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ORIGINAL RESEARCH

Equitable distinguishing chromatic number

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Abstract We introduce the equitable distinguishing chromatic number $\chi_{ED}(G)$ of a graph G as the least number k such that G has an equitable coloring with k colors that is only preserved by the trivial automorphism. The equitable distinguishing chromatic number of some graphs such as paths, cycles, trees, and Cartesian products of two complete graphs are determined.

Keywords Distinguishing chromatic number · Graph automorphism · Equitable coloring · Equitable distinguishing chromatic number

Mathematics Subject Classification 05C15 · 05C25

1 Introduction

Albertson and Collins [1] introduced the distinguishing number of a graph. Let G be an undirected simple graph and let r be a positive integer. A coloring $h : V(G) \to \{1, \ldots, r\}$ of the vertices of G is said to be *r-distinguishing* provided that no nontrivial automorphism of G preserves all of the vertex color. The *distinguishing number* of G, denoted by $D(G)$, is the smallest integer r such that G has an r-distinguishing coloring.

Collins and Trenk [10] studied some properties of distinguishing chromatic number of graphs. The *distinguishing chromatic number* of a graph G, denoted by $\chi_D(G)$, is the minimum r such that G has a proper r-distinguishing coloring. For more results in this direction, the interested reader may like to consult [4–7, 9, 15, 18].

On the other hand, the notion of the equitable colorability was introduced by Meyer [21]. A graph $G = (V, E)$ is said to be *equitably k-colorable*, where k is a positive integer, if V may be partitioned into independent sets $V_1, \ldots, V_k$ such that $||V_i| - |V_j|| \leq 1$ for any i and j with $1 \leq i, j \leq k$ and $i \neq j$. The *equitable chromatic number* of G, denoted by $\chi_e(G)$, is defined as the smallest k such that G is equitably k-

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colorable. Some applications of equitable coloring such as scheduling and constructing timetables were studied in [16, 17, 20].

In this paper, for a given graph G, we introduce the concept of equitable distinguishing chromatic number of G, denoted by $\chi_{ED}(G)$ and determine $\chi_{ED}(G)$ when G is a complete multipartite graph, path, cycle, or tree. We also calculate the equitable distinguishing chromatic number of the Kneser graph $K(n, 2)$ and the equitable distinguishing chromatic number of the Cartesian products of two complete graphs.

2 The equitable distinguishing chromatic number of complete multipartite graphs, paths, and cycles

As mentioned in Introduction, in this paper, we study the concept of equitable distinguishing chromatic number of some classes of graphs. So we begin with the following definition.

Definition 2.1 A coloring of the vertices of a graph G is said to be *proper equitably r-distinguishing* if it is proper r-distinguishing and G is equitably r-colorable. The *equitable distinguishing chromatic number* of G, denoted by $\chi_{ED}(G)$, is the minimum r such that G has a proper equitable r-distinguishing coloring.

It is clear that $\chi_{ED}(G) \geq \max\{\chi_D(G), \chi_e(G)\}$. As we show in the following example, there exist some graphs that their distinguishing chromatic numbers are not equal to their equitable distinguishing chromatic number.

Example 2.2 Consider the graph G that is colored in Figure 1 with six colors. Clearly, this coloring is distinguishing and there is no proper coloring with five colors which is distinguishing. Hence $\chi_D(G) = 6$. Note that this coloring is not equitably distinguishing 6-colorable.

If we use seven colors, then the graph G that is colored in Figure 2, is distinguishing but it is not equitably distinguishing 7-colorable. Note that there is no proper coloring by using seven colors which is equitably distinguishing.

If we use eight colors, then G is equitably distinguishing 8-colorable as in Figure 3. Hence $\chi_{ED}(G) = 8$.

Proposition 2.3 *Let G be a graph. Then $\chi_{ED}(G) = |V(G)|$ if and only if G is a complete multipartite graph.*

Proof Let G be a complete multipartite graph. Then [10, Theorem 2.3] implies that $\chi_D(G) = |V(G)|$. Since $\chi_{ED}(G) \geq \chi_D(G)$, we have $\chi_{ED}(G) = |V(G)|$.

Conversely, assume that G is not a complete multipartite graph and $|V(G)| = n$. Since K_n is a complete n-partite graph, G has nonadjacent vertices. Furthermore, clearly, G must have a pair of nonadjacent vertices, x and y with different neighborhoods. Assign color 1 to both vertices x and y, and color the remaining vertices with $2, 3, \ldots, n-1$, where each gets a distinct color. This is a proper coloring which is equitable $(n-1)$-distinguishing. Therefore $\chi_{ED}(G) \leq n-1$. □

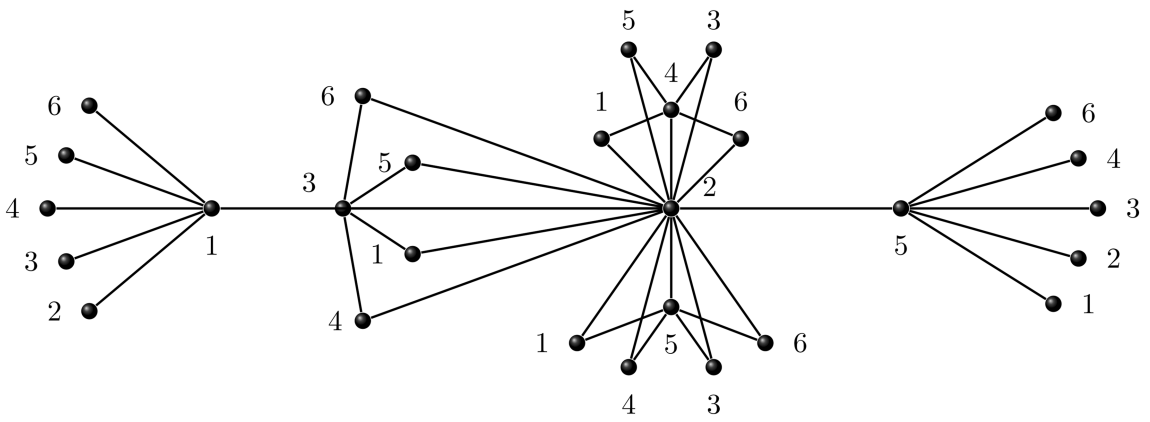

Fig. 1 A proper distinguishing coloring of graph G with six colors which is not equitabely distinguishing 6-colorable

