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ORIGINAL RESEARCH

Ramanujan-type congruences modulo m for (l, m)-regular bipartitions

T. Kathiravan

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Abstract Let $B_{l,m}(n)$ denote the number of (l, m)-regular bipartitions of n. Recently, many authors proved several infinite families of congruences modulo 3, 5 and 11 for $B_{l,m}(n)$. In this paper, we use theta function identities to prove infinite families of congruences modulo m for (l, m)-regular bipartitions, where $m \in \{7, 3, 11, 13, 17\}$.

Keywords Congruence · Regular Bipartition.

Mathematics Subject Classification 11P83 · 05A17

1 Introduction

For a positive integer l, a partition is said to be l-regular if none of its parts is divisible by l. If $b_l(n)$ denote the number of l-regular partitions of n, then the generating function for $b_l(n)$, is given by (see [2])

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{(q^l;q^l)_\infty}{(q;q)_\infty}. \tag{1.1}$$

In recent years, many authors studied arithmetic properties of l-regular partitions [3–6, 8, 9, 11, 13, 17–20, 22].

Recall that, for a positive integers $l > 1$ and $m > 1$, a bipartition (λ, μ) of n is a pair of partitions (λ, μ) such that the sum of all the parts equals n. A (l, m)-regular bipartition of n is a bipartition (λ, μ) of n such that λ is a l-regular partition and μ is a m-regular partition. If $B_{l,m}(n)$ denote the number of (l, m)-regular bipartitions of n, then the generating function $B_{l,m}(n)$, is given by (see [14])

$$\sum_{n=0}^{\infty} B_{l,m}(n)q^n = \frac{(q^l;q^l)_\infty (q^m;q^m)_\infty}{(q;q)_\infty^2}. \tag{1.2}$$

In [7], Dou proved that, for $n \geq 0$ and $\alpha \geq 2$,

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T. Kathiravan (✉)
The Institute of Mathematical Sciences, HBNI, CIT Campus, Taramani, Chennai 600113, India
E-mail: kkathiravan98@gmail.com

$$B_{3,11}\left(3^{\alpha}n+\frac{5\cdot 3^{\alpha-1}-1}{2}\right)\equiv 0\,(\mathrm{mod}\ 11).$$

The reader may refer to [12, 14–16] for related works. Recently, Adiga and Ranganatha [1] proved infinite families of congruences modulo 3 for $B_{3,7}(n)$. More recently, Xia and Yao [21] proved several infinite families of congruences modulo 3 for $B_{3,s}(n)$, modulo 5 for $B_{5,s}(n)$ and modulo 7 for $B_{3,7}(n)$. For example, let s be a positive integer and let $p\geq 5$ be a prime, for $n\geq 0$,

$$B_{3,s}\left(p^{2\alpha+1}n+\frac{(1+s)(p^{2\alpha+2}-1)}{24}\right)\equiv 0\,(\mathrm{mod}\ 3).$$

The aim of this paper is to prove several infinite families of congruences modulo 7 for $B_{3,7}(n)$, modulo 3 for $B_{9,5}(n)$, modulo 11 for $B_{5,11}(n)$, modulo 13 for $B_{5,13}(n)$ and modulo 17 for $B_{81,17}(n)$. The main results of this paper are as follows,

Theorem 1.1 *For all $n\geq 0$ and $m\geq 0$,*

$$B_{3,7}\left(4^{7m}n+\frac{4^{7m}-1}{3}\right)\equiv 3^{m}B_{3,7}(n)\,(\mathrm{mod}\ 7), \tag{1.3}$$

$$B_{3,7}\left(4^{7m+7}n+\frac{10\cdot 4^{7m+6}-1}{3}\right)\equiv 0\,(\mathrm{mod}\ 7). \tag{1.4}$$

Theorem 1.2 *For all $n\geq 0$ and $m\geq 0$,*

$$B_{9,5}\left(5^{4m}n+\frac{5^{4m}-1}{2}\right)\equiv 2^{m}B_{9,5}(n)\,(\mathrm{mod}\ 3), \tag{1.5}$$

$$B_{9,5}\left(5^{4m+4}n+\frac{(2k+1)\cdot 5^{4m+3}-1}{2}\right)\equiv 0\,(\mathrm{mod}\ 3),\ \text{where}\ k\in\{0,4\}. \tag{1.6}$$

Theorem 1.3 *For all $n\geq 0$ and $m\geq 0$,*

$$B_{5,11}\left(5^{12m}n+\frac{7\cdot 5^{12m}-7}{12}\right)\equiv 2^{m}B_{5,11}(n)\,(\mathrm{mod}\ 11), \tag{1.7}$$

$$B_{5,11}\left(5^{12m+12}+\frac{(12k+11)\cdot 5^{12m+11}-7}{12}\right)\equiv 0\,(\mathrm{mod}\ 11),\ \text{where}\ k\in\{0,4\}. \tag{1.8}$$

Theorem 1.4 *For all $n\geq 0$ and $m\geq 0$,*

$$B_{5,13}\left(5^{6m}n+\frac{2\cdot 5^{6m}-2}{3}\right)\equiv 8^{m}B_{5,13}(n)\,(\mathrm{mod}\ 13), \tag{1.9}$$

$$B_{5,13}\left(5^{6m+6}n+\frac{(3k+1)\cdot 5^{6m+5}-2}{3}\right)\equiv 0\,(\mathrm{mod}\ 13),\ \text{where}\ k\in\{0,1\}. \tag{1.10}$$

Theorem 1.5 *For all $n\geq 0$ and $m\geq 0$,*

$$B_{81,17}\left(81\cdot 4^{9}n+81\,\frac{(2\cdot 4^{8}-2)}{3}+50\right)\equiv 0\,(\mathrm{mod}\ 17), \tag{1.11}$$

$$B_{81,17}\left(81\cdot 4^{9m}n+81\,\frac{(2\cdot 4^{9m}-2)}{3}+50\right)\equiv 8^{m}B_{81,17}(81n+50)\,(\mathrm{mod}\ 17), \tag{1.12}$$

$$B_{81,17}\left(162\cdot 4^{8}n+81\,\frac{(5\cdot 4^{8}-2)}{3}+50\right)\equiv B_{81,17}(162n+131)\,(\mathrm{mod}\ 17). \tag{1.13}$$

The paper is organized as follows. In section 2, we state some preliminary results. The proof of the theorems are carried out in sections 3 - 8.

