	1.00400400	0.08536495	1.00400400	0.08536495
0.0125	0.84556420	0.03902393	0.84556420	0.03902393

Table 4 L_2 and L_∞ error norms for $Re = 100$ at $t = 1$

h	$u(x, y, t)$		$v(x, y, t)$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.1	28.26181000	8.45160400	28.26181000	8.45160400
0.05	12.05747000	2.44790600	12.05747000	2.44790600
0.025	5.60840700	0.52988570	5.60840700	0.52988570
0.0125	2.60004600	0.12034020	2.60004600	0.12034020

Table 5 L_2 and L_∞ error norms for $Re = 100$ at $t = 2$

h	$u(x, y, t)$		$v(x, y, t)$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.1	4.16927600	1.23491700	4.16927600	1.23491700
0.05	2.07551400	0.39738000	2.07551400	0.39738000
0.025	1.00400400	0.08536495	1.00400400	0.08536495
0.0125	0.84556420	0.03902393	0.84556420	0.03902393

Table 6 L_2 and L_∞ error norms for $Re = 100$, $h = 0.05$ and different values of k at $t = 1$

K	$u(x,y,t)$		$v(x,y,t)$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
100	64.5753900	1.9529640	64.5753900	1.9529640
200	31.9674900	0.9765417	31.9674900	0.9765417
300	21.0373900	0.6510352	21.0373900	0.6510352
400	15.5730400	0.4882899	15.5730400	0.4882899
500	12.3033200	0.3906187	12.3033200	0.3906187
600	10.1334700	0.3255198	10.1334700	0.3255198
700	8.5932290	0.2790238	8.5932290	0.2790238
800	7.4469640	0.2441459	7.4469640	0.2441459
900	6.5640730	0.2170212	6.5640730	0.2170212
1000	5.8657820	0.1953215	5.8657820	0.1953215

Table 7 L_2 and L_∞ error norms for $u(x,y,t)$ and $v(x,y,t)$ at $t = 0.5$ and $t = 2$

Re	$t = 0.5$		$t = 2$	
	L_2	L_∞	L_2	L_∞
1000	0.5661539	0.1707022	0.4717278	0.0878201
2000	0.7819241	0.2496528	0.9674515	0.1824970
3000	0.8667907	0.2807881	1.3403390	0.2582423
4000	0.9136747	0.2973328	1.6041230	0.3136804
5000	0.9435091	0.3075829	1.7936350	0.3543252

Table 8 Comparison of numerical solutions and exact solutions of $u(x,y,t)$ and $v(x,y,t)$ for $r_2 = 0.0002$

(x,y)	$u(x,y,t)$			$v(x,y,t)$		
	Exact	Present	[14]	Exact	Present	[14]
(0.1,0.1)	0.50000	0.46597	0.50552	1.00000	1.03403	0.99448
(0.5,0.1)	0.50000	0.50019	0.50253	1.00000	0.99981	0.99747
(0.9,0.1)	0.50000	0.50000	0.50012	1.00000	1.00000	0.99988
(0.1,0.5)	0.75000	0.74985	0.74845	0.75000	0.75015	0.75154
(0.5,0.5)	0.50000	0.50146	0.51201	1.00000	0.99854	0.98799
(0.9,0.5)	0.50000	0.49934	0.49869	1.00000	1.00066	1.00131
(0.1,0.9)	0.75000	0.75000	0.74588	0.75000	0.75000	0.75413
(0.9,0.9)	0.50000	0.48056	0.50828	1.00000	1.01944	0.98119

Table 9 L_2 and L_∞ error norms for $r_2 = 0.0002$

t	$u(x,y,t)$		$v(x,y,t)$	
	L_2	L_∞	L_2	L_∞
3	5.257965×10^{-3}	1.365021×10^{-3}	5.257965×10^{-3}	1.365021×10^{-3}
6	8.884519×10^{-6}	4.757535×10^{-6}	8.884519×10^{-6}	4.757535×10^{-6}
9	7.553670×10^{-10}	4.047978×10^{-10}	7.560761×10^{-10}	4.048703×10^{-10}
12	4.207356×10^{-12}	0.902056×10^{-12}	8.763412×10^{-12}	1.964873×10^{-12}
15	1.992412×10^{-12}	0.485723×10^{-12}	4.250264×10^{-12}	0.201261×10^{-12}