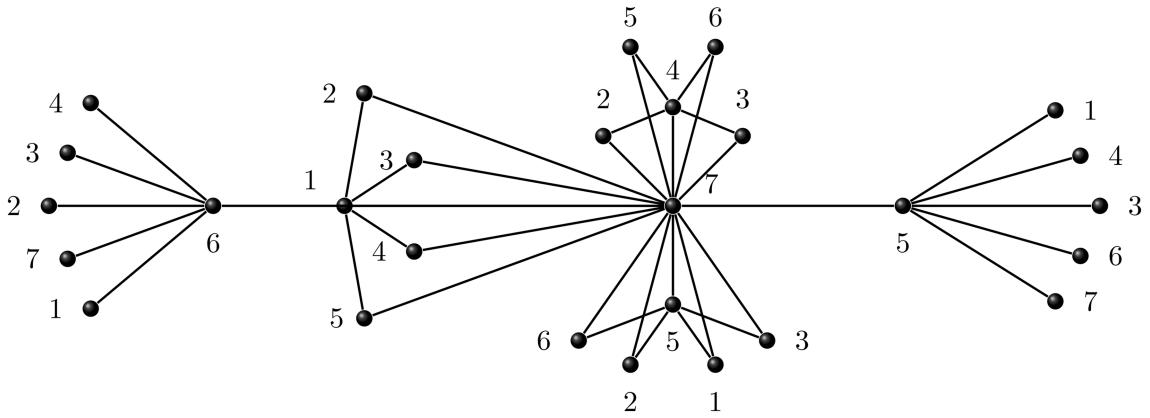

Fig. 2 A proper distinguishing coloring of graph G with seven colors which is not equitabely distinguishing 7-colorable

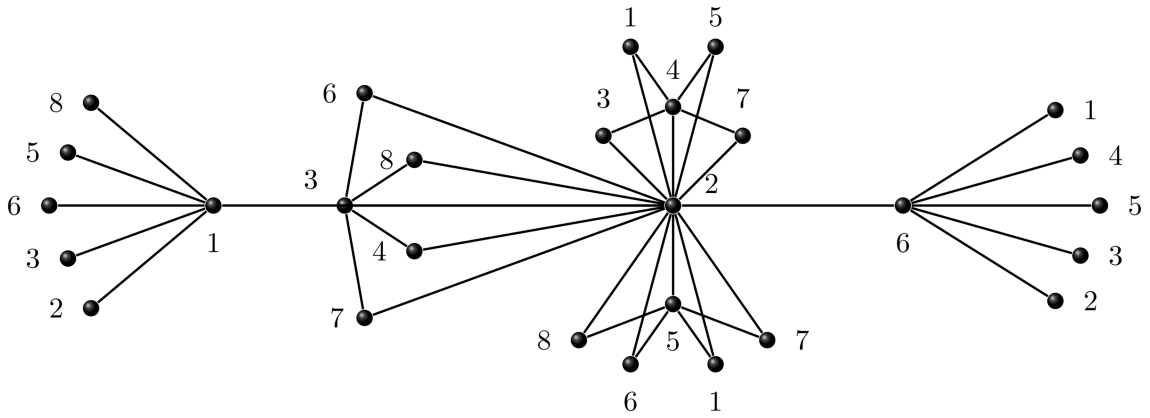

Fig. 3 A proper equitable 8-distinguishing coloring of graph G

Proposition 2.4 *If P_n is a path, then $\chi_{ED}(P_n) = 2$ for even values of $n \geq 2$ and $\chi_{ED}(P_n) = 3$ for odd values of $n \geq 3$.*

Proof Let the vertices of the path P_n be labeled consecutively as $v_1, v_2, \ldots, v_n$. If n is an even integer, then use color 1 for vertices v_i, where $i \equiv 0 \bmod 2$ and color 2 for remaining vertices. Clearly, it is a proper equitably distinguishing 2-coloring. Thus $\chi_{ED}(P_n) = 2$ for even values of n.

If n is odd, then use color 1 for vertices v_i where $i \equiv 0 \bmod 3$, color 2 for vertices v_i where $i \equiv 1 \bmod 3$ and color 3 for vertices v_i where $i \equiv 2 \bmod 3$. Hence it is a proper equitably distinguishing 3-coloring. Thus $\chi_{ED}(P_n) \leq 3$. Note that, by [10, Theorem 2.2], $\chi_D(P_n) = 3$ for odd values of n. Since $3 = \chi_D(P_n) \leq \chi_{ED}(P_n) \leq 3$, then $\chi_{ED}(P_n) = 3$. □

Proposition 2.5 *The equitable distinguishing chromatic number of cycle graphs is given in the following:*

(1) $\chi_{ED}(C_4) = \chi_{ED}(C_6) = 4$ and $\chi_{ED}(C_3) = \chi_{ED}(C_5) = 3$;
(2) $\chi_{ED}(C_n) = 3$ for any $n \geq 7$.