2 Preliminaries

As usual, let

$$\begin{aligned}(a;q)_\infty &= \prod_{n\geq 0}(1-aq^n),\\ f_k &= (q^k;q^k)_\infty,\\ R(q) &:= \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\end{aligned} \tag{2.1}$$

and

$$S = R(q^5) \text{ and } S_1 = R(q^{65}).$$

We require the following 2−dissections.

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.2}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{2.3}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{2.4}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{2.5}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.6}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{2.7}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.8}$$

Let

$$\begin{aligned}v &= \frac{f_1 f_6^3}{f_2 f_3^3},\\ u &= v(q^3).\end{aligned}$$

Then

$$f_1^3 = f_9^3(u^{-1} - 3q + 4q^3u^2), \tag{2.9}$$

and, if ω is a cube root of unity other then 1 and we set q, ωq and $\omega^2 q$ for q in this, multiply the three results and replace q^3 by q, we obtain

$$\frac{f_1^{12}}{f_3^{12}} + 27q = (v^{-1} + 4qv^2)^3. \tag{2.10}$$

The 5−dissection

$$f_1 = f_{25}\left(\frac{1}{S} - q - q^2 S\right) \tag{2.11}$$

as well as the identities

$$\frac{f_1^6}{f_5^6} = \left(\frac{1}{R^5} - 11q - q^2R^5\right) \tag{2.12}$$

and

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(q^8S^4 - q^7S^3 + 2q^6S^2 - 3q^5S + 5q^4 + \frac{3q^3}{S} + \frac{2q^2}{S^2} + \frac{q}{S^3} + \frac{1}{S^4}\right) \tag{2.13}$$

and the congruence

$$f_1^p \equiv f_p \,(\text{mod})p \text{ if p is prime.} \tag{2.14}$$

(2.2) is equivalent to the 2-dissection of $\varphi(q)$, [10, (1.9.4)], (2.3) is equivalent to the 2-dissection of $\varphi(q)^2$, [10, (1.10.1)], (2.42.5) can be obtained by putting $-q$ for q in (2.3), () is [10, (22.6.2)], (2.62.7) can be obtained by putting $-q$ for q in (2.6), (2.8) is [10, (30.12.1)], (2.8) is [10, (30.10.4)], (2.9) is [10, (14.8.5)], (2.11) is [10, (8.4.1)], (2.12) is [10, (8.4.3)] and (2.13) is [10, (8.4.4)]. (2.14) follows from $(1-q)^p \equiv 1 - q^p \,(\text{mod } 9)$.

3 Congruences for (3, 7)-regular bipartitions modulo 7

3.1 Proof of Theorem 1.1

We have

$$\sum_{n=0}^{\infty} B_{3,7}(n)q^n = \frac{f_3f_7}{f_1^2} \tag{3.1}$$

$$\begin{aligned}\sum_{n=0}^{\infty} B_{3,7}(n)q^n = \frac{f_3f_7}{f_1^2} &\equiv \frac{f_3f_1^7}{f_1^2} = f_1^5f_3 = f_1^4 \cdot f_1f_3,\\ &\equiv \left(\frac{f_4^{10}}{f_2^2f_8^4} - 4q\frac{f_2^2f_8^4}{f_4^2}\right)\left(\frac{f_2f_8^2f_{12}^4}{f_4^2f_6f_{24}^2} - q\frac{f_4^4f_6f_{24}^2}{f_2f_8^2f_{12}^2}\right).\end{aligned} \tag{3.2}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{3,7}(2n+1)q^n &\equiv 3\frac{f_1^3f_4^6f_6^4}{f_2^4f_3f_{12}^2} + 6\frac{f_2^{14}f_3f_{12}^2}{f_1^3f_4^6f_6^2},\\ &\equiv 3\frac{f_4^6f_6^4}{f_2^4f_{12}^2}\left(\frac{f_4^3}{f_{12}} - 3q\frac{f_2^2f_{12}^3}{f_4f_6^2}\right) + 6\frac{f_2^{14}f_{12}^2}{f_4^6f_6^2}\left(\frac{f_4^6f_6^3}{f_2^9f_{12}^2} + 3q\frac{f_4^2f_6f_{12}^2}{f_2^7}\right),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\sum_{n=0}^{\infty} B_{3,7}(4n+1)q^n &\equiv 6f_1^5f_3 + 3\frac{f_2^9f_3^4}{f_1^4f_6^3},\\ &\equiv 6\sum_{n=0}^{\infty} B_{3,7}(n)q^n + 3\frac{f_2^9}{f_6^3}\left(\frac{f_6f_{12}^2f_4^4}{f_2^5f_8f_{24}} + 2q\frac{f_6^2f_8f_{24}f_4}{f_2^4f_{12}}\right)^2,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\sum_{n=0}^{\infty} B_{3,7}(8n+5)q^n &\equiv 6\sum_{n=0}^{\infty} B_{3,7}(2n+1)q^n + 5f_2^5f_6,\\ &\equiv 6\sum_{n=0}^{\infty} B_{3,7}(2n+1)q^n + 5\sum_{n=0}^{\infty} B_{3,7}(n)q^{2n},\end{aligned} \tag{3.5}$$

$$\sum_{n=0}^{\infty} B_{3,7}(16n+5)q^n \equiv 6\sum_{n=0}^{\infty} B_{3,7}(4n+1)q^n + 5\sum_{n=0}^{\infty} B_{3,7}(n)q^n$$

and

$$B_{3,7}(16n+5) \equiv 6B_{3,7}(4n+1) + 5B_{3,7}(n), \tag{3.6}$$

$$\begin{aligned} B_{3,7}(64n+21) &\equiv 6B_{3,7}(16n+5) + 5B_{3,7}(4n+1), \\ &\equiv 6(6B_{3,7}(4n+1) + 5B_{3,7}(n)) + 5B_{3,7}(4n+1), \\ &\equiv 6B_{3,7}(4n+1) + 2B_{3,7}(n). \end{aligned} \tag{3.7}$$

Similarly, we find

$$\begin{aligned} B_{3,7}(256n+85) &\equiv 3B_{3,7}(4n+1) + 2B_{3,7}(n), \\ B_{3,7}(1024n+341) &\equiv 6B_{3,7}(4n+1) + B_{3,7}(n), \\ B_{3,7}(4096n+1365) &\equiv 2B_{3,7}(4n+1) + 2B_{3,7}(n) \end{aligned}$$

and

$$B_{3,7}(16384n+5461) \equiv 3B_{3,7}(n). \tag{3.8}$$

From (3.8) using mathematical induction, we have (1.3).