Table 10 Rate of convergence for $t = 0.5$ and different values of N

$N \times N$	$u(x,y,t)$	$v(x,y,t)$
4×4	1.560036425	1.567106028
8×8	1.976098335	1.980768397
16×16	2.037723363	2.040375995
32×32	2.084522722	2.085937642
64×64	2.151895260	2.152626321

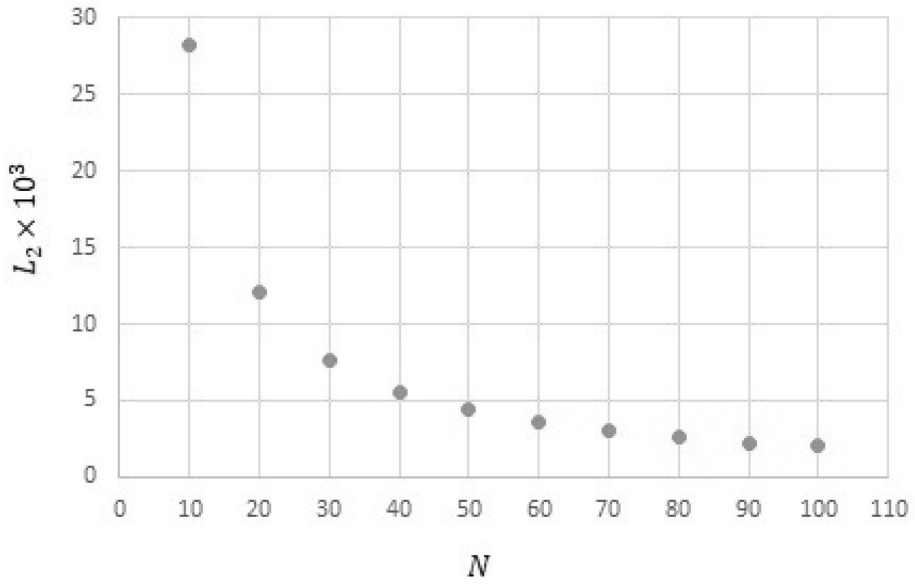


Fig. 1 L_2 error norms for different values of N at $t = 1$

values of L_2 and L_∞ error norms decrease with decrease of h. Table 6 shows L_2 and L_∞ error norms for $Re = 100$, $h = 0.05$ and different values of k at $t = 1$. It is observed from Table 6 that the values of L_2 and L_∞ error norms decrease with decrease of k. Table 7 shows L_2 and L_∞ error norms for different values of Re. To the best of the authors' knowledge, this is the first attempt that the system has been solved up to big Reynolds number $Re = 5000$. It is observed from Table 7 that the values of L_2 and L_∞ error norms increase with increase of Re. Table 8 compare the numerical solutions obtained by present method with exact solutions and numerical solutions of existing method [14] for $Re = 1000$ and $k = 0.001$ at $t = 0.5$. It can be seen from the table that the numerical solutions obtained by present method are better than the numerical solutions given by [14]. Table 9 shows L_2 and L_∞ error norms for $Re = 100$ at different times. It is observed from Table 9 that the values of L_2 and L_∞ error norms decrease as time passes. Table 10 presents rate of convergence for $Re = 100$, $k = 0.0001$ and h changes as N changes at $t = 0.5$. From the table, we observe that the present method is second order accurate in space. Figures 1 and 2 show L_2 and L_∞ error norms for $Re = 100$ and different values of N at $t = 1$, respectively. Obtained L_2 and L_∞ error norms for $u(x, y, t)$ and $v(x, y, t)$ by present method are same. It is observed from Figs. 1 and 2 that the values of L_2 and L_∞ error norms decrease with increase of N. Figures 3 and 4 show absolute errors for $Re = 100$ at $t = 0.5$ and $t = 2$, respectively. Obtained absolute errors for $u(x, y, t)$ and $v(x, y, t)$ by present method are same.