Proof (1) The cycle C_3 is the complete graph K_3 and C_4 is the complete multipartite graph $K_{2,2}$, so $\chi_{ED}(C_3) = 3$ and $\chi_{ED}(C_4) = 4$ by Proposition 2.3. The cycle C_5 with vertices labeled consecutively as v_1, v_2, v_3, v_4, v_5 has $\chi_{ED}(C_5) = 3$; by using color 1 for the vertices v_1 and v_4, color 2 for the vertex v_3 and color 3 for the vertices v_2 and v_5. The cycle C_6 with vertices labeled consecutively as v_1, v_2, v_3, v_4, v_5, v_6 has $\chi_{ED}(C_6) = 4$; by using color 1 for the vertex v_1, color 2 for the vertices v_2 and v_5, color 3 for the vertices v_3 and v_6 and color 4 for the vertex v_4.

(2) Assume that $n \geq 7$ and that the vertices of C_n are labeled consecutively as $v_0, v_1, \ldots, v_{n-1}$. If $n = 3k$ for some $k \in \mathbb{Z}^+$, then use color 1 for the vertices v_{3t+1} with $t = 0, 1, \ldots, k-1$. If $n = 3k + 1$ for some

$k \in \mathbb{Z}^+$, then use color 1 for the vertices v_{3t+1}, where $t = 0, 1, \ldots, \lceil \frac{n}{3} \rceil - 1$. If $n = 3k + 2$ for some $k \in \mathbb{Z}^+$, then use color 1 for the vertices v_{3t+1}, where $t = 0, 1, \ldots, \lceil \frac{n}{3} \rceil$.

Now we use color 2 for vertices v_i which are not labeled before where i is even. Also, use color 3 for vertices v_i where i is odd and v_i is not labeled in the previous cases. Note that this coloring is a proper equitably distinguishing coloring. It follows that $\chi_{ED}(C_n) \leq 3$ for $n \geq 7$. Since $3 = \chi_D(C_n) \leq \chi_{ED}(C_n) \leq 3$, we have $\chi_{ED}(C_n) = 3$ for any $n \geq 7$. □

3 The equitable distinguishing chromatic number of trees

The distinguishing number as well as the distinguishing index have been extensively studied on trees [2, 3, 9, 19]. In this section, we study the equitable distinguishing chromatic number of trees.

Following [11], we first recall some concepts about trees. A *rooted tree* (T, z) is a tree T with a distinguished vertex z which refer to as the *root*. The *depth* or *level* of a vertex v is its distance from the root, and the *height* of a rooted tree is the greatest depth in the tree. The *parent* of v is the vertex immediately following v on the unique path from v to the root. Two vertices are *siblings* if they have the same parent. The *eccentricity* of a vertex v in a graph G is the distance from v to a vertex farthest from v. The *center* of G is the subgraph induced by those vertices with minimum eccentricity. It is well known that the center of a tree is either a single vertex or an edge. A vertex v in a graph G is *pinned* if v is fixed by every automorphism that preserves colors. The notation $T_{k,h}$, for $k \geq 2$, denotes the rooted tree in which all leaves are at the same distance h from the root, and every vertex that is not a leaf has degree k.

Theorem 3.1 *Let T be a tree. Then $\chi_{ED}(T) \leq \triangle(T) + 1$. Furthermore, the equality holds if $T = T_{\triangle,h}$ for some $h \geq 0$.*

Proof If T is a single vertex, then the claim in the theorem holds. We next prove that T has a proper equitable $(\triangle + 1)$-distinguishing labeling using cases based on whether the center of T is a vertex or an edge.

First, consider the case where the center of T is a vertex v, and then let h be the height of the rooted tree (T, v). Since v is the center of T, all automorphisms of T take v to itself, so v is already pinned. Color v with color 1. We color the vertices in level 1 by using colors from the set $\{2, 3, \ldots, \triangle + 1\}$, from left to right, so that each vertex gets color which has not appeared on its siblings. By induction, we may assume that all vertices whose distance to vertex v is less than or equal to i are colored properly equitably and pinned, for $1 \leq i \leq h$, and these vertices are partitioned into $\triangle + 1$ pairwise disjoint independent sets $V_1, \ldots, V_{\triangle+1}$. Let $|V_1| = \cdots = |V_s| = a$ and let $|V_{s+1}| = \cdots = |V_{\triangle+1}| = a + 1$. Set

$$\overline{a} := \{x \in V(G) \mid x \in \cup_{j=1}^{s} V_j\} \quad \text{and}$$
$$\overline{a+1} := \{x \in V(G) \mid x \in \cup_{j=s+1}^{\triangle+1} V_j\}.$$

Assume that $A_1, \ldots, A_t$ are vertices of T which are denominated from left to right in level i. Suppose that the children of $A_1, \ldots, A_l$ $(l < t)$ are colored. It is suffices that we color the children of A_{l+1} to preserve the proper equitable distinguishing coloring.

Set $m := \deg A_{l+1} - 1$. We consider the following cases:

Case 1 In level i and after A_{l+1}, there is a vertex which has at least one child. Let A_{l+j} (for some $2 \leq j \leq t$) be the first vertex after A_{l+1} in level i which has at least one child. So we have the following subcases:

1a $A_{l+j} \in \overline{a}$. If the color of vertex A_{l+1} is deferent from the color of vertex A_{l+j}, then we color the first child of A_{l+1} the same color as A_{l+j} and for coloring the remaining children consider two situations:

1a$_1$ $|\overline{a} - A_{l+1}| \geq m$. Color the other children of A_{l+1} from the set $\{1, 2, \ldots, s\}$ such that each vertex gets the color which has not appeared on its siblings and A_{l+1}.

1a$_2$ $|\overline{a} - A_{l+1}| < m$. First, we use all the colors of elements in $\overline{a} - A_{l+1}$ for coloring children of A_{l+1} and then color $m - |\overline{a} - A_{l+1}|$ other children of A_{l+1} from the set $\{s + 1, s + 2, \ldots, \triangle + 1\}$ such that each vertex gets the color which has not appeared on its siblings. If the color of the vertex A_{l+1} is the same

as the vertex color A_{l+j}, then there are vertices between A_{l+1} and A_{l+j} which have not children. So, we have two cases:

1a$_{2,1}$ The vertex A_{l+1} or A_{l+j} has a sibling in $\overline{a+1}$. In this case, we interchange the color of A_{l+1} or A_{l+j} with the color of one of the its siblings which is in $\overline{a+1}$ and then continue the coloring as above.

