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ORIGINAL RESEARCH

On approximating the arc lemniscate functions

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Abstract This paper deals with the arc lemniscate functions from the point view of bivariate means which have been introduced in [1]. In this study, several optimal bounds for these bivariate means in terms of arithmetic, geometric and quadratic means are established. As a consequence, new bounds for the arc lemniscate functions are also derived, which improve some previously known results.

Keywords Gauss' arc lemniscate functions · Lemniscate mean · Bivariate means · Inequalities

Mathematics Subject Classification 33E05 · 26D07

1 Introduction

Gauss' arc lemniscate sine and the hyperbolic arc lemniscate sine are defined, respectively, as follows:

$$\operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1-t^4}} \tag{1.1}$$

$(|x| \leq 1)$ and

$$\operatorname{arcslh} x = \int_0^x \frac{dt}{\sqrt{1+t^4}} \tag{1.2}$$

$(x \in \mathbb{R})$; see [2, p. 259] and [3, (2.5)-(2.6)]. As we know, the arc length measured from the origin to a point with polar coordinates on the Bernoulli lemniscate $r^2 = \cos 2\theta$ is $\operatorname{arcsl} r$. Further, the *first lemniscate constant* (c.f. [Theorem 1.7] [2] and [4, 19.20.2]) is given by

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$$\omega = \operatorname{arcsl}(1) = \frac{1}{\sqrt{2}}\mathcal{K}(1/\sqrt{2}) = \frac{[\Gamma(1/4)]^2}{4\sqrt{2\pi}} = 1.31103\cdots,$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

is the classical Euler gamma function [5–7] and

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}} = \frac{\pi}{2}F(1/2, 1/2; 1; r^2), \quad (0<r<1)$$

is the complete elliptic integral of the first kind [8–11]. Here $F(a, b; c; x)$ is the Gaussian hypergeometric function for $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$, which defined by

$$F(a, b; c) := {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \quad |x|<1,$$

where $(a, 0) = 1$ for $a \neq 0$ and $(a, n) = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the *shifted factorial function* or the *Pochhammer symbol*. For more information on these functions and recently obtained related results, we refer the reader to [12–25].

An alternative pair of the arc lemniscate functions, following [1, Proposition 3.1], Gauss' arc lemniscate tangent and the hyperbolic arc lemniscate tangent functions are defined by

$$\operatorname{arctl} x = \operatorname{arcsl}\left(\frac{x}{\sqrt[4]{1+x^4}}\right), \quad x \in \mathbb{R} \tag{1.3}$$

and

$$\operatorname{arctlh} x = \operatorname{arcslh}\left(\frac{x}{\sqrt[4]{1-x^4}}\right), \quad |x|<1, \tag{1.4}$$

respectively.

For later use, some other constants are presented as follows,

$$\kappa = \operatorname{arcslh}(+\infty) = \sqrt{2}\omega = 1.85407\cdots, \quad \sigma = \operatorname{arcslh}(1) = \omega/\sqrt{2} = 0.92703\cdots,$$
$$\tau = \operatorname{arctl}(1) = \operatorname{arcsl}(1/\sqrt[4]{2}) = 0.89558\cdots.$$

Let $a, b > 0$ with $a \neq b$. Then the geometric mean $\mathsf{G}(a, b)$, arithmetic mean $\mathsf{A}(a, b)$, quadratic mean $\mathsf{Q}(a, b)$ and lemniscate mean $\mathsf{LM}(a, b)$ [2, 3] are defined by

$$\mathsf{G}(a, b) = \sqrt{ab}, \quad \mathsf{A}(a, b) = \frac{a+b}{2}, \quad \mathsf{Q}(a, b) = \sqrt{\frac{a^2+b^2}{2}} \tag{1.5}$$

and

$$\mathsf{LM}(a, b) = \begin{cases} \dfrac{\sqrt{a^2-b^2}}{\left(\operatorname{arcsl}\sqrt[4]{1-b^2/a^2}\right)^2}, & a > b, \\ \dfrac{\sqrt{b^2-a^2}}{\left(\operatorname{arcslh}\sqrt[4]{b^2/a^2-1}\right)^2}, & a<b, \\ a, & a = b, \end{cases} \tag{1.6}$$

respectively. It is well known that $\mathsf{LM}(a, b) \neq \mathsf{LM}(b, a)$ for $a \neq b$.

A new family of bivariate means was introduced in [1, (6.4)], which is defined by

$$\mathsf{LM}_{\mathsf{X},\mathsf{Y}}(a, b) \equiv \mathsf{LM}\big(\mathsf{X}(a, b), \mathsf{Y}(a, b)\big), \tag{1.7}$$

where $\mathsf{X}(a, b)$ and $\mathsf{Y}(a, b)$ are two symmetric bivariate means of a and b. In what follows, the symbols G, A and Q will stand for the geometric, arithmetic, and the quadratic means of a and b, respectively. Let $\mathsf{X}, \mathsf{Y} \in \{\mathsf{G}, \mathsf{A}, \mathsf{Q}\}$. Then by applying (1.3)–(1.5) to (1.7), explicit formulas for

$\mathsf{LM}_{\mathsf{G},\mathsf{A}}(a,b), \mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b), \mathsf{LM}_{\mathsf{A},\mathsf{Q}}(a,b)$ and $\mathsf{LM}_{\mathsf{Q},\mathsf{A}}(a,b)$ are presented, in terms of four lemniscate functions, as follows

$$\begin{aligned} \mathsf{LM}_{\mathsf{G},\mathsf{A}}(a,b) &:= \frac{|a-b|}{2\left(\operatorname{arctlh}\sqrt{\frac{|a-b|}{a+b}}\right)^2}, \quad \mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b) := \frac{|a-b|}{2\left(\operatorname{arcsl}\sqrt{\frac{|a-b|}{a+b}}\right)^2}, \\ \mathsf{LM}_{\mathsf{A},\mathsf{Q}}(a,b) &:= \frac{|a-b|}{2\left(\operatorname{arcslh}\sqrt{\frac{|a-b|}{a+b}}\right)^2}, \quad \mathsf{LM}_{\mathsf{Q},\mathsf{A}}(a,b) := \frac{|a-b|}{2\left(\operatorname{arctl}\sqrt{\frac{|a-b|}{a+b}}\right)^2}. \end{aligned} \tag{1.8}$$

Using the well-known inequality $\mathsf{G}<\mathsf{A}<\mathsf{Q}$, it has been shown [1, (6.10)] that the inequalities

$$\mathsf{G}<\mathsf{LM}_{\mathsf{G},\mathsf{A}}<\mathsf{LM}_{\mathsf{A},\mathsf{G}}<\mathsf{A}<\mathsf{LM}_{\mathsf{A},\mathsf{Q}}<\mathsf{LM}_{\mathsf{Q},\mathsf{A}}<\mathsf{Q}. \tag{1.9}$$

Motived by (1.9), it makes sense to ask what are the optimal bounds of $\mathsf{LM}_{\mathsf{G},\mathsf{A}}(a,b)$, $\mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b)$, $\mathsf{LM}_{\mathsf{A},\mathsf{Q}}(a,b)$ and $\mathsf{LM}_{\mathsf{Q},\mathsf{A}}(a,b)$ in terms of convex combination of G,A and Q. The goal of this paper is to answer this question and present several new bounds of the arc lemniscate functions, which extend and improve some obtained results in [26, 27].