4.2 Problem 2

We consider the system of eqs. (1) and (2) over the computational domain $D = \{(x, y) : 0 \le x \le 0.5,\ 0 \le y \le 0.5\}$ with initial conditions:

$$\begin{aligned} u(x, y, 0) &= \sin(\pi x) + \cos(\pi y), \quad (x, y) \in D, \\ v(x, y, 0) &= x + y, \quad (x, y) \in D, \end{aligned} \tag{22}$$

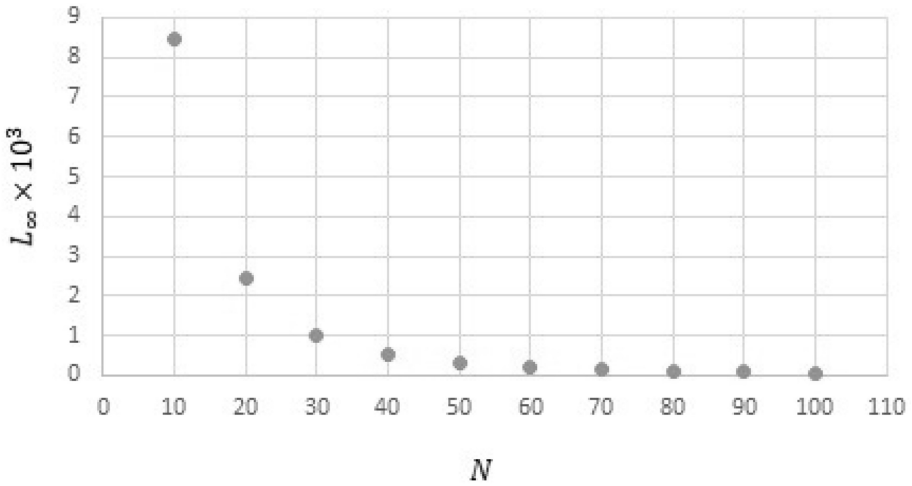
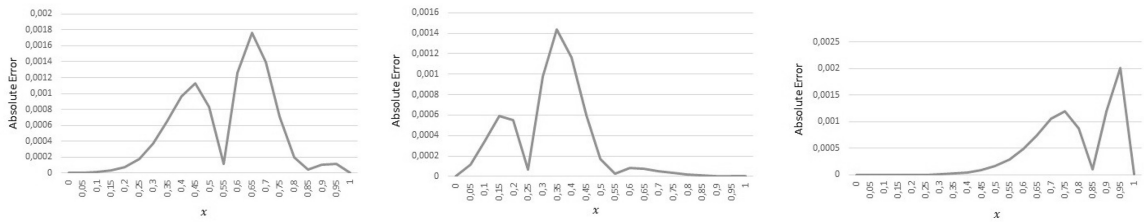
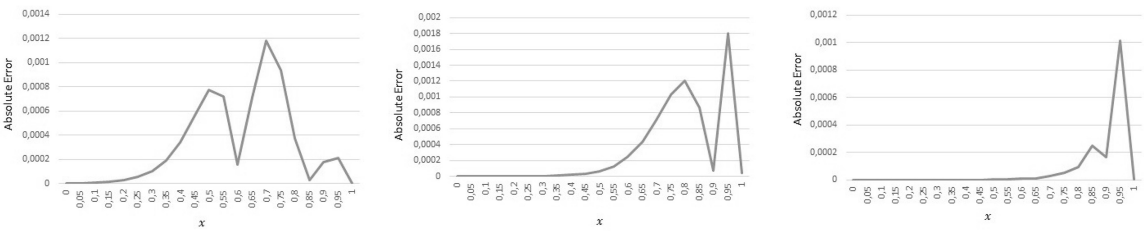