1a$_{2,2}$ The vertices A_{l+1} and A_{l+j} have no sibling which is in $\overline{a+1}$. Suppose that $A_m \in \overline{a+1}$, where $l+1 \leq m < l+j$ such that the color A_m is different from the color parent of A_{l+j} and the color A_{l+j} is different from the color parent of A_m and the color sibling of A_m. In this case, interchange the color A_{l+j} with color A_m and continue the coloring as above (such as subcase 1a).

1b $A_{l+j} \in \overline{a+1}$.

We have two subcases:

1b$_1$ $|\overline{a} - A_{l+1}| \geq m$. Color all the children of A_{l+1} from the set $\{1, 2, \ldots, s\}$ such that each vertex gets the color which has not appeared on its siblings and A_{l+1}.

1b$_2$ $|\overline{a} - A_{l+1}| < m$. First, we use all the colors of elements belonging to $\overline{a} - A_{l+1}$ for coloring children of A_{l+1} and then color $m - |\overline{a} - A_{l+1}|$ other children of A_{l+1} from the set $\{s+1, s+2, \ldots, \triangle + 1\}$ such that each vertex gets the color which has not appeared on its siblings.

Case 2 In level i, vertices which are after A_{l+1} have not a child or $l+1 = t$ ($\deg A_{l+j} = 1$ for all $2 \leq j \leq t$). Let A_k $(k \leq l)$ be the first vertex in level i which has a child ($\deg A_k \geq 2$) and let A_{k1} be the first child of A_k such that $\deg A_{k1} \geq 2$ and $A_{k1} \in \overline{a}$. Then we color the first child of A_{l+1} the same as color A_{k1} and color the other children of A_{l+1} similar to Case 1. If there is no such vertex, then color all children of A_{l+1} similar to Case 1.

By [10, Lemma 3.2], this coloring is a $(\triangle + 1)$-distinguishing coloring. As vertices of T can be partitioned into independent sets $V_1, \ldots, V_{\triangle+1}$ such that for any $i \neq j$, $||V_i| - |V_j|| \leq 1$, so it is an equitable $(\triangle + 1)$-distinguishing coloring.

Finally, consider the case where the center of T is an edge, $\{x, y\}$. Any automorphism of T must either fix both x and y, or interchange them. Color x with 1 and y with 2. This pins x and y. Remove the edge between x and y, and consider x as the root of its subtree and y as the root of its subtree, and proceed to color each subtree as in the preceding case. Now we will never use the color $\triangle + 1$, since x (respectively, y) has at most $\triangle - 1$ children in its subtree. Again by [10, Lemma 3.2], this is a $\triangle$-distinguishing coloring. Thus it is an equitable $\triangle$-distinguishing coloring.

This completes the proof of $\chi_{ED}(T) \leq \triangle(T) + 1$.

For the final claim of the theorem, assume that $T = T_{\triangle,h}$ for some $h \geq 0$. Since $\chi_{ED}(T) \geqslant \max\{\chi_D(T), \chi_e(T)\}$ and, by [10, Theorem 3.4], $\chi_D(T_{\triangle,h}) = \triangle + 1$, it follows that $\chi_{ED}(T_{\triangle,h}) = \triangle + 1$. □

Remark 3.2 In the proof of the above theorem, assume that $\deg(A_{l+2}) = 1$ and $A_{l+2} \in \overline{a}$. If we color the child of A_{l+1} the same as vertex A_{l+2}, then it is possible that we do not have an equitably distinguishing coloring. To illustrate this, consider the following example.

Example 3.3 (1) In coloring of graph G in Figure 4, the color of vertex B is the same as the color assigning to vertex A, and so it is not equitably distinguishing colorable.

(2) If we use the proof of Theorem 3.1 for coloring G in part (1), then we have an equitable distinguishing coloring as we see in Figure 5.

Remark 3.4 In the proof of Theorem 3.1, for coloring the last vertex which has a child in level i, we use the first child color of the vertex in level i. Otherwise, we may not have an equitably distinguishing coloring as we show in the next example.

Example 3.5 In coloring of graph G in Figure 6, we use the color of the vertex B for coloring the child of A. It is easy to see that this coloring is not equitably distinguishing colorable.

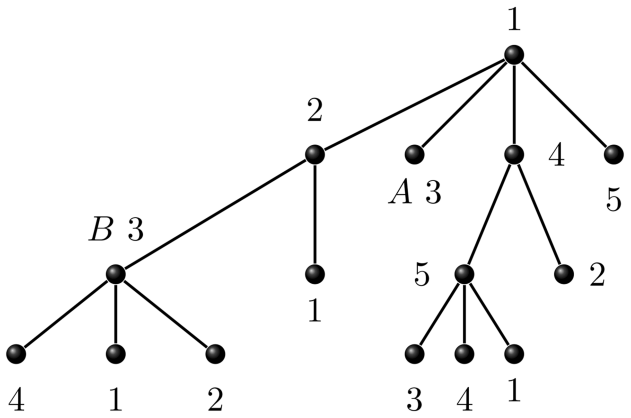

Fig. 4 Configuration used in the Example 3.3 (1)