2 Preliminaries

In order to facilitate computation, we now recall the derivative formulas of the arc lemniscate functions, by the definitions and the chain rule, as follows:

$$\begin{aligned} \frac{d\operatorname{arcsl}x}{dx} &= (1-x^4)^{-\frac{1}{2}}, \quad |x|<1, \\ \frac{d\operatorname{arcslh}x}{dx} &= (1+x^4)^{-\frac{1}{2}}, \quad x\in\mathbb{R}, \\ \frac{d\operatorname{arctl}x}{dx} &= (1+x^4)^{-\frac{3}{4}}, \quad x\in\mathbb{R}, \\ \frac{d\operatorname{arctlh}x}{dx} &= (1-x^4)^{-\frac{3}{4}}, \quad |x|<1. \end{aligned}$$

Taylor series for the arc lemniscate functions have been shown in [28, Lemmas 1 and 2], which are presented in the following.

Lemma 2.1

(i) For $|x|<1$, we have

$$\operatorname{arctlh}x = \sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})(4n+1)\cdot n!}x^{4n+1}, \tag{2.1}$$

$$\operatorname{arcsl}x = \sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!}x^{4n+1}. \tag{2.2}$$

(ii) For $0<x<1$, we have

$$\operatorname{arcslh}x = \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!}x^{4n+1}, \tag{2.3}$$

$$\operatorname{arctl}x = \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(n+\frac{3}{4})}{\Gamma(\frac{3}{4})(4n+1)\cdot n!}x^{4n+1}. \tag{2.4}$$

Lemma 2.2 *Let $p\in(0,1)$ and*

$$f_p(t) = -p(1-p+p^2) + (3-7p-2p^2+2p^3)t + (3-7p+p^2-p^3)t^2, \tag{2.5}$$

$$g_p(t) = 4-8p+2p^2-p^3 + (4-5p-4p^2+2p^3)t + (1-p)(1-p+p^2)t^2. \tag{2.6}$$

Then the following statements are true:

(i) $f_p(t)<0$ for all $t \in (0,1)$ if and only if $p \geq 2/5$;
(ii) $g_p(t)<0$ for all $t \in (0,1)$ if and only if $p \geq 3/5$;
(iii) if $p<2/5$, then $f_p(t)$ is strictly increasing on (0, 1). In particular, there exists $t_1 \in (0,1)$ such that $f_p(t)<0$ for $t \in (0,t_1)$ and $f_p(t)>0$ for $t \in (t_1,1)$;
(iv) if $p \leq p_0$, then $g_p(t)>0$ for $t \in (0,1)$; if $p_0<p<3/5$, then there exists $t_2 \in (0,1)$ such that $g_p(t)<0$ for $t \in (0,t_2)$ and $g_p(t)>0$ for $t \in (t_2,1)$, where $p_0 = 0.55576\cdots$ is a unique root of $4-8p+2p^2-p^3=0$ for $0<p<3/5$.

Proof Differentiations of $f_p(t)$ and $g_p(t)$ with respect to p yield

$$\frac{\partial f_p(t)}{\partial p} = -(1-t)^2\left[2p^2+(p-1)^2\right] - 3t(2t+3) < 0,$$

$$\frac{\partial g_p(t)}{\partial p} = -(1-t)^2\left[p^2+2(p-1)^2\right] - 3(3t+2) < 0$$

for $t \in (0,1)$ and $p \in (0,1)$. This gives

$$f_p(t) \leq f_{2/5}(t) = -\frac{(1-t)(38+37t)}{125} < 0 \quad \text{if } p \geq 2/5,$$

$$g_p(t) \leq g_{3/5}(t) = -\frac{(1-t)(37+38t)}{125} < 0 \quad \text{if } p \geq 3/5.$$

On the other hand, it follows from (2.5) and (2.6) that

$$f_p(1) = 3(5p-2), \quad g_p(1) = 3(5p-3). \tag{2.7}$$

By continuity, (2.7) leads to the conclusion that there exist small enough $\varepsilon_1, \varepsilon_2 \in (0,1)$ such that $f_p(t)>0$ for $t \in (1-\varepsilon_1, 1)$ if $p<2/5$ and $g_p(t)>0$ for $t \in (1-\varepsilon_2,1)$ if $p<3/5$.

This completes the proof of (i) and (ii).

(iii) Differentiations of $f_p(t)$ gives

$$f_p'(t) = \frac{1}{8}\left[(2-5p)(11+(1-p)(1+3p))+p^3\right](1+2t) + 6p^2(1-p)t > 0$$

for $t \in (0,1)$ if $p<2/5$. In this case, $f_p(0) = -p(1-p+p^2)<0$ and (2.7) enable us to know that there exists $t_1 \in (0,1)$ such that $f_p(t)<0$ for $t \in (0,t_1)$ and $f_p(t)>0$ for $t \in (t_1,1)$;

(iv) We divide the proof into two cases.

Case A: $4-8p+2p^2-p^3 \geq 0$. In this case, we obtain

$$4-5p-4p^2+2p^3 = 4-8p+2p^2-p^3+3p(1-p)^2 > 0.$$

This in conjunction with (2.6) yields $g_p(t)>0$ for $t \in (0,1)$.

Case B: $4-8p+2p^2-p^3<0$. Then we clearly see that $g_p(0)<0$. As a quadratic function, the graph of $g_p(t)$ is an opening upwards parabola. Combining this with $g_p(1) = 3(3-5p)>0$ yields there exists $t_2 \in (0,1)$ such that $g_p(t)<0$ for $t \in (0,t_2)$ and $g_p(t)>0$ for $t \in (t_2,1)$.

