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ORIGINAL RESEARCH

Averages of shifted convolutions of general divisor sums involving Hecke eigenvalues

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Abstract Suppose $\sum_{n=1}^{\infty} a(n)n^{-s}$ be a Dirichlet series in the Selberg class of degree d and let $E(x)$ be the arithmetical error term of $\sum_{n\leqslant x} a(n)$. By the truncated Tong-type formula of $E(x)$, we can get two kinds of the mean square estimates of $E(x)$ in short intervals of Jutila's type. Using the estimates, we are able to improve some previous results established by Lü and Wang [9].

Keywords Shifted convolutions · Hecke eigenvalue · Mean square estimate in short interval · Truncated Tong-type formula

Mathematics Subject Classification 11F30 · 11F37 · 11M41 · 11N37

1 Introduction and main results

1.1 Mean square in short intervals of Jutila's type

Let $a(n)$ be an arithmetical function and let $E(x)$ be the arithmetical error term defined by

$$E(x) = \sum_{n\leqslant x}{}' a(n) - T(x),$$

where the prime in the summation symbol means that the last term $a(x)$ should be halved when x is an integer and $T(x)$ is the main term. Jutila [6] first studied the mean square of $E(x)$ in short intervals of the following type

$$\int_{T}^{T+H} |E(x+U) - E(x)|^2 dx$$

under certain conditions on U and H, he mainly studied the divisor function $d(n)$.

As we all know that $\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s)$ and the error term $\Delta(x)$ of the summatory function of $d(n)$ is given by

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$$\Delta(x) = \sum_{n\leqslant x}{}' d(n) - x(\log x + 2\gamma - 1) - \frac{1}{4},$$

where γ is the Euler constant. He proved that if $T \geqslant 2$ and $1 \leqslant U \ll T^{1/2} \ll H \leqslant T$, then

$$\begin{aligned}\int_T^{T+H} (\Delta(x+U) - \Delta(x))^2 dx &= \frac{1}{4\pi^2} \sum_{n\leqslant T/2U} \frac{d^2(n)}{n^{3/2}} \int_T^{T+H} x^{1/2} \left| \exp\left(2\pi i \left(\frac{n}{x}\right)^{1/2} U\right) - 1 \right|^2 dx \\ &\quad + O(T^{1+\epsilon} + HU^{1/2}T^{\epsilon}),\end{aligned}$$

which implies that

$$\int_T^{T+H} (\Delta(x+U) - \Delta(x))^2 dx \ll HU \log^3\left(\frac{T^{1/2}}{U}\right) + T^{1+\epsilon}$$

holds for $T^{\epsilon} \ll U \leqslant T^{1/2} \ll H \leqslant T$.

1.2 The truncated Tong-type formula of $E(x)$

This formula was first obtained by Tong [10] in 1956 in the case of $\zeta^k(s)$. Cao, Tanigawa and Zhai [1] reconstructed the truncated Tong-type formula for Dirichlet series in the extended Selberg class. Next we recall the definition of Selberg class.

1.2.1 Selberg class

Let $\mathcal{N}$ be the set of non-vanishing Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

which satisfies the following hypotheses.

I. Ramanujan's conjecture: For all $n, a(n) \ll n^{\epsilon}$, where $\epsilon > 0$ is an arbitrarily small positive number.

II. Analytic continuation: There exists a non-negative integer m_l such that $(s-1)^{m_l} L(s)$ is an entire function of finite order.

III. Functional equation: Correction in R.H.S. of the modified functional equation namely we have,

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^{L} \Gamma(\alpha_j s + \beta_j)$$

with positive real numbers Q, α_j, and complex numbers β_j, ω, with Re $\beta_j \geqslant 0$ and $|\omega| = 1$. The number

$$d := 2\sum_{j=1}^{L} \alpha_j$$

is called the degree of $L(s)$.

IV. Euler product: $L(s)$ satisfies

$$L(s) = \prod_p \exp\left(\sum_{n\geqslant 1} \frac{b(p^n)}{p^{ns}}\right)$$

with suitable coefficients $b(p^n)$ satisfying $b(p^n) \ll p^{nc}$ for some $c < 1/2$.