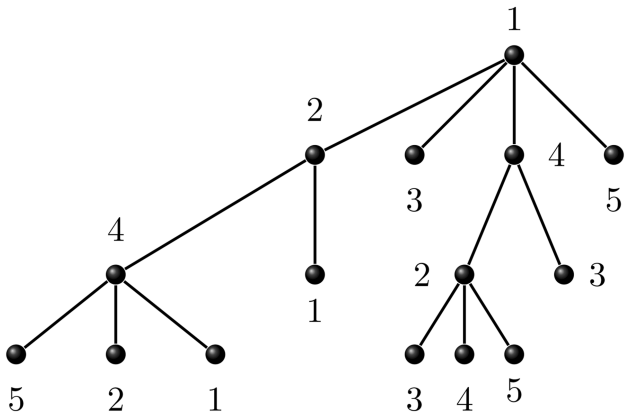

Fig. 5 Configuration used in the Example 3.3 (2)

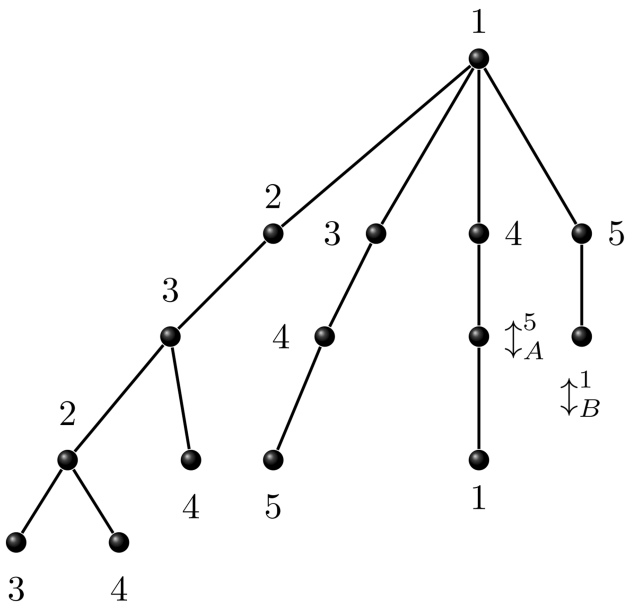

Fig. 6 Configuration used in the Example 3.5

Remark 3.6 (1) In the proof of the above theorem we have to check that the color of vertex A_{l+1} is the same as color of vertex A_{l+j} or not. In Example 3.7(1) the color A_{l+1} is the same as A_{l+j} and we have not an equitably distinguishing coloring.

(2) Let the color of vertex A_{l+1} be the same as A_{l+j}. If we interchange the color A_{l+j} or A_{l+1} with the color which we assigned to one of its siblings which is in $\overline{a}$, then we do not have an equitably distinguishing coloring as we see in Example 3.7(2).

(3) In Example 3.7(3), we interchange the color of A_{l+j} with the color which we assigned to one of its siblings which is in $\overline{a+1}$, and so we have an equitable distinguishing coloring.

Example 3.7 (1) In the graph G shown in Figure 7, the color of vertices A and D are same, and so the coloring is not an equitable distinguishing coloring.

(2) It can easily be checked that if we interchange the color of C with D in the graph G in Figure 7, then the coloring is not equitable distinguishing. Note that $D \in \overline{a}$.

(3) Let G be the graph that is shown in Figure 7. If we interchange the color of the vertex D with B of G, then resulting coloring is not equitable distinguishing (see Figure 8). Also, if we interchange the color of the vertex A with E, then we have the equitable distinguishing coloring (see Figure 9).

Remark 3.8 In the proof of the above theorem, it is possible that the vertices A_{l+1} and A_{l+j} have the same color, and also neither A_{l+j} nor A_{l+1} have any sibling in $\overline{a+1}$ (see Example 3.9(1)).

Example 3.9 (1) In coloring of the graph G that is shown in Figure 10, the color of vertices A and C are the same and neither A nor C have any sibling. Note that this coloring is not equitably distinguishing.

(2) In the graph that is shown in Figure 10, we interchange the color of the vertex B with C and we can get an equitably distinguishing coloring (see Figure 11).

4 The equitable distinguishing chromatic number of the Kneser graph $K(n, 2)$ when $n \geq 5$

Let n and k be positive integers. The Kneser graph $K(n, k)$ is the graph whose vertices correspond to the k-element subsets of a set of n elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint.

For the convenience of the reader, we introduce some notation. For positive integers i and j with $i<j$, let $[i, j]$ be the set of integers $i, i+1, \ldots, j$, and also we put $[n] = [1, n]$. If X is a set, then the collection of all k-subsets of X is denoted by $\binom{X}{k}$. Hence, the vertex set $V(K(n, k))$ is denoted by $\binom{[n]}{k}$ and $|V(K(n,k))| =$

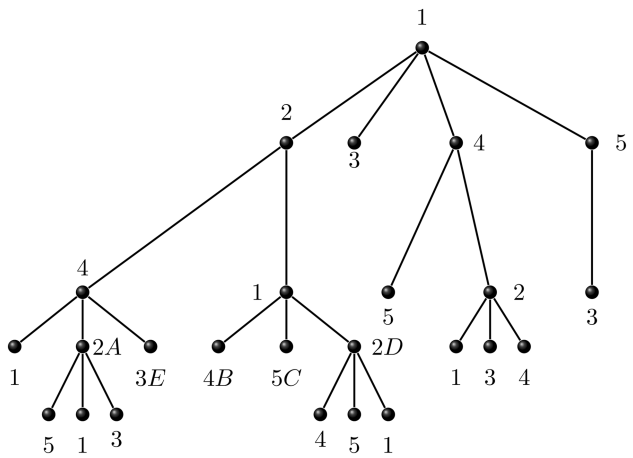

Fig. 7 Configuration used in the Example 3.7 (1)

