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ORIGINAL RESEARCH

Wiener index of generalized odd complete graphs

KM. Kathiresan · C. Parameswaran

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Abstract The Wiener index $W(G)$ of a graph G is the sum of distances between all pairs of vertices of a connected graph. Motivated by the definition of an odd graph, in this paper we define generalized odd complete graphs and we obtain an explicit expression for the Wiener index of the generalized odd complete graphs and hence we derive the Wiener index of complete graphs with odd number of vertices.

Keywords Generalized odd complete graphs · Wiener index · odd graphs

Mathematics Subject Classification 05C12

1 Introduction

Let $G = (V, E)$ be a simple connected undirected graph with $|V(G)| = n$ and $|E(G)| = m$. Given two vertices u, v of G, let $d(u, v)$ denote the distance between u and v. The Wiener index $W(G)$ of the graph G is defined by $W(G) = \sum_{\{u,v\}} d(u, v)$, where the sum is taken over all unordered pairs $\{u, v\}$ of distinct vertices in G. The Wiener index was first proposed by Harold Wiener [7] as an aid to determine the boiling point of paraffin. Since then, the index has been shown to correlate with a host of other properties of molecules (viewed as graphs). For more information about the Wiener index in Chemistry and Mathematics see [6] and [2], respectively. Two recent survays on the topic are [3] and [4].

For a positive integer $k \geq 2$, let X be any set of cardinality $2k - 1$ and V, the collection of all $(k - 1)$-subsets of X. The odd graph O_k has V as its vertex set, and two vertices of O_k are adjacent if and only if the corresponding $(k-1)$-subsets are disjoint. It is well known that O_3 is the Petersen graph. Odd graphs have found applications in the design of interconnection networks for high performance parallel computing systems [5]. Recently, Balakrishnan et al. [1] derived an explicit expression for the Wiener index of an odd graph. Motivated by the definition of an odd graph, we define a new family of graphs namely generalized odd complete graphs and we obtain an explicit expression for the Wiener index of generalized odd complete graphs and hence we deduce the Wiener index of complete graphs with odd number of vertices.

K. M. Kathiresan · C. Parameswaran (✉)
Centre for Research and Post Graduate Studies in Mathematics, Ayya Nadar Janaki Ammal College, Sivakasi, Tamil Nadu 626124, India
E-mail: kathir2esan@yahoo.com

C. Parameswaran
E-mail: parames65_c@yahoo.com

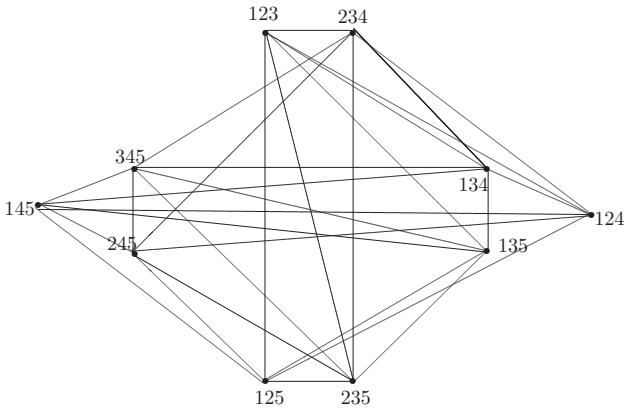

Fig. 1 Graph of $G_{2,2}$

2 Generalized odd complete graphs

Definition 2.1 For a positive integer k, fix an integer n such that $1 \leq n \leq 2k-1$. Let X be any set of cardinality $2k+1$ and V, the collection of all $(n+1)$-subsets of X. The graph $G_{k,n}$ has V as its vertex set and two vertices are adjacent if and only if the cardinality of the intersection of the corresponding $(n+1)$ subsets is n (Fig. 1).

Example 2.2 Let $k = 2$ and $n = 2$. Let $X = \{1, 2, 3, 4, 5\}$. The graph of $G_{2,2}$ is given below. Here abc means the set $\{a, b, c\}$.

Lemma 2.3 *Fix $A_0 \in V(G_{k,n})$. Then for any $A \in V(G_{k,n})$, the distance $d(A_0, A) = i$ if and only if $|A_0 \cap A| = n - i + 1$.*

Proof If $k = 1$, then $G_{k,n} = C_3$ and the result is true. So assume that $k \geq 2$. We prove the result by induction on $d(A_0, A)$. Now $d(A_0, A) = 0$ if and only if $A_0 = A$ and $d(A_0, A) = 1$ if and only if $|A_0 \cap A| = n$. Hence the result is true for $i = 0, i = 1$.

Now consider the case when $d(A_0, A) = 2$. Then there exists $A_1 \in V$ such that A_0A_1A is a path of length 2 in $G_{k,n}$. This means that $|A_0 \cap A_1| = n = |A_1 \cap A|$. Now $|A_0 \cup A| = n + 3$, since A_0 and A differ only by n elements. Therefore $|A_0 \cap A| = |A_0| + |A| - |A_0 \cup A| = (n+1) + (n+1) - (n+3) = n - 1$. Conversely assume that $|A_0 \cap A| = n - 1$. Set $Y = A_0 \cap A$. Then $A_0 = Y \cup \{y_0, y_1\}$ and $A = Y \cup \{x_0, x_1\}$ for distinct $x_0, x_1, y_0, y_1 \in X - Y$. Let $B = Y \cup \{y_0, x_0\}$. Then $B \in V(G_{k,n})$ and $|A \cap B| = n = |A_0 \cap B|$. Hence A_0BA is a path of length 2 in $G_{k,n}$ and hence $d(A_0, A) \leq 2$. But $|A_0 \cap A| \neq n$ and so A is non-adjacent to A_0 in $G_{k,n}$. Thus $d(A_0, A) = 2$.