To this end, it remains to analysis the sign of $\zeta(p) = 4-8p+2p^2-p^3$. Differentiation yields $\zeta'(p) = -[2p^2+4+(2-p)^2]<0$ for $p \in (0,1)$. This in conjunction with $\zeta(0)=4$ and $\zeta(3/5) = -37/125$ implies that $\zeta(p)$ has a unique root on (0, 3/5). Numerical experiment results allow us to take $p_0 = 0.55576\cdots$ as an approximation of $\zeta(p)=0$.

□

Lemma 2.3 *Let $p \in (0,1)$ and*

$$h_p(t) = 3 - 7p + p^2 - p^3 + (3 - 7p - 2p^2 + 2p^3)t - p(1 - p + p^2)t^2, \tag{2.8}$$

$$j_p(t) = (1-p)(1 - p + p^2) + (4 - 5p - 4p^2 + 2p^3)t + (4 - 8p + 2p^2 - p^3)t^2. \tag{2.9}$$

Then the following statements are true:

(i) $h_p(t) < 0$ for all $t \in (1,\infty)$ if and only if $p \geq 2/5$;
(ii) $j_p(t) < 0$ for all $t \in (1,\infty)$ if and only if $p \geq 3/5$;
(iii) if $0 < p < 2/5$, then there exists $t_3 \in (1,\infty)$ such that $h_p(t) > 0$ for $t \in (1, t_3)$ and $h_p(t) < 0$ for $t \in (t_3,\infty)$;
(iv) if $0 < p \leq p_0$, then $j_p(t) > 0$ for $t \in (1,\infty)$; if $p_0 < p < 3/5$, then there exists $t_4 \in (1,\infty)$ such that $j_p(t) > 0$ for $t \in (1, t_4)$ and $j_p(t) < 0$ for $t \in (t_4,\infty)$, where p_0 is defined as in Lemma 2.2(iv).

Proof Observe from (2.8) and (2.9) that

$$h_p(t) = t^2 f_p(1/t), \quad j_p(t) = t^2 g_p(1/t).$$

This in conjunction with Lemma 2.2(i)(ii) completes the proof of (i) and (ii).

(iii) As a quadratic function, (2.8) enables us to know that the graph of $h_p(t)$ is an opening downwards parabola. Combining this with $h_p(1) = 3(2 - 5p) > 0$ yields that there exists $t_3 \in (1,\infty)$ such that $h_p(t) > 0$ for $t \in (1, t_3)$ and $h_p(t) < 0$ for $t \in (t_3,\infty)$.

(iv) As in proof of Lemma 2.2(iv), we clearly see that $4 - 8p + 2p^2 - p^3 \geq 0$ if $0 < p \leq p_0$, which in conjunction with (2.8) yields $j_p(t) > 0$ for $t \in (1,\infty)$.

If $p_0 < p < 3/5$, then we obtain $4 - 8p + 2p^2 - p^3 < 0$. In this case, the graph of $j_p(t)$ is an opening downwards parabola. This in conjunction with $j_p(1) = 3(3 - 5p) > 0$ yields there exists $t_4 \in (1,\infty)$ such that $j_p(t) > 0$ for $t \in (1, t_4)$ and $j_p(t) < 0$ for $t \in (t_4,\infty)$.

□

3 Bounds for $\mathsf{LM}_{\mathsf{G,A}}(a,b), \mathsf{LM}_{\mathsf{A,G}}(a,b), \mathsf{LM}_{\mathsf{A,Q}}(a,b)$ and $\mathsf{LM}_{\mathsf{Q,A}}(a,b)$

By the symmetry and homogenousness for all bivariate means, we assume that $a > b > 0$ in this section. Let $x = \sqrt{(a-b)/(a+b)} \in (0,1)$. Then from (1.1) and (1.8) we obtain

$$\mathsf{G}(a,b) = \mathsf{A}\sqrt{1 - x^4}, \quad \mathsf{Q}(a,b) = \mathsf{A}\sqrt{1 + x^4} \tag{3.1}$$

and

$$\begin{aligned} \mathsf{LM}_{\mathsf{G,A}}(a,b) &:= \mathsf{A}\Big(\frac{x}{\operatorname{arctlh} x}\Big)^2, \quad \mathsf{LM}_{\mathsf{A,G}}(a,b) := \mathsf{A}\Big(\frac{x}{\operatorname{arcsl} x}\Big)^2, \\ \mathsf{LM}_{\mathsf{A,Q}}(a,b) &:= \mathsf{A}\Big(\frac{x}{\operatorname{arcslh} x}\Big)^2, \quad \mathsf{LM}_{\mathsf{Q,A}}(a,b) := \mathsf{A}\Big(\frac{x}{\operatorname{arctl} x}\Big)^2. \end{aligned} \tag{3.2}$$

We are now in a position to state and prove our main results.

Theorem 3.1 *The double inequality*

$$\alpha_1 \mathsf{A}(a,b) + (1-\alpha_1)\mathsf{G}(a,b) < \mathsf{LM}_{\mathsf{G,A}}(a,b) < \beta_1 \mathsf{A}(a,b) + (1-\beta_1)\mathsf{G}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/\kappa^2 = 0.290901\cdots$ and $\beta_1 \geq 2/5$.

Proof Let $p \in (0,1)$. Then from (3.1) and (3.2) we clearly see that

$$\begin{aligned} &\mathsf{LM}_{\mathsf{G,A}}(a,b) - p\mathsf{A}(a,b) - (1-p)G(a.b) \\ &= \mathsf{A}\Big[\Big(\frac{x}{\operatorname{arctlh} x}\Big)^2 - p - (1-p)\sqrt{1-x^4}\Big] = \mathsf{A}\frac{[p + (1-p)\sqrt{1-x^4}]}{\operatorname{arctlh}^2 x} F_p(x), \end{aligned} \tag{3.3}$$

where

$$F_p(x) = \frac{x^2}{p + (1-p)\sqrt{1-x^4}} - \operatorname{arctlh}^2 x.$$

Differentiation and elementary calculations lead to

$$F_p(0^+) = 0, \quad F_p(1^-) = \frac{1}{p} - \kappa^2, \tag{3.4}$$

$$\frac{(1-x^4)^{3/4}}{2} F_p'(x) = \frac{x\sqrt[4]{1-x^4}(1-p+p\sqrt{1-x^4})}{\left[p+(1-p)\sqrt{1-x^4}\right]^2} - \operatorname{arctlh} x \triangleq \hat{F}_p(x), \tag{3.5}$$

$$\hat{F}_p(0^+) = 0, \quad \hat{F}_p(1^-) = -\kappa, \tag{3.6}$$

$$\hat{F}_p'(x) = \frac{(1-\sqrt{1-x^4}) f_p(\sqrt{1-x^4})}{(1-x^4)^{3/4}[p+(1-p)\sqrt{1-x^4}]^3}, \tag{3.7}$$

where $f_p(t)$ is defined as in (2.5).