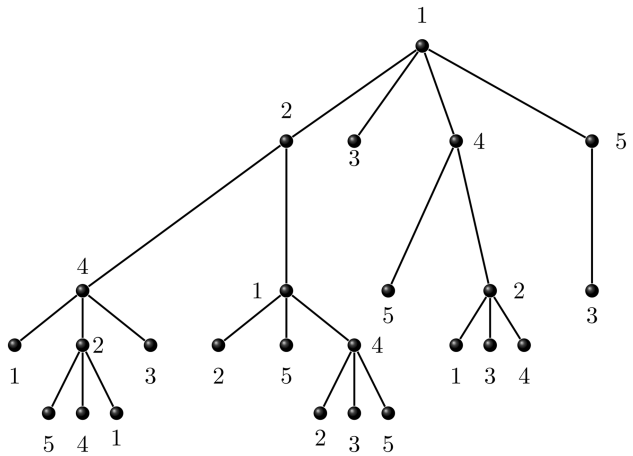

Fig. 8 Configuration used in the Example 3.7 (3)

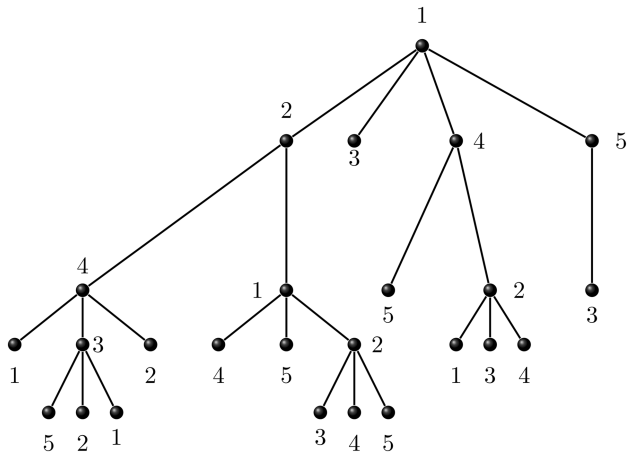

Fig. 9 Configuration used in the Example 3.7 (3)

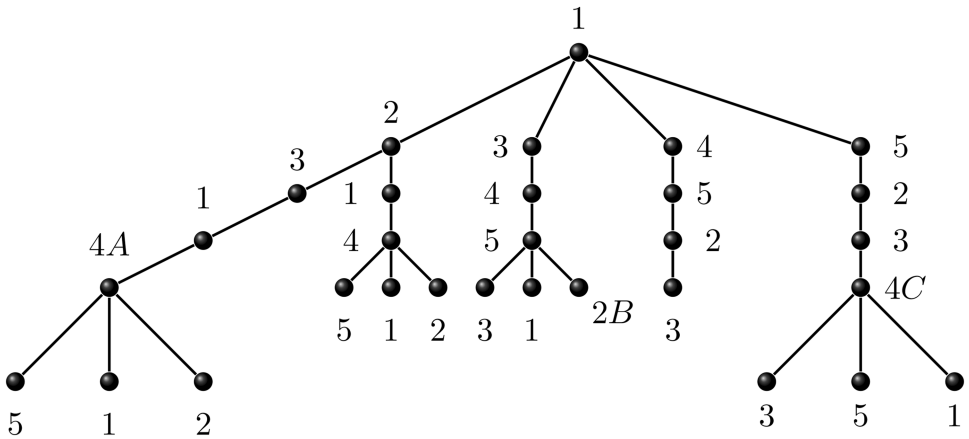

Fig. 10 Configuration used in the Example 3.9 (1)

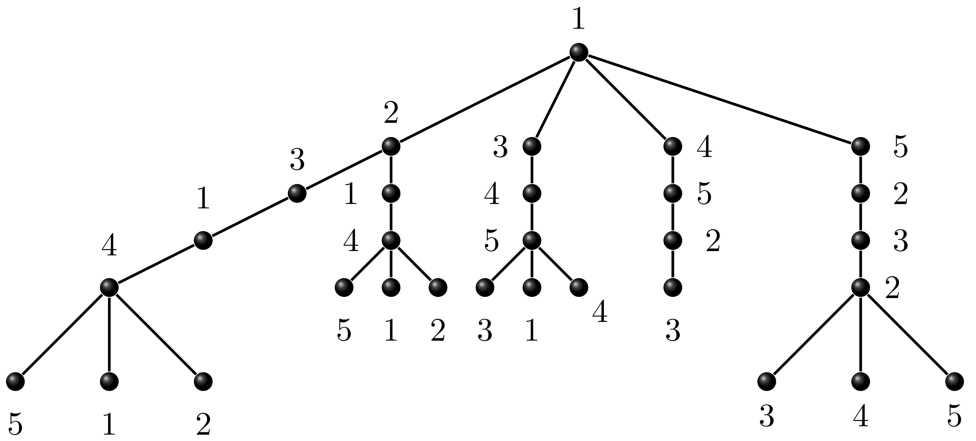

Fig. 11 Configuration used in the Example 3.9 (2)

$C(n,k) = \binom{n}{k}$. An i-flower $\mathcal{F}$ of $\binom{X}{k}$ is a subcollection of $\binom{X}{k}$ in which all k-subsets have a common element i, that is, $i \in \bigcap_{A\in\mathcal{F}} A$. It is clear that every i-flower is an independent set of $K(n, k)$. An independent set $\mathcal{F}$ of $K(n, k)$ is also called an intersection family of $\binom{[n]}{k}$, that is, $A \cap B \neq \emptyset$ for all A and B in $\mathcal{F}$.