We have

$$\sum_{n=0}^{\infty} B_{3,7}(4096n+1365)q^n \equiv 2\sum_{n=0}^{\infty} B_{3,7}(4n+1)q^n + 2\sum_{n=0}^{\infty} B_{3,7}(n)q^n, \tag{3.9}$$

from which it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} B_{3,7}(8192n+5461)q^n &\equiv 2\sum_{n=0}^{\infty} B_{3,7}(8n+5)q^n + 2\sum_{n=0}^{\infty} B_{3,7}(2n+1)q^n, \\ &\equiv 2\left(6\sum_{n=0}^{\infty} B_{3,7}(2n+1)q^n + 5\sum_{n=0}^{\infty} B_{3,7}(n)q^{2n}\right) + 2\sum_{n=0}^{\infty} B_{3,7}(2n+1)q^n, \\ &\equiv 3\sum_{n=0}^{\infty} B_{3,7}(n)q^{2n} \end{aligned}$$

and

$$B_{3,7}(16384n+13653) \equiv 0. \tag{3.10}$$

Employing (3.10) in (1.3), we arrive at the congruence (1.4).

4 Congruences for (9, 5)-regular bipartitions modulo 3

4.1 Proof of Theorem 1.2

We have

$$\begin{aligned} \sum_{n=0}^{\infty} B_{9,5}(n)q^n &= \frac{f_5 f_9}{f_1^2} \equiv \frac{f_5 f_1^9}{f_1^2} = f_1^7 f_5, \\ &\equiv f_5 f_{25}^7 \left(\frac{1}{S} - q - q^2 S\right)^7, \\ &\equiv f_5 f_{25}^7 \left(\frac{1}{S^7} + \frac{2q}{S^6} + \frac{2q^2}{S^5} + \frac{q^3}{S^4} + \frac{2q^4}{S^3} + \frac{2q^5}{S^2} + \frac{2q^6}{S} + q^7 + q^8 S + 2q^9 S^2\right. \\ &\quad \left. + q^{10} S^3 + q^{11} S^4 + q^{12} S^5 + 2q^{13} S^6 + 2q^{14} S^7\right). \end{aligned} \tag{4.1}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{9,5}(5n+2)q^n &\equiv f_1 f_5^7\left(2\left(\frac{1}{R^5}-11q-q^2R^5\right)+2q\right),\\ &\equiv f_1 f_5^7\left(2\frac{f_1^6}{f_5^6}+2q\right),\\ &\equiv 2f_1^7 f_5+2qf_1f_5^7.\end{aligned} \tag{4.2}$$

Substituting (2.11) into (4.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{9,5}(5n+2)q^n \equiv 2f_5f_{25}^7\Bigg(&\frac{1}{S^7}+\frac{2q}{S^6}+\frac{2q^2}{S^5}+\frac{q^3}{S^4}+\frac{2q^4}{S^3}+\frac{2q^5}{S^2}+\frac{2q^6}{S}\\ &+q^7+q^8S+2q^9S^2+q^{10}S^3+q^{11}S^4+q^{12}S^5+2q^{13}S^6+2q^{14}S^7\Bigg)\\ &+2f_5^7f_{25}q\left(\frac{1}{S}-q-q^2S\right).\end{aligned} \tag{4.3}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{9,5}(25n+12)q^n &\equiv 2f_1f_5^7\left(2\left(\frac{1}{R^5}-11q-q^2R^5\right)+2q\right)+f_1^7f_5,\\ &\equiv 2f_1f_5^7\left(2\frac{f_1^6}{f_5^6}+2q\right)+f_1^7f_5,\\ &\equiv 2f_1^7f_5+qf_1f_5^7,\\ &\equiv 2\sum_{n=0}^{\infty} B_{9,5}(5n+2)q^n+\sum_{n=0}^{\infty} B_{9,5}(n)q^n\end{aligned} \tag{4.4}$$

and

$$B_{9,5}(25n+12) \equiv 2B_{9,5}(5n+2)+B_{9,5}(n). \tag{4.5}$$

It follows that

$$\begin{aligned}B_{9,5}(125n+62) &\equiv 2B_{9,5}(25n+12)+B_{9,5}(5n+2),\\ &\equiv 2\big(2B_{9,5}(5n+2)+B_{9,5}(n)\big)+B_{9,5}(5n+2),\\ &\equiv 2B_{9,5}(5n+2)+2B_{9,5}(n),\\ B_{9,5}(625n+312) &\equiv 2B_{9,5}(25n+12)+2B_{9,5}(5n+2),\\ &\equiv 2\big(2B_{9,5}(5n+2)+B_{9,5}(n)\big)+2B_{9,5}(5n+2),\\ &\equiv 2B_{9,5}(n).\end{aligned} \tag{4.6}$$

From (4.6) using mathematical induction, we have (1.5). We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{9,5}(125n+62)q^n &\equiv 2\sum_{n=0}^{\infty} B_{9,5}(5n+2)q^n+2\sum_{n=0}^{\infty} B_{9,5}(n)q^n,\\ &\equiv 2(2f_1^7f_5+2qf_1f_5^7)+2f_1^7f_5,\\ &\equiv qf_1f_5^7.\end{aligned} \tag{4.7}$$

Substituting (2.11) into (4.7), we have

$$\sum_{n=0}^{\infty} B_{9,5}(125n+62)q^n \equiv f_5^7f_{25}q\left(\frac{1}{S}-q-q^2S\right). \tag{4.8}$$

There are no terms on the right in which the power of q is 0 or 4 modulo 5, so

$$B_{9,5}(625n + 62) \equiv 0 \tag{4.9}$$

and

$$B_{9,5}(625n + 562) \equiv 0. \tag{4.10}$$

Employing (4.9) and (4.10) in (1.5), we arrive at the congruence (1.6).