Fig. 2 L_∞ error norms for different values of N at $t = 1$



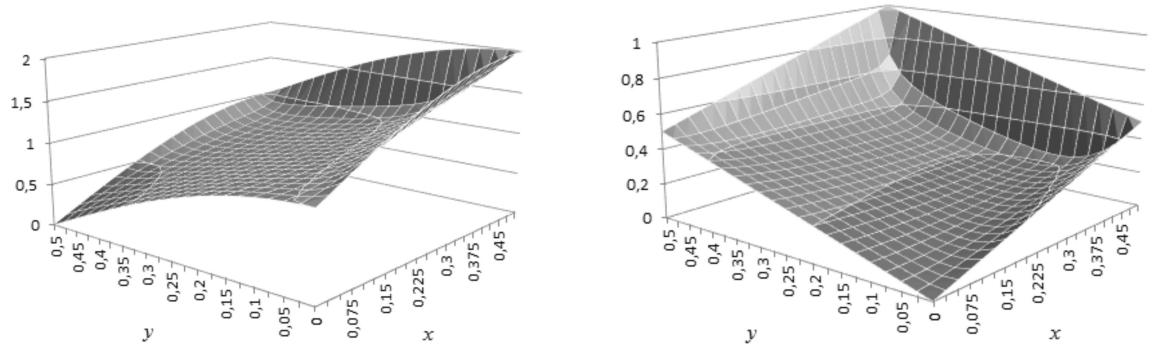
(a) $y = 0.2$ **(b)** $y = 0.5$ **(c)** $y = 0.8$

Fig. 3 Absolute errors for $u(x, y, t)$ and $v(x, y, t)$ for $r_2 = 0.0002$ at $t = 0.5$



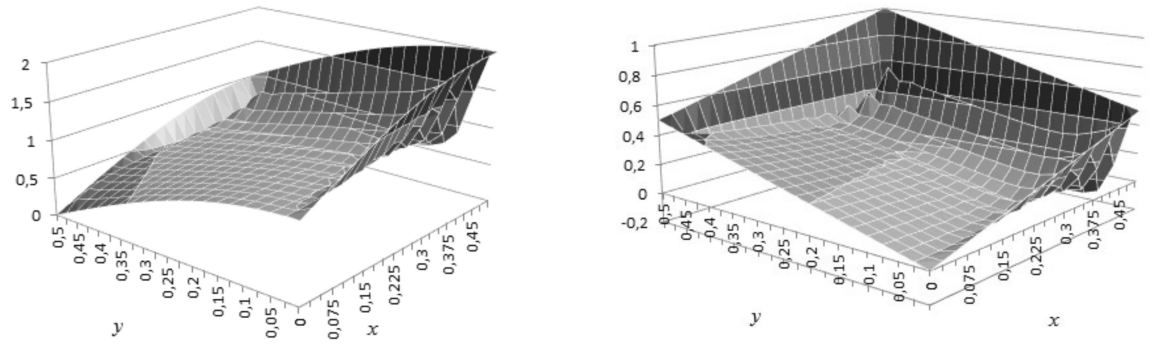
(a) $y = 0.2$ **(b)** $y = 0.5$ **(c)** $y = 0.8$

Fig. 4 Absolute errors for $u(x, y, t)$ and $v(x, y, t)$ for $r_2 = 0.0002$ at $t = 2$



(a) $u(x, y, t)$ **(b)** $v(x, y, t)$

Fig. 5 Numerical solutions for $r_2 = 0.0016$ at $t = 0.625$



(a) $u(x, y, t)$ **(b)** $v(x, y, t)$

Fig. 6 Numerical solutions for $r_2 = 0.00016$ at $t = 0.625$

and boundary conditions:

$$\begin{cases} u(0, y, t) = \cos(\pi y), \\ u(0.5, y, t) = 1 + \cos(\pi y), \\ u(x, 0, t) = 1 + \sin(\pi x), \\ u(x, 0.5, t) = \sin(\pi x), \\ v(0, y, t) = y, \\ v(0.5, y, t) = 0.5 + y, \\ v(x, 0, t) = x, \\ v(x, 0.5, t) = 0.5 + x, \end{cases} \quad t > 0. \tag{23}$$