We divide the proof into three cases.

Case 1: $p \geq 2/5$. Lemma 2.2(i) and (3.4)–(3.7) enable us to obtain that $F_p(x) < 0$ for $x \in (0,1)$, which in conjunction with (3.1) implies that the inequality

$$\mathsf{LM}_{\mathsf{G},\mathsf{A}}(a,b) < p\mathsf{A}(a,b) + (1-p)\mathsf{G}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

Case 2: $p \leq 1/\kappa^2$. From (3.4) we obtain

$$F_p(0^+) = 0, \quad F_p(1^-) \geq 0 \tag{3.8}$$

It follows from Lemma 2.2(iii) and (3.7) that there exists $x_1 \in (0,1)$ such that $\hat{F}_p(x)$ is strictly increasing on $(0, x_1)$ and strictly decreasing on $(x_1, 1)$. Further, combining this with (3.5) and (3.6) yields that there exists $x_2 \in (0,1)$ such that $F_p(x)$ is strictly increasing on $(0, x_2)$ and strictly decreasing on $(x_2, 1)$. This in conjunction with (3.3) and (3.8) implies that the inequality

$$\mathsf{LM}_{\mathsf{G},\mathsf{A}}(a,b) > p\mathsf{A}(a,b) + (1-p)\mathsf{G}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

Case 3 $1/\kappa^2 < p < 2/5$. Lemma 2.2(iii) together with the same argument as in Case 2 leads to the conclusion that there exists $x_3 \in (0,1)$ such that $F_p(x)$ is strictly increasing on $(0, x_3)$ and strictly decreasing on $(x_3, 1)$. In this case, from (3.4) we clearly see that

$$F_p(0^+) = 0, \quad F_p(1^-) < 0.$$

Combining this with the piecewise monotonicity of $F_p(x)$ enables us to know that there exist $x_4, x_5 \in (0,1)$ such that

$$\mathsf{LM}_{\mathsf{G},\mathsf{A}}(a,b) > p\mathsf{A}(a,b) + (1-p)\mathsf{G}(a,b)$$

holds for $a > b > 0$ with $\sqrt{(a-b)/(a+b)} < x_4$ and

$$\mathsf{LM}_{\mathsf{G},\mathsf{A}}(a,b) < p\mathsf{A}(a,b) + (1-p)\mathsf{G}(a,b)$$

holds for $a > b > 0$ with $1 - x_5 < \sqrt{(a-b)/(a+b)} < 1$.

□

Theorem 3.2 *The double inequality*

$$\alpha_2\mathsf{A}(a,b) + (1-\alpha_2)\mathsf{G}(a,b) < \mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b) < \beta_2\mathsf{A}(a,b) + (1-\beta_2)\mathsf{G}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 1/\omega^2 = 0.5818\cdots$ and $\beta_2 \geq 3/5$.

Proof Let $p \in (0,1)$. Then from (3.1) and (3.2) we obtain

$$\begin{aligned}&\mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b)-p\mathsf{A}(a,b)-(1-p)\mathsf{G}(a.b)\\&=\mathsf{A}\Big[\Big(\frac{x}{\operatorname{arcsl}x}\Big)^2-p-(1-p)\sqrt{1-x^4}\Big]=\mathsf{A}\frac{[p+(1-p)\sqrt{1-x^4}]}{\operatorname{arcsl}^2x}G_p(x),\end{aligned}\tag{3.9}$$

where

$$G_p(x)=\frac{x^2}{p+(1-p)\sqrt{1-x^4}}-\operatorname{arcsl}^2x.$$

Elementary calculations yield

$$G_p(0^+)=0,\quad G_p(1^-)=\frac{1}{p}-\omega^2,\tag{3.10}$$

$$\frac{\sqrt{1-x^4}}{2}G_p'(x)=\frac{x(1-p+p\sqrt{1-x^4})}{\big[p+(1-p)\sqrt{1-x^4}\big]^2}-\operatorname{arcsl}x\triangleq\hat{G}_p(x),\tag{3.11}$$

$$\hat{G}_p(0^+)=0,\quad \hat{G}_p(1^-)=\frac{1-p}{p^2}-\omega,\tag{3.12}$$

$$\hat{G}_p'(x)=\frac{(1-\sqrt{1-x^4})g_p(\sqrt{1-x^4})}{\sqrt{1-x^4}[p+(1-p)\sqrt{1-x^4}]^3},\tag{3.13}$$

where $g_p(t)$ is defined as in (2.6).

Lemma 2.2(ii) and (3.10)-(3.13) enable us to obtain that $G_{3/5}(x)<0$ for $x\in(0,1)$, which in conjunction with (3.9) implies that the inequality

$$\mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b)<\frac{3}{5}\mathsf{A}(a,b)+\frac{2}{5}\mathsf{G}(a,b)\tag{3.14}$$

holds for all $a,b>0$ with $a\neq b$.

On the other hand, Lemma 2.2(iv) and (3.13) together with $p_0<1/\omega<3/5$ lead to the conclusion that there exists $x_6\in(0,1)$ such that $\hat{G}_{1/\omega^2}(x)$ is strictly increasing on $(0,x_6)$ and strictly decreasing on $(x_6,1)$. From (3.12) we obtain

$$\hat{G}_{1/\omega^2}(0^+)=0,\quad \hat{G}_{1/\omega^2}(1^-)=\omega(\omega^3-\omega-1)=-0.075564\cdots.$$

This in conjunction with (3.11) and the piecewise monotonicity of $\hat{G}_{1/\omega^2}(x)$ implies that there exists $x_7\in(0,1)$ such that $G_{1/\omega^2}(x)$ is strictly increasing $(0,x_7)$ and strictly decreasing on $(x_7,1)$. Further, it follows from (3.10) that $G_{1/\omega^2}(0^+)=G_{1/\omega^2}(1^-)=0$. Combining this with (3.9) together with the piecewise monotonicity of $G_{1/\omega^2}(x)$ yields the inequality

$$\frac{1}{\omega^2}\mathsf{A}(a,b)+\left(1-\frac{1}{\omega^2}\right)\mathsf{G}(a,b)<\mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b)\tag{3.15}$$

holds for all $a,b>0$ with $a\neq b$.