5 Congruences for (5, 11)-regular bipartitions modulo 11

5.1 Proof of Theorem 1.3

We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,11}(n)q^n &= \frac{f_5 f_{11}}{f_1^2} \equiv \frac{f_5 f_1^{11}}{f_1^2} = f_1^9 f_5,\\ &\equiv f_5 f_{25}^9 \left(\frac{1}{S} - q - q^2 S\right)^9,\\ &\equiv f_5 f_{25}^9 \left(\frac{1}{S^9} + \frac{2q}{S^8} + \frac{5q^2}{S^7} + \frac{10q^3}{S^6} + \frac{9q^4}{S^5} + \frac{5q^5}{S^4} + \frac{5q^6}{S^3} + \frac{9q^7}{S^2} + \frac{4q^8}{S}\right.\\ &\quad + 2q^9 + 7q^{10}S + 9q^{11}S^2 + 6q^{12}S^3 + 5q^{13}S^4 + 2q^{14}S^5\\ &\quad \left. + 10q^{15}S^6 + 6q^{16}S^7. \qquad + 2q^{17}S^8 + 10q^{18}S^9\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,11}(5n + 4)q^n &\equiv f_1 f_5^9 \left(\frac{9}{R^5} + 2q + 2qR^5\right),\\ &\equiv f_1 f_5^9 \left(9\left(\frac{1}{R^5} - 11q - 2qR^5\right) + 2q\right),\\ &\equiv f_1 f_5^9 \left(9\frac{f_1^6}{f_5^6} + 2q\right),\\ &\equiv 9 f_1^7 f_5^3 + 2q f_1 f_5^9.\end{aligned} \tag{5.1}$$

Substituting (2.11) into (5.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,11}(5n + 4)q^n &\equiv 9 f_5^3 f_{25}^7 \left(\frac{1}{S} - q - q^2 S\right)^7 + 2 f_5^9 f_{25} q \left(\frac{1}{S} - q - q^2 S\right),\\ &\equiv 9 f_5^3 f_{25}^7 \left(\frac{1}{S^7} + \frac{4q}{S^6} + \frac{3q^2}{S^5} + \frac{7q^3}{S^4} + \frac{6q^4}{S^3} + \frac{3q^5}{S^2}\right.\\ &\quad \left. + 4q^7 + 3q^9 S^2 + 5q^{10} S^3 + 7q^{11} S^4 + 8q^{12} S^5 + 4q^{13} S^6 + 10q^{14} S^7\right)\\ &\quad + 2 f_5^9 f_{25} q \left(\frac{1}{S} - q - q^2 S\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,11}(25n+14)q^n &\equiv 9f_1^3f_5^7\left(\frac{3}{R^5}+4q+8q^2R^5\right)+9f_1^9f_5,\\ &\equiv 9f_1^3f_5^7\left(3\left(\frac{1}{R^5}-11q-q^2R^5\right)+4q\right)+9f_1^9f_5,\\ &\equiv 9f_1^3f_5^7\left(3\frac{f_1^6}{f_5^6}+4q\right)+9f_1^9f_5,\\ &\equiv 3f_5f_1^9+3qf_5^7f_1^3,\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(n)q^n+3qf_5^7f_1^3.\end{aligned} \tag{5.2}$$

Substituting (2.11) in (5.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,11}(25n+14)q^n &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(n)q^n+3f_5^7f_{25}^3q\left(\frac{1}{S}-q-q^2S\right)^3,\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(n)q^n+3f_5^7f_{25}^3q\left(\frac{1}{S^3}+\frac{8q}{S^2}+5q^3+8q^5S^2+10q^6S^3\right),\end{aligned}$$

from which it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,11}(125n+114)q^n &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(5n+4)q^n+4f_1^7f_5^3,\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(5n+4)q^n+4f_5^3f_{25}^7\left(\frac{1}{S^7}+\frac{4q}{S^6}+\frac{3q^2}{S^5}+\frac{7q^3}{S^4}+\frac{6q^4}{S^3}+\frac{3q^5}{S^2}\right.\\ &\quad\left.+4q^7+3q^9S^2+5q^{10}S^3+7q^{11}S^4+8q^{12}S^5+4q^{13}S^6+10q^{14}S^7\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,11}(625n+364)q^n &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(25n+14)q^n+4f_1^3f_5^7\left(\frac{3}{R^5}+4q+8q^2R^5\right),\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(25n+14)q^n+4f_1^3f_5^7\left(3\left(\frac{1}{R^5}-11q-q^2R^5\right)+4q\right),\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,11}(25n+14)q^n+4f_1^3f_5^7\left(3\frac{f_1^6}{f_5^6}+4q\right),\\ &\equiv 3(3f_1^9f_5+3qf_1^3f_5^7)+f_1^9f_5+5qf_1^3f_5^7,\\ &\equiv 10f_1^9f_5+3qf_1^3f_5^7,\\ &\equiv \left(3f_1^9f_5+3qf_1^3f_5^7\right)+7f_1^9f_5,\\ &\equiv \sum_{n=0}^{\infty} B_{5,11}(25n+14)q^n+7\sum_{n=0}^{\infty} B_{5,11}(n)q^n.\end{aligned} \tag{5.3}$$

It follows that

$$
\begin{aligned}
B_{5,11}(625n+364) &\equiv B_{5,11}(25n+14)+7B_{5,11}(n),\\
B_{5,11}(3125n+2864) &\equiv B_{5,11}(125n+114)+7B_{5,11}(5n+4),\\
B_{5,11}(15625n+9114) &\equiv B_{5,11}(625n+364)+7B_{5,11}(25n+14),\\
&\equiv \big(B_{5,11}(25n+14)+7B_{5,11}(n)\big)+7B_{5,11}(25n+14),\\
&\equiv 8B_{5,11}(25n+14)+7B_{5,11}(n),\\
B_{5,11}(78125n+71614) &\equiv 8B_{5,11}(125n+114)+7B_{5,11}(5n+4),\\
B_{5,11}(390625n+227864) &\equiv 4B_{5,11}(25n+14)+B_{5,11}(n),\\
B_{5,11}(1953125n+1790364) &\equiv 4B_{5,11}(125n+114)+B_{5,11}(5n+4),\\
B_{5,11}(9765625n+5696614) &\equiv 5B_{5,11}(25n+14)+6B_{5,11}(n),\\
B_{5,11}(48828125n+44759114) &\equiv 5B_{5,11}(125n+114)+6B_{5,11}(5n+4)
\end{aligned}
$$