Theorem 4.1 *Let $n \geq 5$. Then $\chi_{ED}(K(n,2)) = n - 1 = \chi(K(n,2)) + 1$.*

Proof Since $\chi_{ED}(K(n,2)) \geqslant \max\{\chi_D(K(n,2)), \chi_e(K(n,2))\}$, in view of [7, Theorem 5] and [8, Theorem 16], we have $\chi_{ED}(K(n,2)) \geq n-1$. Hence, it is enough for us to show that $\chi_{ED}(K(n,2)) \leq n-1$. We will partition $\binom{[n]}{2}$ into $n-1$ subcollections $V_1, \ldots, V_{n-1}$ with $a_i = |V_i| = \lceil \frac{C(n,2)-i+1}{n-1} \rceil$, for $1 \leq i \leq n-1$, such that V_i is contained in a $\pi(i)$-flower, where $\pi(i) = i+1$ for $1 \leq i \leq n-1$. To achieve this, we construct a bipartite graph $G = G(X, Y)$ with bipartition (X, Y), where X is the disjoint union of the sets $X_i = \{x_{ij} \mid 1 \leq j \leq a_i\}$ for $1 \leq i \leq n-1$ and $Y = \binom{[n]}{2}$. Two vertices $x_{ij} \in X$ and $A \in Y$ are adjacent if and only if $\pi(i) \in A$. It is easy to see that $|X| = |Y| = \binom{n}{2}$. We have to show that G has a perfect matching $M = \{\{x_{i,j}, A_{i,j}\} \mid 1 \leq i \leq n-1, 1 \leq j \leq a_i\}$. By [8, Theorem 6], G has a perfect matching, if $|N(S)| \geq |S|$ for all $S \subseteq X$. Hence, we need to show the inequality $|N(S)| \geq |S|$. Suppose $S \subseteq X$. Let $I(S) = \{\pi(j) \mid S \cap X_j \neq \emptyset\}$. Note that if $|I(S)| = n-1$, then $N(S) = Y$ and $|N(S)| \geq |S|$. For $|I(S)| = i \leq n-2$, let $S_i = \cup_{\pi(j)=2}^{i+1} X_j$. Then $|S| \leq |S_i|$. Moreover, the set $I(S_i) = \{\pi(j) \mid S_i \cap X_j \neq \emptyset\}$ has the same size as $I(S)$. It follows that $|N(S)| = |N(S_i)| = C(n,2) - C(n-i,2)$, and then $|N(S)| - |S| \geq |N(S_i)| - |S_i|$. If $i = n-2$, then $|S_i| = |X| - |X_{n-1}|$. Thus $|N(S_i)| - |S_i| \geq C(n,2) - 1 - (C(n,2) - \lfloor \frac{C(n,2)}{n-1} \rfloor) = \lfloor \frac{n}{2} \rfloor - 1 > 0$. If $i \leq n-3$, then $|N(S_i)| - |S_i| \geq C(n,2) - (C(n-i,2) - i\lfloor \frac{C(n,2)}{n-1} \rfloor) - i \geq \frac{i}{2}(n-i-3) \geq 0$. Therefore G has a perfect matching. Now, assume that $V_i = \{A_{ij} \mid 1 \leq j \leq a_i\}$, $1 \leq i \leq n-1$. Then the partition $V_1, \ldots, V_{n-1}$ forms an equitable $(n-1)$-coloring of $K(n, 2)$. Let ψ be a proper $(n-1)$-coloring of $K(n, 2)$ with color classes V_i for $1 \leq i \leq n-1$; then we claim that ψ is distinguishing. Note that all color classes of ψ are contained in a $\pi(i)$-flower, where $\pi(i) = i+1$. Since the automorphism group of $K(n, 2)$ is S_n by [12], we can consider an automorphism of $K(n, 2)$ as a permutation of $[n]$. Let σ be an automorphism of $K(n, 2)$ which preserves the above color classes. To preserve the color class V_i, σ fixes the integer $i+1$ ($1 \leq i \leq n-1$). It follows that σ fixes the integer 1, and so σ is trivial. □

We recall that the *Petersen graph* is the Kneser graph $K(5, 2)$.

Corollary 4.2 *If P is the Petersen graph, then $\chi_{ED}(P) = 4$.*

5 The equitable distinguishing chromatic number of Cartesian products of two complete graphs

Recall that the Cartesian product of two graphs G and H, denoted by $G\Box H$, is a graph with the vertex set $V(G) \times V(H)$ and two distinct vertices (g, h) and (g', h') are adjacent if they are equal in one coordinate and adjacent in the other. For a vertex g in G, the subgraph of $G\Box H$ induced by $g \times V(H)$ is isomorphic to H, called an *H-fiber*. Similarly *G-fibers* are defined. Let $V(K_k) = \{x_1, x_2, \ldots, x_k\}$ and let $V(K_n) = \{y_1, y_2, \ldots, y_n\}$. A d-coloring of the vertices of the graph $K_k\Box K_n$ corresponds to n by k matrix L with entries from $\{0, 1, \ldots, d-1\}$. The i, j entry of the matrix L is m whenever the vertex (x_i, y_j) in $K_k\Box K_n$ is colored with m. For $k \neq n$ every automorphism of $K_k\Box K_n$ preserves the set of K_k-fibers and the set of K_n-fibers. In this case every automorphism ϕ of $K_k\Box K_n$ is determined by a permutation $\pi \in S_n$ of K_k-fibers and a permutation $\psi \in S_k$ of K_n-fibers. Let P_π be the permutation matrix representing permutation $\pi \in S_n$ and let P_ψ be the permutation matrix representing permutation $\psi \in S_k$. Then ϕ preserves the coloring L if $L = P_\pi L P_\psi$. Moreover, L is a d-distinguishing coloring if $L = P_\pi L P_\psi$ implies $\pi = id$ and $\psi = id$. If $k = n$, then every automorphism of $K_n\Box K_n$ is generated by the automorphisms of the factors and transpositions of isomorphic factors. In this situation, ϕ preserves the coloring L if $L = P_\pi L P_\psi$ or $L = P_\pi L^T P_\psi$, where P_π and P_ψ are defined as before.

See [13, 14, 22] for further properties on Cartesian products of graphs and automorphisms in Cartesian products of graphs.

The following proposition gives the equitable distinguishing chromatic number of the Cartesian product of two complete graphs. The proof of Proposition 5.1 relies heavily on ideas in [18, Theorem 1.1], and we bring it for the convenience of the reader.