Therefore, the sufficient condition of Theorem 3.2 follows from inequalities (3.6) and (3.7) together with the monotonicity of $p\mathsf{A}(a,b)+(1-p)\mathsf{G}(a,b)$ with respect to p.

We now prove $1/\omega^2$ and 3/5 are the best possible constants valid for Theorem 3.2.

Let $1/\omega^2<p<3/5$ and $0<x<1$. Then from (3.1) and (3.2) we obtain

$$\frac{\mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b)-\mathsf{G}(a,b)}{\mathsf{A}(a,b)-\mathsf{G}(a,b)}=\frac{(\frac{x}{\operatorname{arcsl}x})^2-\sqrt{1-x^4}}{1-\sqrt{1-x^4}}.\tag{3.16}$$

By Taylor series (2.2), one has

$$\lim_{x\to 0^+}\frac{(\frac{x}{\operatorname{arcsl} x})^2-\sqrt{1-x^4}}{1-\sqrt{1-x^4}}=\lim_{x\to 0^+}\frac{(1+\sqrt{1-x^4})\big[1-(\frac{\operatorname{arcsl} x}{x})^2\sqrt{1-x^4}\big]}{x^2\operatorname{arcsl}^2 x}$$
$$=2\lim_{x\to 0^+}\frac{1-\big[1+\frac{x^4}{10}+o(x^4)\big]^2\big[1-\frac{x^4}{2}+o(x^4)\big]}{x^4}=\frac{3}{5}$$

and

$$\lim_{x\to 1^-}\frac{(\frac{x}{\operatorname{arcsl} x})^2-\sqrt{1-x^4}}{1-\sqrt{1-x^4}}=\frac{1}{\omega^2}.$$

This in conjunction with (3.16) yields that there exist small enough $\delta_1\in(0,1)$ such that

$$\mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b)>p\mathsf{A}(a,b)+(1-p)\mathsf{G}(a,b)$$

holds for $a>b>0$ with $0<\sqrt{(a-b)/(a+b)}<\delta_1$ and

$$\mathsf{LM}_{\mathsf{A},\mathsf{G}}(a,b)<p\mathsf{A}(a,b)+(1-p)\mathsf{G}(a,b)$$

holds for $a>b>0$ with $1-\delta_1<\sqrt{(a-b)/(a+b)}<1$. □

Theorem 3.3 *The double inequality*

$$\alpha_3\mathsf{Q}(a,b)+(1-\alpha_3)\mathsf{A}(a,b)<\mathsf{LM}_{\mathsf{A},\mathsf{Q}}(a,b)<\beta_3\mathsf{Q}(a,b)+(1-\beta_3)\mathsf{A}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_3\le(\sqrt{2}+1)(1/\sigma^2-1)=0.394977\cdots$ and $\beta_3\ge 2/5$.

Proof Let $\alpha_3^*=(\sqrt{2}+1)(1/\sigma^2-1)$ and $p\in(0,1)$. Then (3.1) and (3.2) give rise to

$$\begin{aligned}&\mathsf{LM}_{\mathsf{A},\mathsf{Q}}(a,b)-p\mathsf{Q}(a,b)-(1-p)\mathsf{A}(a,b)\\&=\mathsf{A}\Big[\Big(\frac{x}{\operatorname{arcslh} x}\Big)^2-p\sqrt{1+x^4}-(1-p)\Big]=\mathsf{A}\frac{[p\sqrt{1+x^4}+(1-p)]}{\operatorname{arcslh}^2 x}H_p(x),\end{aligned}\tag{3.17}$$

where

$$H_p(x)=\frac{x^2}{p\sqrt{1+x^4}+(1-p)}-\operatorname{arcslh}^2 x.$$

From (3.17), it suffices to show $H_{2/5}(x)<0$ and $H_{\alpha_3^*}(x)>0$ for $x\in(0,1)$ since $H_p(x)$ is strictly decreasing for $p\in\mathbb{R}$.

Differentiation and simple calculations lead to

$$H_p(0^+)=0,\quad H_p(1^-)=\frac{1}{(\sqrt{2}-1)p+1}-\sigma^2,\tag{3.18}$$

$$\frac{\sqrt{1+x^4}}{2}H_p'(x)=\frac{x\big[p+(1-p)\sqrt{1+x^4}\big]}{\big[p\sqrt{1+x^4}+1-p\big]^2}-\operatorname{arcslh} x\triangleq\hat{H}_p(x),\tag{3.19}$$

$$\hat{H}_p(0^+)=0,\quad \hat{H}_p(1^-)=\frac{\sqrt{2}(1-p)+p}{\big[(\sqrt{2}-1)p+1\big]^2}-\sigma,\tag{3.20}$$

$$\hat{H}_p'(x)=\frac{(\sqrt{1+x^4}-1)h_p(\sqrt{1+x^4})}{\sqrt{1+x^4}\big(p\sqrt{1+x^4}+1-p\big)^3},\tag{3.21}$$

where $h_p(t)$ is defined as in (2.8).

Lemma 2.3(i) and (3.18)-(3.21) yield $H_{2/5}(x)<0$ for $x\in(0,1)$.

On the other hand, from (3.18) we clearly see that

$$H_{\alpha_3^*}(0^+)=0,\quad H_{\alpha_3^*}(1^-)=0.\tag{3.22}$$

Lemma 2.3(iv) and (3.21) together with $p_0<\alpha_3^*<2/5$ and $h_{\alpha_3^*}(\sqrt{2})=-0.20603\cdots$ lead to the conclusion that there exists $x_8\in(0,1)$ such that $\hat{H}_{\alpha_3^*}(x)$ is strictly increasing on $(0,x_8)$ and strictly decreasing on $(x_8,1)$. Combining this with (3.19) and (3.22) together with $\hat{H}_{\alpha_3^*}(0)=0,\hat{H}_{\alpha_3^*}(1)=-0.00338\cdots$ yields $H_{\alpha_3^*}(x)>0$ for $x\in(0,1)$.

It remains to show that 2/5 and α_3^* are the best possible constants.