and

$$B_{5,11}(244140625n+142415364)\equiv 2B_{5,11}(n). \tag{5.4}$$

We have

$$
\begin{aligned}
\sum_{n=0}^{\infty}B_{5,11}(48828125n+44759114)q^n &\equiv 5\sum_{n=0}^{\infty}B_{5,11}(125n+114)q^n+6\sum_{n=0}^{\infty}B_{5,11}(5n+4)q^n,\\
&\equiv 5\left(3\sum_{n=0}^{\infty}B_{5,11}(5n+4)q^n+4f_1^7f_5^3\right)+6\sum_{n=0}^{\infty}B_{5,11}(5n+4)q^n,\\
&\equiv 10\sum_{n=0}^{\infty}B_{5,11}(5n+4)q^n+9f_1^7f_5^3,\\
&\equiv 10\big(9f_1^7f_5^3+2qf_1f_5^9\big)+9f_1^7f_5^3,\\
&\equiv 9qf_1f_5^9.
\end{aligned}
\tag{5.5}
$$

Substituting (2.11) into (5.5), we have

$$\sum_{n=0}^{\infty}B_{5,11}(48828125n+44759114)q^n\equiv 9f_5^9f_{25}q\left(\frac{1}{S}-q-q^2S\right).$$

There are no terms on the right in which the power of q is 0 or 4 modulo 5, so

$$B_{5,11}(244140625n+44759114)\equiv 0\,(\mathrm{mod}\ 11) \tag{5.6}$$

and

$$B_{5,11}(244140625n+240071614)\equiv 0\,(\mathrm{mod}\ 11). \tag{5.7}$$

Equation (1.7) follow from (5.4) by mathematical induction. Empolying (5.6) and (5.7) in (1.7), we obtain (1.8).

6 Congruences for (5, 13)-regular bipartitions modulo 13

6.1 Proof of Theorem 1.4

We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(n)q^n &= \frac{f_5 f_{13}}{f_1^2} \equiv \frac{f_5 f_1^{13}}{f_1^2} = f_1^{11} f_5,\\ &\equiv f_5 f_{25}^{11}\left(\frac{1}{S} - q - q^2 S\right)^{11},\\ &\equiv f_5 f_{25}^{11}\left(\frac{1}{S^{11}} + \frac{2q}{S^{10}} + \frac{5q^2}{S^9} + \frac{10q^3}{S^8} + \frac{7q^4}{S^7} + \frac{12q^5}{S^6} + \frac{6q^6}{S^5} + \frac{q^8}{S^3} + \frac{4q^9}{S^2} + \frac{3q^{10}}{S}\right.\\ &\quad + 8q^{11} + 10q^{12}S + 4q^{13}S^2 + 12q^{14}S^3 + 7q^{16}S^5 + 12q^{17}S^6 + 6q^{18}S^7 + 10q^{19}S^8\\ &\quad \left. + 8q^{20}S^9 + 2q^{21}S^{10} + 12q^{22}S^{11}\right).\end{aligned} \tag{6.1}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n &\equiv f_1 f_5^{11}\left(\frac{2}{R^{10}} + \frac{6q}{R^5} + 8q^2 + 7q^3R^5 + 2q^4R^{10}\right),\\ &\equiv f_1 f_5^{11}\left(2\left(\frac{1}{R^5} - 11q - q^2R^5\right)^2 + 11q\left(\frac{1}{R^5} - 11q - q^2R^5\right) + 8q^2\right),\\ &\equiv f_1 f_5^{11}\left(2\frac{f_1^{12}}{f_5^{12}} + 11q\frac{f_1^6}{f_5^6} + 8q^2\right),\\ &\equiv 2\frac{f_{13}}{f_5} + 11q f_1^7 f_5^5 + 8q^2 f_1 f_5^{11}.\end{aligned} \tag{6.2}$$

Substituting (2.11) in (6.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n &\equiv 2\frac{f_{325}}{f_5}\left(\frac{1}{S_1} - q^{13} - q^{26}S_1\right) + 11f_5^5 f_{25}^7 q\left(\frac{1}{S} - q - q^2S\right)^7\\ &\quad + 8f_5^{11} f_{25} q^2\left(\frac{1}{S} - q - q^2S\right),\\ &\equiv 2\frac{f_{325}}{f_5}\left(\frac{1}{S_1} - q^{13} - q^{26}S_1\right) + 11f_5^5 f_{25}^7 q\left(\frac{1}{S^7} + \frac{6q}{S^6} + \frac{q^2}{S^5} + \frac{7q^3}{S^4} + \frac{3q^4}{S^3}\right.\\ &\quad + \frac{q^5}{S^2} + \frac{12q^6}{S} + 10q^7 + q^8S + q^9S^2 + 10q^{10}S^3 + 7q^{11}S^4 + 12q^{12}S^5\\ &\quad \left. + 6q^{13}S^6 + 12q^{14}S^7\right) + 8f_5^{11} f_{25} q^2\left(\frac{1}{S} - q - q^2S\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(25n+16)q^n &\equiv 5f_1^{11}f_5 + 11f_1^5f_5^7\left(\frac{1}{R^5} + 10q + 12q^2R^5\right) + 11q^2\frac{f_{65}}{f_1},\\ &\equiv 5f_1^{11}f_5 + 11f_1^5f_5^7\left(\left(\frac{1}{R^5} - 11q - q^2R^5\right) + 8q\right) + 11q^2\frac{f_{65}}{f_1},\\ &\equiv 5f_1^{11}f_5 + 11f_1^5f_5^7\left(\frac{f_1^6}{f_5^6} + 8q\right) + 11q^2\frac{f_{65}}{f_1},\\ &\equiv 3f_1^{11}f_5 + 10qf_1^5f_5^7 + 11q^2\frac{f_{65}}{f_1}.\end{aligned} \tag{6.3}$$