In numerical computations present by Tables 11, 12, 13 and 14 and Figs. 5 and 6 , we used $h_x = h_y = h = 0.025$ and $k = 0.0001$. Tables 11 and 12 present the comparison of computed values of $u(x, y, t)$and $v(x, y, t)$ with those given by [4, 6, 12, 15–19] for $Re = 50$. As seen from the Tables 11 and 12, the obtained results by present method are in good agreement with the existing methods [4, 6, 12, 15–19] in the literature. The numerical solutions obtained by present method are compared with the other numerical solutions obtained by existing methods [4, 6, 12, 16, 19] for $Re = 500$ in Tables 13 and 14. It can be seen from the tables that computed values obtained by present method are in good agreement with those available in the literature. Also, we could get solutions for $k = 0.001$, Reynolds numbers $Re = 1000$ and $Re = 1200$ (Tables 15, 16). Figures 5 and 6 show $u(x, y, t)$ and $v(x, y, t)$ numerical solutions for $Re = 50, 500$ at $t = 0.625$.

Table 11 Comparison of numerical solutions of $u(x, y, t)$ for $r_2 = 0.0016$ at $t = 0.625$

(x, y)	Present	[4]	[6]	[12]	[15]	[16]	[17]	[18]	[19]
(0.1,0.1)	0.97238	0.96688	0.97146	0.97258	0.97056	0.97146	0.97146	0.97146	0.971461
(0.3,0.1)	1.15606	1.14827	1.15282	1.16214	1.15152	1.15280	1.15280	1.15280	1.152820
(0.2,0.2)	0.86497	0.85911	0.87310	0.86281	0.86244	0.86307	0.86308	0.86308	0.863072
(0.4,0.2)	0.98290	0.97637	0.97980	0.96483	0.98078	0.97981	0.97985	0.97985	0.979813
(0.1,0.3)	0.66462	0.66019	0.66316	0.66318	0.66336	0.66316	0.66316	0.66316	0.663157
(0.3,0.3)	0.77482	0.76932	0.77229	0.77030	0.77226	0.77230	0.77233	0.77233	0.772297
(0.2,0.4)	0.58403	0.57966	0.60488	0.58070	0.58273	0.58180	0.58181	0.58181	0.581799
(0.4,0.4)	0.75856	0.75678	0.75853	0.74435	0.76179	0.75856	0.75862	0.75862	0.758558

Table 12 Comparison of numerical solutions of $v(x, y, t)$ for $r_2 = 0.0016$ at $t = 0.625$

(x, y)	Present	[4]	[6]	[12]	[15]	[16]	[17]	[18]	[19]
(0.1,0.1)	0.09834	0.09824	0.09869	0.09773	0.09842	0.09869	0.09869	0.09869	0.098688
(0.3,0.1)	0.14170	0.14112	0.14158	0.14039	0.14107	0.14158	0.14158	0.14158	0.141582
(0.2,0.2)	0.16672	0.16681	0.16754	0.16660	0.16732	0.16754	0.16754	0.16754	0.167542
(0.4,0.2)	0.17002	0.17065	0.17109	0.17397	0.17223	0.17110	0.17111	0.17111	0.171095
(0.1,0.3)	0.26305	0.26261	0.26378	0.26294	0.26380	0.26378	0.26378	0.26378	0.263781
(0.3,0.3)	0.22495	0.22576	0.22654	0.22463	0.22653	0.22654	0.22655	0.22655	0.226539
(0.2,0.4)	0.32579	0.32745	0.32851	0.32402	0.32935	0.32851	0.32851	0.32851	0.328508
(0.4,0.4)	0.31815	0.32441	0.32499	0.31822	0.32884	0.32500	0.32502	0.32502	0.324997

Table 13 Comparison of numerical solutions of $u(x, y, t)$ for $r_2 = 0.00016$ at $t = 0.625$

(x, y)	Present	[4]	[6]	[12]	[16]	[19]
(0.15,0.1)	0.96873	0.96650	0.96870	0.95691	0.96870	0.968969
(0.3,0.1)	1.03595	1.02970	1.03202	0.95616	1.03200	1.032020
(0.1,0.2)	0.84741	0.84449	0.84619	0.84257	0.86178	0.846187
(0.2,0.2)	0.88109	0.87631	0.87814	0.86399	0.87814	0.878141
(0.1,0.3)	0.68095	0.67809	0.67920	0.67667	0.67920	0.679202
(0.3,0.3)	0.80671	0.79792	0.79947	0.76876	0.79947	0.799471
(0.15,0.4)	0.54785	0.54601	0.54674	0.54408	0.66036	0.546743
(0.2,0.4)	0.59115	0.58874	0.58959	0.58778	0.58959	0.589589