Let $\alpha_3^*<p<2/5$ and $0<x<1$. Then by Taylor series (2.3), one has

$$\lim_{x\to0^+}\frac{\left(\frac{x}{\operatorname{arcslh}x}\right)^2-1}{\sqrt{1+x^4}-1}=\lim_{x\to0^+}\frac{(\sqrt{1+x^4}+1)\left[1-\left(\frac{\operatorname{arcslh}x}{x}\right)^2\right]}{x^2\operatorname{arcslh}^2x}$$
$$=2\lim_{x\to0^+}\frac{1-\left[1-\frac{x^4}{10}+o(x^4)\right]^2}{x^4}=\frac{2}{5}$$

and

$$\lim_{x\to1^-}\frac{\left(\frac{x}{\operatorname{arcslh}x}\right)^2-1}{\sqrt{1+x^4}-1}=\alpha_3^*.$$

This in conjunction with (3.17) implies that there exist small enough $\delta_2\in(0,1)$ such that

$$\mathsf{LM}_{\mathsf{A},\mathsf{Q}}(a,b)>p\mathsf{Q}(a,b)+(1-p)\mathsf{A}(a,b)$$

holds for $a>b>0$ with $0<\sqrt{(a-b)/(a+b)}<\delta_2$ and

$$\mathsf{LM}_{\mathsf{A},\mathsf{Q}}(a,b)<p\mathsf{Q}(a,b)+(1-p)\mathsf{A}(a,b)$$

holds for $a>b>0$ with $1-\delta_2<\sqrt{(a-b)/(a+b)}<1$.

□

Theorem 3.4 *The double inequality*

$$\alpha_4\mathsf{Q}(a,b)+(1-\alpha_4)\mathsf{A}(a,b)<\mathsf{LM}_{\mathsf{Q},\mathsf{A}}(a,b)<\beta_4\mathsf{Q}(a,b)+(1-\beta_4)\mathsf{A}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_4\le(\sqrt{2}+1)(1/\tau^2-1)=0.595785\cdots$ and $\beta_4\ge3/5$.

Proof Let $\alpha_4^*=(\sqrt{2}+1)(1/\tau^2-1)$ and $p\in(0,1)$. Then (3.1) and (3.2) lead to

$$\begin{aligned}&\mathsf{LM}_{\mathsf{Q},\mathsf{A}}(a,b)-p\mathsf{Q}(a,b)-(1-p)\mathsf{A}(a,b)\\&=\mathsf{A}\left[\left(\frac{x}{\operatorname{arctl}x}\right)^2-p\sqrt{1+x^4}-(1-p)\right]=\mathsf{A}\frac{[p\sqrt{1+x^4}+(1-p)]}{\operatorname{arctl}^2x}J_p(x),\end{aligned}\tag{3.23}$$

where

$$J_p(x)=\frac{x^2}{p\sqrt{1+x^4}+(1-p)}-\operatorname{arctl}^2x.$$

Differentiation and elementary calculations lead to

$$J_p(0^+)=0,\quad J_p(1^-)=\frac{1}{(\sqrt{2}-1)p+1}-\tau^2,\tag{3.24}$$

$$\frac{(1+x^4)^{3/4}}{2}J_p'(x)=\frac{x\sqrt[4]{1+x^4}\left[p+(1-p)\sqrt{1+x^4}\right]}{\left[p\sqrt{1+x^4}+1-p\right]^2}-\operatorname{arctl}x\triangleq\hat{J}_p(x),\tag{3.25}$$

$$\hat{J}_p(0^+)=0,\quad \hat{J}_p(1^-)=\frac{2^{3/4}(1-p)+\sqrt[4]{2}p}{\left[(\sqrt{2}-1)p+1\right]^2}-\tau,\tag{3.26}$$

$$\hat{J}_p'(x)=\frac{(\sqrt{1+x^4}-1)j_p(\sqrt{1+x^4})}{\sqrt{1+x^4}\left(p\sqrt{1+x^4}+1-p\right)^3},\tag{3.27}$$

where $J_p(t)$ is defined as in (2.9).

Further, by Taylor series (2.4), we have

$$\begin{aligned}\lim_{x\to 0^+}\frac{\left(\frac{x}{\operatorname{arctl} x}\right)^2-1}{\sqrt{1+x^4}-1}&=\lim_{x\to 0^+}\frac{\left(\sqrt{1+x^4}+1\right)\left[1-\left(\frac{\operatorname{arctl} x}{x}\right)^2\right]}{x^2\operatorname{arctl}^2 x}\\&=2\lim_{x\to 0^+}\frac{1-\left[1-\frac{3x^4}{20}+o(x^4)\right]^2}{x^4}=\frac{3}{5},\end{aligned}\tag{3.28}$$

$$\lim_{x\to 1^-}\frac{\left(\frac{x}{\operatorname{arcslh} x}\right)^2-1}{\sqrt{1+x^4}-1}=\alpha_4^*.\tag{3.29}$$

By the same argument as in Theorem 3.3, Lemma 2.3(ii), (iv) and (3.23)–(3.29) give rise to the proof of Theorem 3.4.

□

4 New bounds for the arc lemniscate functions

In this section, we give a brief overview of known results which relate to the main results of this paper.

Utilizing the inequality

$$(a^3b^2)^{1/3}<\mathsf{LM}(a,b)<\frac{3a+2b}{5}$$

(see [1, Lemma 4.1]) with (3.1) and (3.2), simple lower and upper bounds for the arc lemniscate functions in the following proposition is due to Neuman.

Proposition 4.1 ([29, Theorem 5.1]). Let $0<|x|<1$. Then

$$\begin{aligned}\left(\frac{5}{3\sqrt{1-x^4}+2}\right)^{1/2}&<\frac{\operatorname{arctlh} x}{x}<\frac{1}{(1-x^4)^{3/20}},\\\left(\frac{5}{3+2\sqrt{1-x^4}}\right)^{1/2}&<\frac{\operatorname{arcsl} x}{x}<\frac{1}{(1-x^4)^{1/10}},\\\left(\frac{5}{3+2\sqrt{1+x^4}}\right)^{1/2}&<\frac{\operatorname{arcslh} x}{x}<\frac{1}{(1+x^4)^{1/10}},\\\left(\frac{5}{3\sqrt{1+x^4}+2}\right)^{1/2}&<\frac{\operatorname{arctl} x}{x}<\frac{1}{(1+x^4)^{3/20}}.\end{aligned}$$

Many remarkable inequalities for the arc lemniscate functions have attracted the attention of several researchers; see [28, 30, 31]. In particular, the arc lemniscate counterparts of Shafer-Fink's inequality have been established by Deng et al. [26], and Wei et al. [27], which are stated explicitly in the following.