Substituting (2.11) and (2.13) in (6.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(25n+16)q^n &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(n)q^n + 10f_5^7f_{25}^5q\left(\frac{1}{S} - q - q^2S\right)^5 + 11\frac{f_{65}f_{25}^5}{f_5^6}q^2\left(\frac{1}{S^4} + \frac{q}{S^3} + \frac{2q^2}{S^2}\right.\\ &\quad\left. + \frac{3q^3}{S} + 5q^4 - 3q^5S + 2q^6S^2 - q^7S^3 + q^8S^4\right),\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(n)q^n + 10f_5^7f_{25}^5q\left(\frac{1}{S^5} + \frac{8q}{S^4} + \frac{5q^2}{S^3} + \frac{10q^3}{S^2} + \frac{11q^4}{S} + 2q^5\right.\\ &\quad\left. + 2q^6S + 10q^7S^2 + 8q^8S^3 + 8q^9S^4 + 12q^{10}S^5\right)\\ &\quad + 11\frac{f_{65}f_{25}^5}{f_5^6}q^2\left(\frac{1}{S^4} + \frac{q}{S^3} + \frac{2q^2}{S^2} + \frac{3q^3}{S} + 5q^4 - 3q^5S\right.\\ &\quad\left. + 2q^6S^2 - q^7S^3 + q^8S^4\right),\end{aligned}$$

from which it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(125n+41)q^n &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n + 10f_1^7f_5^5\left(\frac{1}{R^5} + 2q + 12q^2R^5\right) + 3q\frac{f_{13}f_5^5}{f_1^6},\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n + 10f_1^7f_5^5\left(\frac{f_1^6}{f_5^6}\right) + 3q\frac{f_1^{13}f_5^5}{f_1^6},\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n + 10\frac{f_{13}}{f_5} + 3qf_1^7f_5^5.\end{aligned} \tag{6.4}$$

Substituting (2.11) in (6.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(125n+41)q^n &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n + 10\frac{f_{325}}{f_5}\left(\frac{1}{S_1} - q^{13} - q^{26}S_1\right)\\ &\quad + 3qf_5^5 f_{25}^7\left(\frac{1}{S} - q - q^2 S\right)^7,\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n + 10\frac{f_{325}}{f_5}\left(\frac{1}{S_1} - q^{13} - q^{26}S_1\right)\\ &\quad + 3qf_5^5 f_{25}^7\left(\frac{1}{S^7} + \frac{6q}{S^6} + \frac{q^2}{S^5} + \frac{7q^3}{S^4} + \frac{3q^4}{S^3} + \frac{q^5}{S^2}\right.\\ &\quad + \frac{12q^6}{S} + 10q^7 + q^8 S + q^9 S^2 + 10q^{10}S^3 + 7q^{11}S^4\\ &\quad \left. + 12q^{12}S^5 + 6q^{13}S^6 + 12q^{14}S^7\right).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} B_{5,13}(625n+416)q^n &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(25n+16)q^n + 3q^2\frac{f_{65}}{f_1} + 3f_1^5 f_5^7\left(\frac{1}{R^5} + 10q + 12q^2R^5\right),\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(25n+16)q^n + 3f_1^5 f_5^7\left(\left(\frac{1}{R^5} - 11q - q^2R^5\right) + 8q\right) + 3q^2\frac{f_{65}}{f_1},\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(25n+16)q^n + 3f_1^5 f_5^7\left(\frac{f_1^6}{f_5^6} + 8q\right) + 3q^2\frac{f_{65}}{f_1},\\ &\equiv 3\sum_{n=0}^{\infty} B_{5,13}(25n+16)q^n + 3f_1^{11}f_5 + 11qf_1^5 f_5^7 + 3q^2\frac{f_{65}}{f_1},\\ &\equiv 3\left(3f_1^{11}f_5 + 10qf_1^5 f_5^7 + 11q^2\frac{f_{65}}{f_1}\right) + 3f_1^{11}f_5 + 11qf_1^5 f_5^7 + 3q^2\frac{f_{65}}{f_1},\\ &\equiv 12f_1^{11}f_5 + 2qf_1^5 f_5^7 + 10q^2\frac{f_{65}}{f_1},\\ &\equiv 8\left(3f_1^{11}f_5 + 10qf_1^5 f_5^7 + 11q^2\frac{f_{65}}{f_1}\right) + f_1^{11}f_5,\\ &\equiv 8\sum_{n=0}^{\infty} B_{5,13}(25n+16)q^n + \sum_{n=0}^{\infty} B_{5,13}(n)q^n.\end{aligned}$$

It follows that

$$\begin{aligned}B_{5,13}(625n+416) &\equiv 8B_{5,13}(25n+16) + B_{5,13}(n),\\ \sum_{n=0}^{\infty} B_{5,13}(3125n+1041)q^n &\equiv 8\sum_{n=0}^{\infty} B_{5,13}(125n+41)q^n + \sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n,\\ &\equiv 8\left(3\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n + 10\frac{f_{13}}{f_5} + 3qf_1^7 f_5^5\right) + \sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n,\\ &\equiv 12\sum_{n=0}^{\infty} B_{5,13}(5n+1)q^n + 2\frac{f_{13}}{f_5} + 11qf_1^7 f_5^5,\\ &\equiv 12\left(2\frac{f_{13}}{f_5} + 11qf_1^7 f_5^5 + 8q^2 f_1 f_5^{11}\right) + 2\frac{f_{13}}{f_5} + 11qf_1^7 f_5^5,\\ &\equiv 5q^2 f_1 f_5^{11}.\end{aligned} \tag{6.5}$$

Substituting (2.11) into (6.5), we have

$$\sum_{n=0}^{\infty} B_{5,13}(3125n+1041)q^n \equiv 5f_5^{11}f_{25}q^2\left(\frac{1}{S}-q-q^2S\right). \tag{6.6}$$

There are no terms on the right in which the power of q is 0 or 1 modulo 5, so

$$B_{5,13}(15625n+1041) \equiv 0 \,(\text{mod } 13) \tag{6.7}$$

and

$$B_{5,13}(15625n+4166) \equiv 0 \,(\text{mod } 13). \tag{6.8}$$

From (6.6), we extract those terms in which the power of q is congruent to 3 modulo 5, divide by q^3 and replace q^5 by q, we have

$$B_{5,13}(15625n+10416) \equiv 8B_{5,13}(n) \,(\text{mod } 13). \tag{6.9}$$

Equation (1.9) follow from (6.9) by mathematical induction. Employing (6.7) and (6.8) in (1.9), we obtain (1.10).