Therefore $d(A_0, A) = 2$ if and only if $|A_0 \cap A| = n - 1$. So assume that $d(A_0, A) = i > 2$ and the result is true if $d(A_0, A) \leq i - 1$. There exists vertices $A_1, A_2, \ldots, A_{i-1}$ of $G_{k,n}$ such that $A_0A_1A_2 \ldots A_{i-1}A$ is a path of length i in $G_{k,n}$. As $d(A_0, A_{i-1}) = d(A_1, A) = i - 1$, by our induction assumption, $|A_0 \cap A_{i-1}| = |A_1 \cap A| = n - (i-1) + 1 = n - i + 2$. Now $|A_0 \cap A| = |A_0 \cap A_{i-1}| - 1 = n - i + 2 - 1 = n - i + 1$.

Conversely assume that

$$|A_0 \cap A| = n - i + 1 \tag{1}$$

Set $Y = A_0 \cap A$. Let $A_0 = Y \cup \{a_1, a_2, \ldots, a_i\}$ and $A = Y \cup \{b_1, b_2, \ldots, b_i\}$ where $\{a_1, a_2, \ldots, a_i\} \cap \{b_1, b_2, \ldots, b_i\} = \emptyset$. Let $Z_0 = Y \cup \{a_1, b_2, \ldots, b_i\}$. Then $|Z_0| = n - i + 1 + i = n + 1$. Therefore $Z_0 \in V(G_{k,n})$. Further $|A_0 \cap Z_0| = n - i + 2$ and $|A \cap Z_0| = (n - i + 1) + (i - 1) = n$. By assumption $d(A_0, Z_0) = i - 1, d(A, Z_0) = 1$. Therefore $d(A_0, A) \leq d(A_0, Z_0) + d(Z_0, A) = i - 1 + 1 = i$. By (1) and by induction hypothesis, $d(A_0, A) \geq i$.

Therefore $d(A_0, A) = i$. □

The following table shows the distance $d(A_0, A)$ and the corresponding $|A_0 \cap A|$.

when $n < k$		when $n \geq k$	
$d(A_0, A)$	$\lvert A_0 \cap A\rvert$	$d(A_0, A)$	$\lvert A_0 \cap A\rvert$
0	$n+1$	0	$n+1$
1	n	1	n
2	$n-1$	2	$n-1$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
n	1	$2k-n-1$	$2n-2k+2$
$n+1$	0	$2k-n$	$2n-2k+1$

From the above table we have the following lemma

Lemma 2.4 $diam(G_{k,n}) = \begin{cases} n+1 & if\, n < k \\ 2k-n & if\, n \geq k \end{cases}$

Remark 2.5 Fix $A_0 \in V(G_{k,n})$. Then the number of vertices of B $G_{k,n}$ such that $|A_0 \cap B| = i$ is $\binom{n+1}{j}\binom{2k-n}{n-j+1}$.

Theorem 2.6

$$W(G_{k,n}) = \begin{cases} \frac{1}{2}\binom{2k+1}{n+1}\left[\sum_{j=0}^{n}(n-j+1)\binom{n+1}{j}\binom{2k-n}{n-j+1}\right] & if\, n < k \\ \frac{1}{2}\binom{2k+1}{n+1}\left[\sum_{j=2n-2k+j}^{n}(n-j+1)\binom{n+1}{j}\binom{2k-n}{n-j+1}\right] & if\, n \geq k \end{cases}$$

Proof By Remark 2.5, the number of vertices B of $G_{k,n}$ such that $|A_0 \cap B| = j$ is $\binom{n+1}{j}\binom{2k-n}{n-j+1}$. If $n < k$, the distance from a fixed vertex A_0 to all other vertices is $\sum_{j=0}^{n}(n-j+1)\binom{n+1}{j}\binom{2k-n}{n-j+1}$. If $n \geq k$, the distance from a fixed A_0 to all other vertices is

$$\sum_{j=2n-2k+1}^{n}(n-j+1)\binom{n+1}{j}\binom{2k-n}{n-j+1}.$$

There are $\binom{2k+1}{n+1}$ vertices in G. Therefore

$$W(G_{k,n}) = \begin{cases} \frac{1}{2}\binom{2k+1}{n+1}\left[\sum_{j=0}^{n}(n-j+1)\binom{n+1}{j}\binom{2k-n}{n-j+1}\right] & \text{if } n < k \\ \frac{1}{2}\binom{2k+1}{n+1}\left[\sum_{j=2n-2k+1}^{n}(n-j+1)\binom{n+1}{j}\binom{2k-n}{n-j+1}\right] & \text{if } n \geq k \end{cases}$$

□

Corollary 2.7 $W(K_{2k+1}) = W(G_{k,2k-1}) = \binom{2k+1}{2}$.

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