Proposition 4.2 ([26]). The following inequalities

$$\frac{a_1}{\frac{7}{3}+\sqrt{1-x^4}}<\frac{\operatorname{arctlh} x}{x}<\frac{b_1}{\frac{7}{3}+\sqrt{1-x^4}},\quad(0<|x|<1)\tag{4.1}$$

$$\frac{a_2}{4+\sqrt{1-x^4}}<\frac{\operatorname{arcsl} x}{x}<\frac{b_2}{4+\sqrt{1-x^4}},\quad(0<|x|<1)\tag{4.2}$$

$$\frac{a_3}{\frac{17}{3}+(1+x^4)^{1/4}}<\frac{\operatorname{arcslh} x}{x}<\frac{b_3}{\frac{17}{3}+(1+x^4)^{1/4}},\quad(|x|>0)\tag{4.3}$$

$$\frac{a_4}{\frac{2}{3}+(1+x^4)^{1/4}}<\frac{\operatorname{arctl} x}{x}<\frac{b_4}{\frac{2}{3}+(1+x^4)^{1/4}},\quad(|x|>0)\tag{4.4}$$

hold with the best possible constants $a_1 = 10/3, b_1 = 7\kappa/3 = 4.32617\cdots, a_2 = 5, b_2 = 4\omega = 5.24412\cdots$, $a_3 = \kappa = 1.85407\cdots, b_3 = 20/3$ and $a_4 = \omega = 1.31103\cdots, b_4 = 5/3$.

Proposition 4.3 ([27]). The following inequalities

$$\frac{1}{\lambda_1 + (1-\lambda_1)\sqrt{1-x^4}} < \frac{\operatorname{arctlh} x}{x} < \frac{1}{\mu_1 + (1-\mu_1)\sqrt{1-x^4}}, \quad (0<|x|<1) \tag{4.5}$$

$$\frac{1}{\lambda_2 + (1-\lambda_2)\sqrt{1-x^4}} < \frac{\operatorname{arcsl} x}{x} < \frac{1}{\mu_2 + (1-\mu_2)\sqrt{1-x^4}}, \quad (0<|x|<1) \tag{4.6}$$

$$\frac{1}{\lambda_3 + (1-\lambda_3)\sqrt{1+x^4}} < \frac{\operatorname{arcslh} x}{x}, (|x|>0) \tag{4.7}$$

$$\frac{1}{\lambda_4 + (1-\lambda_4)\sqrt{1+x^4}} < \frac{\operatorname{arctl} x}{x}, (|x|>0) \tag{4.8}$$

hold with the best possible constants $\lambda_1 = \lambda_4 = 7/10$, $\lambda_2 = \lambda_3 = 4/5$ and $\mu_1 = 1/(\sqrt{2}\omega)$, $\mu_2 = 1/\omega$.

We are now in a position to present new bounds for the arc lemniscate functions, which are derived from Theorems 3.1–3.4.

Theorem 4.4 *Let $0<|x|<1$. Then the following inequalities*

$$\left[\frac{1}{\beta_1 + (1-\beta_1)\sqrt{1-x^4}}\right]^{1/2} < \frac{\operatorname{arctlh} x}{x} < \left[\frac{1}{\alpha_1 + (1-\alpha_1)\sqrt{1-x^4}}\right]^{1/2}, \tag{4.9}$$

$$\left[\frac{1}{\beta_2 + (1-\beta_2)\sqrt{1-x^4}}\right]^{1/2} < \frac{\operatorname{arcsl} x}{x} < \left[\frac{1}{\alpha_2 + (1-\alpha_2)\sqrt{1-x^4}}\right]^{1/2}, \tag{4.10}$$

$$\left[\frac{1}{\beta_3\sqrt{1+x^4} + (1-\beta_3)}\right]^{1/2} < \frac{\operatorname{arcslh} x}{x} < \left[\frac{1}{\alpha_3\sqrt{1+x^4} + (1-\alpha_3)}\right]^{1/2}, \tag{4.11}$$

$$\left[\frac{1}{\beta_4\sqrt{1+x^4} + (1-\beta_4)}\right]^{1/2} < \frac{\operatorname{arctl} x}{x} < \left[\frac{1}{\alpha_4\sqrt{1+x^4} + (1-\alpha_4)}\right]^{1/2} \tag{4.12}$$

hold with the best possible constants $\alpha_1 = 1/\kappa^2 = 0.290901\cdots$, $\beta_1 = 2/5$, $\alpha_2 = 1/\omega^2 = 0.581802\cdots$, $\beta_2 = 3/5$, $\alpha_3 = (\sqrt{2}+1)(1/\sigma^2 - 1) = 0.394977\cdots, \beta_3 = 2/5$ and $\alpha_4 = (\sqrt{2}+1)(1/\tau^2 - 1) = 0.595785\cdots, \beta_4 = 3/5$.

Further, as in the proof of Theorem 3.3 and 3.4, Lemma 2.3 allows us to extend the range of x from $0<|x|<1$ to $|x|>0$ and establish new bounds of $\operatorname{arcslh} x/x$ and $\operatorname{arctl} x/x$ in the following.

Theorem 4.5 *Let $|x|>0$. Then the following inequalities*

$$\left[\frac{1}{\beta_3'\sqrt{1+x^4} + (1-\beta_3')}\right]^{1/2} < \frac{\operatorname{arcslh} x}{x} < \left[\frac{1}{\alpha_3'\sqrt{1+x^4} + (1-\alpha_3')}\right]^{1/2}, \tag{4.13}$$

$$\left[\frac{1}{\beta_4'\sqrt{1+x^4} + (1-\beta_4')}\right]^{1/2} < \frac{\operatorname{arctl} x}{x} < \left[\frac{1}{\alpha_4'\sqrt{1+x^4} + (1-\alpha_4')}\right]^{1/2} \tag{4.14}$$

hold with the best possible constants $\alpha_3' = 1/\kappa^2 = 0.290901\cdots$, $\beta_3' = 2/5$ and $\alpha_4' = 1/\omega^2 = 0.581802\cdots$, $\beta_4' = 3/5$.