7 Congruences for (81, 17)-regular bipartitions modulo 17

In this section we shall develop some results to prove our main Theorem 1.5.

Theorem 7.1 *For $n \geq 0$, we have*

$$B_{81,17}(27(3n+k)+23) \equiv 0 \,(\text{mod } 17), \text{where} k \in \{0,2\} \tag{7.1}$$

and

$$\sum_{n=0}^{\infty} B_{81,17}(81n+50)q^n \equiv 5f_1^{16} \,(\text{mod } 17). \tag{7.2}$$

Proof We have

$$\sum_{n=0}^{\infty} B_{81,17}(n)q^n = \frac{f_{81}f_{17}}{f_1^2}. \tag{7.3}$$

From (2.14) and (7.3), we have

$$\sum_{n=0}^{\infty} B_{81,17}(n)q^n \equiv f_{81}f_1^{15} \,(\text{mod } 17). \tag{7.4}$$

Substituting (2.9) into (7.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(n)q^n \equiv f_{81}f_9^{15}\Bigg(&\frac{1}{u^5}+\frac{2q}{u^4}+\frac{5q^2}{u^3}+\frac{5q^3}{u^2}+\frac{12q^4}{u}+4q^5+6q^6u+10q^7u^2+2q^8u^3\\ &+9q^9u^4+2q^{10}u^5+14q^{11}u^6+5q^{12}u^7+2q^{13}u^8+4q^{15}u^{10}\Bigg).\end{aligned} \tag{7.5}$$

If we extract those terms in which the power of q is congruent to 2 modulo 3, divide by q^2 and replace q^3 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(3n+2)q^n &\equiv f_{27}f_3^{15}\left(\frac{5}{v^3}+4q+2q^2v^3+14q^3v^6\right),\\ &\equiv f_{27}f_3^{15}\left(5\left(\frac{1}{v}+4qv^2\right)^3+12q\right).\end{aligned} \tag{7.6}$$

Substituting (2.10) into (7.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(3n+2)q^n &\equiv f_{27}f_3^{15}\left(5\left(\frac{f_1^{12}}{f_3^{12}}+27q\right)+12q\right),\\ &\equiv 5f_1^{12}f_{27}f_3^3+11qf_{27}f_3^{15}.\end{aligned} \tag{7.7}$$

Substituting (2.9) into (7.7), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(3n+2)q^n \equiv 5f_9^{12}f_{27}f_3^3\Big(&\frac{1}{u^4}+\frac{5q}{u^3}+\frac{3q^2}{u^2}+\frac{10q^3}{u}+5q^4+7q^5u+4q^6u^2+2q^7u^3+14q^8u^4\\ &+q^9u^5+14q^{10}u^6+q^{12}u^8\Big)+11qf_{27}f_3^{15}.\end{aligned} \tag{7.8}$$

If we extract those terms in which the power of q is congruent to 1 modulo 3, divide by q and replace q^3 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(9n+5)q^n &\equiv 5f_3^{12}f_9f_1^3\left(\frac{5}{v^3}+5q+2q^2v^3+14q^3v^6\right)+11f_9f_1^{15},\\ &\equiv 5f_3^{12}f_9f_1^3\left(5\left(\frac{1}{v}+4qv^2\right)^3+13q\right)+11f_9f_1^{15}.\end{aligned} \tag{7.9}$$

Substituting (2.10) into (7.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(9n+5)q^n &\equiv 5f_3^{12}f_9f_1^3\left(5\left(\frac{f_1^{12}}{f_3^{12}}+27q\right)+13q\right)+11f_9f_1^{15},\\ &\equiv 2f_9f_1^{15}+9qf_3^{12}f_9f_1^3.\end{aligned} \tag{7.10}$$

Substituting (2.9) into (7.10), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(9n+5)q^n \equiv 2f_9^{16}\Big(&\frac{1}{u^5}+\frac{2q}{u^4}+\frac{5q^2}{u^3}+\frac{5q^3}{u^2}+\frac{12q^4}{u}+4q^5+6q^6u+10q^7u^2+2q^8u^3\\ &+9q^9u^4+2q^{10}u^5+14q^{11}u^6+5q^{12}u^7+2q^{13}u^8+4q^{15}u^{10}\Big)\\ &+9f_3^{12}f_9^4\left(qu^{-1}-3q^2+4q^4u^2\right).\end{aligned} \tag{7.11}$$

If we extract those terms in which the power of q is congruent to 2 modulo 3, divide by q^2 and replace q^3 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(27n+23)q^n &\equiv 2f_3^{16}\left(\frac{5}{v^3}+4q+2q^2v^3+14q^3v^6\right)+7f_1^{12}f_3^4,\\ &\equiv 2f_3^{16}\left(5\left(\frac{1}{v}+4qv^2\right)^3+12q\right)+7f_1^{12}f_3^4.\end{aligned} \tag{7.12}$$

Substituting (2.10) into (7.12), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{81,17}(27n+23)q^n &\equiv 2f_3^{16}\left(5\left(\frac{f_1^{12}}{f_3^{12}}+27q\right)+12q\right)+7f_1^{12}f_3^4,\\ &\equiv 5qf_3^{16}.\end{aligned} \tag{7.13}$$

This completes the proof of equation (7.1) and (7.2), follow from (7.13). □

Theorem 7.2 *For $n \geq 0$, we have*

$$b_{17}\left(4^9n+\frac{2(4^8-1)}{3}\right)\equiv 0\,(\mathrm{mod}\ 17), \tag{7.14}$$

$$b_{17}\left(4^9n+\frac{2(4^9-1)}{3}\right)\equiv 8b_{17}(n)\,(\mathrm{mod}\ 17) \tag{7.15}$$

and

$$b_{17}\left(2\cdot 4^8n+\frac{5\cdot 4^8-2}{3}\right)\equiv b_{17}(2n+1)\,(\mathrm{mod}\ 17) \tag{7.16}$$