Table 14 Comparison of numerical solutions of $v(x, y, t)$ for $r_2 = 0.00016$ at $t = 0.625$

(x, y)	Present	[4]	[6]	[12]	[16]	[19]
(0.15,0.1)	0.09014	0.09020	0.09043	0.10177	0.09043	0.092303
(0.3,0.1)	0.11131	0.10690	0.10728	0.13287	0.10728	0.107275
(0.1,0.2)	0.17981	0.17972	0.18010	0.18503	0.17295	0.180103
(0.2,0.2)	0.16878	0.16777	0.16816	0.18169	0.16816	0.168157
(0.1,0.3)	0.26215	0.26222	0.26268	0.26560	0.26268	0.262677
(0.3,0.3)	0.23938	0.23497	0.23550	0.25142	0.23550	0.235501
(0.15,0.4)	0.31804	0.31753	0.31799	0.32084	0.29019	0.317991
(0.2,0.4)	0.30460	0.30371	0.30419	0.30927	0.30419	0.304187

5 Stability Analysis

To investigate the stability of schemes (7)–(8) and (11)–(12), we will use the von Neumann stability analysis in which the growth factor of a typical Fourier mode is defined as follows:

$$u_{i,j}^n = u_{p,q}^n = \gamma^n e^{i\alpha ph} e^{i\beta qh} \tag{24}$$

where $i = \sqrt{-1}$.

von Neumann stability analysis is used to analyze the stability of finite difference schemes applied to linear partial differential equations. In the presented method, local constants can be written instead of

Table 15 Comparison of numerical solutions of $u(x,y,t)$ and $v(x,y,t)$ for $Re = 1000$, $k = 0.001$ and different values of h at $t = 0.625$

(x,y)	$u(x,y,t)$			$v(x,y,t)$		
	$h = 0.025$	$h = 0.0125$	$h = 0.00625$	$h = 0.025$	$h = 0.0125$	$h = 0.00625$
(0.15,0.1)	1.02515	0.96241	0.96217	0.12262	0.08620	0.08633
(0.3,0.1)	1.19681	0.97426	0.96945	0.19975	0.07824	0.07608
(0.3,0.3)	0.91651	0.77841	0.77641	0.32704	0.21689	0.21602
(0.15,0.4)	0.55907	0.54967	0.54946	0.35156	0.31431	0.31437
(0.2,0.4)	0.61874	0.59015	0.58976	0.35706	0.29856	0.29866

Table 16 Comparison of numerical solutions of $u(x,y,t)$ and $v(x,y,t)$ for $Re = 1200$, $k = 0.001$ and different values of h at $t = 0.625$

(x,y)	$u(x,y,t)$			$v(x,y,t)$		
	$h = 0.025$	$h = 0.0125$	$h = 0.00625$	$h = 0.025$	$h = 0.0125$	$h = 0.00625$
(0.15,0.1)	1.05304	0.96291	0.96238	0.13857	0.08638	0.08633
(0.3,0.1)	1.25335	0.98048	0.96975	0.23047	0.08148	0.07607
(0.3,0.3)	0.95702	0.78128	0.77670	0.36005	0.21898	0.21604
(0.15,0.4)	0.56746	0.54986	0.54974	0.36885	0.31448	0.31434
(0.2,0.4)	0.63468	0.59046	0.59002	0.38203	0.29886	0.29865

convective coefficients, since the known values of convective coefficients from the previous time step are used in each time step. Therefore, in order to analyze the stability of the numerical schemes (7)–(8), the nonlinear terms uu_x and vu_y in the two-dimensional Burgers equation (1) have been linearized by replacing the quantities u and v by local constants. Thus the nonlinear terms uu_x and vu_y convert into $\hat{U}u_x$ and $\hat{V}u_y$ in equation (1) and in that case eq. (1) turns into

$$u_t + \hat{U}u_x + \hat{V}u_y = \frac{1}{Re}\left(u_{xx} + u_{yy}\right). \tag{25}$$