Remark 4.6

(i) The lower bounds as shown in Proposition 4.1 are the same as in (4.9)–(4.12) but they are not specified as the optimal lower bounds in certain sense;

(ii) There is no strict comparison between two upper bounds in Proposition 4.1 and Theorem 4.4. For instance, by Taylor series, we have

$$\frac{1}{(1-x^4)^{3/20}}-\left[\frac{\kappa^2}{1+(\kappa^2-1)\sqrt{1-x^4}}\right]^{1/2}=\frac{1}{4}\left(\frac{1}{\kappa^2}-\frac{2}{5}\right)x^4+o(x^4)$$

as $x\to 0^+$, which in conjunction with $1/\kappa^2<2/5$ yields the upper bound of $\operatorname{arctlh} x/x$ in Proposition 4.1 is better than that of (4.9) for small $x>0$. On the other hand,

$$\frac{1}{(1-x^4)^{3/20}}-\left[\frac{\kappa^2}{1+(\kappa^2-1)\sqrt{1-x^4}}\right]^{1/2}\to\infty,\quad (x\to 1^-)$$

gives the upper bound of $\operatorname{arctlh} x/x$ in (4.9) is better than the one in Proposition 4.1 for sufficient large $x>0$. Similar for other upper bounds.

Remark 4.7 We give the comparison between the bounds of the arc lemniscate functions in Theorems 4.4, 4.5 and those in Propositions 4.2, 4.3.

Let $u=(1-x^4)^{1/2}$ and $v=(1+x^4)^{1/4}$.

(i) Notice that the inequalities (4.1) and (4.2) are the same as (4.5) and (4.6), respectively. Elementary computations give

$$\begin{aligned}
\frac{5}{2+3\sqrt{1-x^4}}-\left(\frac{10}{7+3\sqrt{1-x^4}}\right)^2&=\frac{45(1-u)^2}{(2+3u)(7+3u)^2}>0,\\
\frac{\kappa^2}{1+(\kappa^2-1)\sqrt{1-x^4}}-\left(\frac{10\kappa}{7+3\sqrt{1-x^4}}\right)^2&=-\frac{\kappa^2u\left[49\kappa^2-100+9(1-u)\right]}{(7+3u)^2\left[1+(\kappa^2-1)u\right]}<0,\\
\frac{5}{3+2\sqrt{1-x^4}}-\left(\frac{10}{4+\sqrt{1-x^4}}\right)^2&=\frac{5(1-u)^2}{(4+u)^2(3+2u)}>0,\\
\frac{\omega^2}{1+(\omega^2-1)\sqrt{1-x^4}}-\left(\frac{4\omega}{4+\sqrt{1-x^4}}\right)^2&=-\frac{\omega^2u\left[16\omega^2-25+(1-u)\right]}{(4+u)^2\left[1+(\omega^2-1)u\right]}<0,
\end{aligned}$$

which yield inequalities (4.9) and (4.10) are better than inequalities (4.1) and (4.2) (or (4.5) and (4.6)), respectively.

(ii) As mentioned in [26, remark 2.2] together with

$$\begin{aligned}
&\frac{5}{2\sqrt{1+x^4}+3}-\left(\frac{5}{4+\sqrt{1+x^4}}\right)^2=\frac{5(1-\sqrt{1+x^4})^2}{(4+\sqrt{1+x^4})^2(3+2\sqrt{1+x^4})}>0,\\
&\frac{1}{\alpha_3^*\sqrt{1+x^4}+1-\alpha_3^*}-\left[\frac{20}{17+3(1+x^4)^{1/4}}\right]^2\\
&=-\frac{(v-1)\left[400\alpha_3^*-111+(400\alpha_3^*-9)v\right]}{(17+3v)^2(\alpha_3^*v^2+1-\alpha_3^*)}<0,
\end{aligned}\tag{4.15}$$

we conclude that inequality (4.11) is better than (4.3) for $0<|x|<1$ and inequality (4.13) is sharper than (4.7) for $|x|>0$. It is worth to pointing out that the upper bound of (4.11) is sharper than the one in (4.3) for $|x|\geq 1$ by replacing α_3^* with $1/\kappa^2$ in (4.15). But there is no strict comparison between the two lower bounds in (4.3) and (4.11) since $1445-27\kappa^2>0$, $45-18\kappa^2=-16.8767\cdots$ and

$$\frac{5}{2\sqrt{1+x^4}+3}-\left[\frac{3\kappa}{17+3(1+x^4)^{1/4}}\right]^2=\frac{(1445-27\kappa^2)+510v+(45-18\kappa^2)v^2}{(17+3v)^2(3+2v^2)}.$$

(iii) Inequality (4.14) is sharper than (4.8) and the lower bound of (4.12) is better than (4.4) for $0<|x|<1$ following from

$$\frac{5}{3\sqrt{1+x^4}+2}-\left(\frac{10}{7+3\sqrt{1+x^4}}\right)^2=\frac{45(1-\sqrt{1+x^4})^2}{(2+3\sqrt{1+x^4})(7+3\sqrt{1+x^4})^2}>0$$

and

$$\begin{aligned}\frac{5}{3\sqrt{1+x^4}+2}-\left[\frac{3\omega}{2+3(1+x^4)^{1/4}}\right]^2&=\frac{2(10-9\omega^2)+60v+9(5-3\omega^2)v^2}{(2+3v)^2(2+3v^2)}\\&>\frac{2(10-9\omega^2)+60+9(5-3\omega^2)\sqrt{5^4+1}}{(2+3v)^2(2+3v^2)}>\frac{13}{(2+3v)^2(2+3v^2)}>0\end{aligned}\tag{4.16}$$

for $0<x<5$. On the other hand, combining (4.16) with $5-3\omega^2=-0.156389\cdots$ yields

$$\frac{5}{3\sqrt{1+x^4}+2}<\left[\frac{3\omega}{2+3(1+x^4)^{1/4}}\right]^2$$

for sufficient large $x>5$. In other words, there is no strict comparison between the two lower bounds of (4.4) and (4.14) for $|x|\geq 1$. Further, there is also no strict comparison between the two upper bounds of (4.4) and (4.14) for $|x|>0$ but almost the upper bound of (4.14) is better than the one in (4.4), which follows from

$$\frac{1}{\alpha_4^*\sqrt{1+x^4}+1-\alpha_4^*}-\left[\frac{5}{2+3(1+x^4)^{1/4}}\right]^2=\frac{(v-1)\left[(21-25\alpha_4^*)+(9-25\alpha_4^*)v\right]}{(2+3v)^2(av^2+1-a)}$$

$$\begin{cases}>0, & 0<x<\left[\left(\frac{21-25\alpha_4^*}{25\alpha_4^*-9}\right)^4-1\right]^{1/4}=0.623236\cdots\\ \leq 0, & x\geq\left[\left(\frac{21-25\alpha_4^*}{25\alpha_4^*-9}\right)^4-1\right]^{1/4}.\end{cases}$$

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Declarations

Competing interest The authors declare that they have no competing interest.

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