Proposition 5.1 *The following results hold:*

(1) $\chi_{ED}(K_2\Box K_2) = 4$;
(2) $\chi_{ED}(K_3\Box K_3) = \chi_{ED}(K_4\Box K_4) = 5$;
(3) *Assume that* $1 \leq k \leq n$. *Then* $\chi_{ED}(K_k\Box K_n) = n$ *for* $k = n = 1$, $k = n \geq 5$ *and* $k < n$.

Proof (1) Since $K_2\Box K_2 = C_4$, we have $\chi_{ED}(K_2\Box K_2) = 4$.

(2) In view of [18, Theorem 1.1], $\chi_D(K_3\Box K_3) = 5$, and so $\chi_{ED}(K_3\Box K_3) \geq 5$. Therefore it is enough to show that $\chi_{ED}(K_3\Box K_3) \leq 5$. Consider the matrix $L = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 4 & 0 \end{pmatrix}$ as a representation of 5-coloring of the vertices of the graph $K_3\Box K_3$. It is clear that with this labeling $K_3\Box K_3$ is equitably 5-colorable. So we need only to show that L is a 5-distinguishing labeling. Let P_{σ_i} be the permutation matrix representing permutation $\sigma_i \in S_3$ $(1 \leq i \leq 6)$. It is easy to verify that if $L = P_{\sigma_i} L P_{\sigma_j}$ or $L = P_{\sigma_i} L^T P_{\sigma_j}$, then $\sigma_i = \sigma_j = id$ for all $1 \leq i, j \leq 6$. Thus L is a 5-distinguishing coloring. Thus $\chi_{ED}(K_3\Box K_3) = 5$.

Similarly, one can show that $\chi_{ED}(K_4\Box K_4) = 5$. By [18, Theorem 1.1], $\chi_D(K_4\Box K_4) = 5$, and so $\chi_{ED}(K_4\Box K_4) \geq 5$. Now, consider the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 2 & 4 & 0 & 1 \\ 3 & 2 & 4 & 0 \end{pmatrix}$. It is straightforward to see that A is proper equitable distinguishing 5-colorable, as the previous case. Thus $\chi_{ED}(K_4\Box K_4) = 5$.

(3) Clearly $K_1\Box K_1 = K_1$, thus $\chi_{ED}(K_1\Box K_1) = 1$.

Now let $k = n \geq 6$ and let n be an even integer. We show that $\chi_{ED}(K_n\Box K_n) = n$. Suppose that $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ and define $\alpha : V(K_n\Box K_n) \to \{0, 1, \ldots, n-1\}$ as follows:

$$\alpha(x_i, x_j) = \begin{cases} n-1, & i = j, \\ i-2, & j = 1, i \neq 1, \\ j-2, & i = 1, j \neq 1, \\ (j+i-3) \ mod \ (n-1), & i < j, i \neq 1, j \neq 2, n, \\ (j+i-4) \ \text{mod} \ \ (n-1), & i > j, j \neq 1, 2, \\ (2i-4) \ mod \ (n-1), & j = 2, i \geq 3, \\ (2i-3) \ mod \ (n-1), & j = n, 2 \leq i \leq n-1. \end{cases}$$

By the proof of [18, Theorem 1.1], α is a proper n-distinguishing vertex coloring. Let A be the $n \times n$ matrix which corresponds to the coloring α. Let R_i be the set of colors from the ith row of A and let C_j be the set of colors from the jth column of A for $i,j \in \{1,2,\ldots,n\}$. Then, in view of [18, Lemma 4.1], $R_i = \{0,1,\ldots,n-1\}$ for every $1 \le i \le n$ and, by [18, Lemma 4.2], $C_j = \{0,1,\ldots,n-1\}$ for every $1 \le j \le n$. Thus, for every color class V_t, we have $|V_t| = n$ where $0 \le t \le n-1$. Hence $K_n \square K_n$ is equitably distinguishing n-colorable. Therefore $\chi_{ED}(K_n \square K_n) = n$ where $n \ge 6$ and n is an even integer.

Finally, let $k = n \ge 5$ and let n be an odd integer. We show that $\chi_{ED}(K_n \square K_n) = n$. Define $\beta : V(K_n \square K_n) \to \{0,1,\ldots,n-1\}$ as follows:

$$\beta(x_i,x_j) = \begin{cases} n-1, & i=j, \\ i-2, & j=1, i \neq 1, \\ j-2, & i=1, j \neq 1, \\ (j+i-3) \ mod \ (n-1), & i<j, i \neq 1, j \neq 2, n, \\ (j+i-4) \ mod \ (n-1), & i>j, j \neq 1, 2, \\ (2i-4) \ mod \ (n-1), & 3 \le i \le (n-1)/2, j=2 \ or \ i=n, j=2 \ or \\ & (n+1)/2 \le i \le n-1, j=n, \\ (2i-3) \ mod \ (n-1), & (n+1)/2 \le i \le n-1, j=2 \ or \\ & 2 \le i \le (n-1)/2, j=n. \end{cases}$$

By the proof of [18, Theorem 1.1], β is a proper n-distinguishing vertex coloring. Similar to the previous case, assume that B is the $n \times n$ matrix which corresponds to the coloring β. Let R_i be the set of colors from the ith row of B and let C_j be the set of colors from the jth column of B for $i,j \in \{1,2,\ldots,n\}$. By [18, Lemma 5.1], $R_i = \{0,1,\ldots,n-1\}$ for every $1 \le i \le n$ and, by [18, Lemma 5.2], $C_j = \{0,1,\ldots,n-1\}$ for every $1 \le j \le n$. It is easy to see that $K_n \square K_n$ is equitably n-colorable. Therefore $\chi_{ED}(K_n \square K_n) = n$ where $n \ge 5$ and n is an odd integer. □

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