If we write the present scheme for this linearized equation, newly approximations are taken as:

$$\begin{aligned} &\left(-r_1\widehat{V} - r_2\right)u_{i,j-1}^{n+\frac{1}{2}} + (1+2r_2)u_{i,j}^{n+\frac{1}{2}} + \left(r_1\widehat{V} - r_2\right)u_{i,j+1}^{n+\frac{1}{2}} \\ &\quad = \left(r_1\widehat{U} + r_2\right)u_{i-1,j}^{n} + (1-2r_2)u_{i,j}^{n} + \left(-r_1\widehat{U} + r_2\right)u_{i+1,j}^{n} \end{aligned} \tag{26}$$

$$\begin{aligned} &\left(-r_1\widehat{U} - r_2\right)u_{i-1,j}^{n+1} + (1+2r_2)u_{i,j}^{n+1} + \left(r_1\widehat{U} - r_2\right)u_{i+1,j}^{n+1} \\ &\quad = \left(r_1\widehat{V} + r_2\right)u_{i,j-1}^{n+\frac{1}{2}} + (1-2r_2)u_{i,j}^{n+\frac{1}{2}} + \left(-r_1\widehat{V} + r_2\right)u_{i,j+1}^{n+\frac{1}{2}} \end{aligned} \tag{27}$$

then substituting the (24) equality into the newly numerical schemes (26) and (27) we get the following growth factors:

$$\begin{aligned} \frac{\gamma^{n+\frac{1}{2}}}{\gamma^{n}} &= \frac{1-\alpha_1}{1+\alpha_2} \\ \frac{\gamma^{n+1}}{\gamma^{n+\frac{1}{2}}} &= \frac{1-\alpha_2}{1+\alpha_1} \end{aligned} \tag{28}$$

where

$$\begin{aligned} \alpha_1 &= 4r_2\sin^2\frac{\alpha h}{2} + i\left(2r_1\hat{U}\sin\alpha h\right) \\ \alpha_2 &= 4r_2\sin^2\frac{\beta h}{2} + i\left(2r_1\hat{V}\sin\beta h\right). \end{aligned} \tag{29}$$

Hence

$$\frac{\gamma^{n+1}}{\gamma^n} = \frac{1-\alpha_2}{1+\alpha_1}\frac{1-\alpha_1}{1+\alpha_2}. \tag{30}$$

The growth factor from n to $n+1$ must be less than 1 in absolute value for unconditionally stability.

$$\left|\frac{\gamma^{n+1}}{\gamma^n}\right| = \sqrt{\frac{1+a_1^2-2a_1+b_1^2}{1+a_1^2+2a_1+b_1^2}}\sqrt{\frac{1+a_2^2-2a_2+b_2^2}{1+a_2^2+2a_2+b_2^2}} \tag{31}$$

where $a_1 = 4r_2\sin^2\frac{\alpha h}{2}$, $b_1 = 2r_1\hat{U}\sin\alpha h$, $a_2 = 4r_2\sin^2\frac{\beta h}{2}$ and $b_2 = 2r_1\hat{V}\sin\beta h$. Since $a_1, a_2 \geq 0$ it can be seen that $\left|\frac{\gamma^{n+1}}{\gamma^n}\right| \leq 1$. Hence schemes (7) and (8) are unconditionally stable.

Similarly we can prove the unconditional stability of the numerical schemes (11) and (12).

6 Conclusion

A new approximation is presented for numerical solutions of the system of two-dimensional Burgers equations. Two test problems are used to illustrate the accuracy of the present method. Comparisons are made with the existing methods in the literature. The method is analyzed by von-Neumann stability analysis method and it is displayed that the method is unconditionally stable. The method is shown to be consistent and second order accurate in time and space. The obtained results are acceptable and compatible with the results available in the literature. The obtained results show that the present method is a successful method to solve the system of two-dimensional Burgers equations. The present method can be applied without having to solve large systems; it is a simple solution process. The proposed algorithm is reasonable, easy to implement and reliable. Therefore, the method used in this study can strongly be advised to obtain approximate solutions of several other systems of two-dimensional non-linear equations commonly used in the literature.

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