Proof Setting $l=17$ in (1.1), we have

$$\sum_{n=0}^{\infty}b_{17}(n)q^n=\frac{f_{17}}{f_1}\equiv f_1^{16}\,(\mathrm{mod}\ 17). \tag{7.17}$$

Substituting (2.42.5) into (7.17), we have

$$\begin{aligned}\sum_{n=0}^{\infty}b_{17}(n)q^n&\equiv\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)^4,\\&\equiv\frac{f_4^{40}}{f_2^8f_8^{16}}+q\frac{f_4^{28}}{f_2^4f_8^8}+11q^2f_4^{16}+16q^3f_2^4f_4^4f_8^8+q^4\frac{f_2^8f_8^{16}}{f_4^8}.\end{aligned} \tag{7.18}$$

It follows that

$$\sum_{n=0}^{\infty}b_{17}(2n)q^n\equiv\frac{f_2^{40}}{f_1^8f_4^{16}}+11qf_2^{16}+q^2\frac{f_1^8f_4^{16}}{f_2^8} \tag{7.19}$$

and

$$\sum_{n=0}^{\infty}b_{17}(2n+1)q^n\equiv\frac{f_2^{28}}{f_1^4f_4^8}+16qf_1^4f_4^8f_2^4 \tag{7.20}$$

Substituting (2.3), (2.42.5) into (7.19), we have

$$\sum_{n=0}^{\infty}b_{17}(2n)q^n\equiv\frac{f_2^{40}}{f_4^{16}}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2+11qf_2^{16}+q^2\frac{f_4^{16}}{f_2^8}\left(\frac{f_4^{10}}{f_2^2f_8^4}-4q\frac{f_2^2f_8^4}{f_4^2}\right)^2 \tag{7.21}$$

If we extract those terms in which the power of q is congruent to 1 modulo 2, divide by q and replace q^2 by q, we have

$$\begin{aligned}\sum_{n=0}^{\infty}b_{17}(4n+2)q^n&\equiv 2f_1^{16}+9q\frac{f_2^{24}}{f_1^8},\\&\equiv 2\sum_{n=0}^{\infty}b_{17}(n)q^n+9q\frac{f_2^{24}}{f_1^8}.\end{aligned} \tag{7.22}$$

Substituting (2.3) into (7.22), we have

$$\sum_{n=0}^{\infty}b_{17}(4n+2)q^n\equiv 2\sum_{n=0}^{\infty}b_{17}(n)q^n+9qf_2^{24}\left(\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2.$$

If we extract those terms in which the power of q is congruent to 0 modulo 2 and replace q^2 by q, we have

$$\sum_{n=0}^{\infty}b_{17}(8n+2)q^n\equiv 2\sum_{n=0}^{\infty}b_{17}(2n)q^n+4\sum_{n=0}^{\infty}b_{17}(n)q^{2n+1}.$$

It follows that

$$\sum_{n=0}^{\infty} b_{17}(16n+10)q^n \equiv 2\sum_{n=0}^{\infty} b_{17}(4n+2)q^n + 4\sum_{n=0}^{\infty} b_{17}(n)q^n$$

and

$$b_{17}(16n+10) \equiv 2b_{17}(4n+2) + 4b_{17}(n).$$

It follows that

$$\begin{aligned} b_{17}(64n+42) &\equiv 2b_{17}(16n+10) + 4b_{17}(4n+2), \\ &\equiv 2(2b_{17}(4n+2) + 4b_{17}(n)) + 4b_{17}(4n+2), \\ &\equiv 8b_{17}(4n+2) + 8b_{17}(n). \end{aligned} \tag{7.23}$$

Similarly, we find

$$\begin{aligned} b_{17}(256n+170) &\equiv 7b_{17}(4n+2) + 15b_{17}(n), \\ b_{17}(1024n+682) &\equiv 12b_{17}(4n+2) + 11b_{17}(n), \\ b_{17}(4096n+2730) &\equiv b_{17}(4n+2) + 14b_{17}(n), \\ b_{17}(16384n+10922) &\equiv 16b_{17}(4n+2) + 4b_{17}(n), \\ b_{17}(65536n+43690) &\equiv 2b_{17}(4n+2) + 13b_{17}(n), \\ b_{17}(262144n+174762) &\equiv 8b_{17}(n). \end{aligned} \tag{7.24}$$

We have

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(65536n+43690)q^n &\equiv 2\sum_{n=0}^{\infty} b_{17}(4n+2)q^n + 13\sum_{n=0}^{\infty} b_{17}(n)q^n, \\ &\equiv 2\left(2f_1^{16} + 9q\frac{f_2^{24}}{f_1^8}\right) + 13f_1^{16}, \\ &\equiv q\frac{f_2^{24}}{f_1^8}. \end{aligned} \tag{7.25}$$

substituting (2.3) into (7.25), we have

$$\begin{aligned} \sum_{n=0}^{\infty} b_{17}(65536n+43690)q^n &\equiv 5qf_2^{24}\left(\frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}\right)^2, \\ &\equiv q\frac{f_4^{28}}{f_2^4f_8^8} + 8q^2f_4^{16} + q^3f_2^4f_4^4f_8^8. \end{aligned} \tag{7.26}$$

It follows that

$$\sum_{n=0}^{\infty} b_{17}(131072n+43690)q^n \equiv 8qf_2^{16} \tag{7.27}$$

and

$$\sum_{n=0}^{\infty} b_{17}(131072n+109226)q^n \equiv \frac{f_2^{28}}{f_1^4f_4^8} + 16qf_1^4f_2^4f_4^8. \tag{7.28}$$

This completes the proof of equation (7.14), (7.15) and (7.16), follow from (7.27) and (7.28). □

7.1 Proof of Theorem 1.5

From equation (7.2) and (7.17), we have

$$B_{81,17}(81n + 50) \equiv 5b_{17}(n). \tag{7.29}$$

By the mathematical induction equation (7.15), can be written as

$$b_{17}\left(4^{9m}n + \frac{2(4^{9m} - 1)}{3}\right) \equiv 8^m b_{17}(n). \tag{7.30}$$

Now, replacing $n = 4^{9m}n + \frac{2 \cdot 4^{9m} - 2}{3}$ in (7.29) and compare the equation (7.30), we have (1.12). The results (1.11) and (1.13) are from (7.14), (7.16) and (7.29).

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