Let $1/2 \leqslant \sigma^* < 1$ denote a number such that the estimate

$$\sigma^* := \inf\left\{\sigma | \int_0^T |L(\sigma+it)|^2 dt \ll T^{1+\epsilon}\right\}$$

holds for any $\epsilon > 0$. σ^* also satisfies the condition

$$\sigma^* \leqslant \frac{d+1}{2d}.$$

The number σ^* which plays an important role in Tong's theory.

Let $L(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ be a Dirichlet series of degree d which satisfies conditions I, II and III, $a(n) \in \mathbb{R}$ and $L(s)$ has no poles in $\sigma \leqslant \sigma^*$.

Let

$$b(n) = \omega Q^{-1} a(n), \quad \mu_n = Q^{-2} n,$$

and

$$L^*(s) = \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s},$$

then the functional equation of $L(s)$ can be written as

$$L(s)\prod_{j=1}^{L}\Gamma(\alpha_j s + \beta_j) = L^*(1-s)\prod_{j=1}^{L}\Gamma(\alpha_j(1-s)+\beta_j).$$

Let $\widehat{Q}(x)$ be the sum of residues over all poles of $L^*(s)x^s/s$ except the pole $s = 0$, and let

$$\widehat{E}(x) = \sum_{\mu_n \leqslant x} b(n) - \widehat{Q}(x).$$

Let $y \geqslant 1, \frac{1}{2} \leqslant M \leqslant N < \infty$ and λ a real number. Define

$$I(\lambda, M, N, y) = 2\pi i \int_M^N u^{\lambda}\widehat{E}(u)\exp(-ih(uy)^{1/d})du,$$

where h is defined by

$$h = d\left(\prod_{j=1}^{L}\alpha_j^{\alpha_j}\right)^{-2/d}.$$

In Tong's theory a certain averaging method plays an important role. Let x be a large parameter and let k be a suitable large integer. We write

$$\int_{E_k} g(\widetilde{y})dY_k = \int_0^1 \cdots \int_0^1 g(\widetilde{y})dy_1 \cdots dy_k,$$

where $g(y)$ is an integrable function and

$$\widetilde{y} = y + \frac{1}{x}(y_1 + \cdots + y_k).$$

1.2.2 The truncated Tong-type formula of E(x) for Selberg class

Let $1 \leqslant x \leqslant y \leqslant (1+\delta)x$, where $\delta > 0$ is a fixed constant and let $N = [x^{2d-1-\epsilon}]$ and $J = [(d^2+2d)\epsilon^{-1}]$. We also let $h' = hQ^{-2/d}$. Then in every subinterval $[t, t+Bt^{1-1/d}] \subset [1, \sqrt{N}]$, there exists a real number M such that the following truncated Tong-type formula holds:

$$E(y) = \sum_{j=1}^{7} R_j(y),$$

where

$$R_1(y) = \kappa_0 y^{\frac{d-1}{2d}} \sum_{n \leqslant M'} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos(h'(ny)^{1/d} + c_0\pi),$$

$$R_2(y) = y^{\frac{d+1}{2d}} Re(c_{00} I(\lambda_0, M, N, y)),$$

$$R_3(y) = \sum_{\substack{0 \leqslant l,m \leqslant J \\ l+m>0}} Re\left(c_{lm} I\left(\lambda_0 + \frac{l-m}{d}, M, N, y\right)\right) x^{-l} y^{-l+\frac{1}{2}+\frac{1}{2d}+\frac{l-m}{d}},$$

$$R_4(y) = \sum_{0 \leqslant l,m \leqslant k} Re\left(c'_{lm} I\left(\lambda_0 - \frac{k+m}{d}, N, \infty, y + \frac{l}{x}\right)\right) x^k \left(y + \frac{l}{x}\right)^{k+\frac{1}{2}+\frac{1}{2d}-\frac{k+m}{d}},$$

$$R_5(y) \ll x^{\frac{1}{2}-\frac{3}{2d}} M^{\max(\frac{1}{2}-\frac{3}{2d},0)} \log^A x,$$

$$R_6(y) \begin{cases} = 0, \text{if} \quad a(n) \geqslant 0, \\ \ll x^{\frac{1}{2}-\frac{1}{2d}} M^{-\frac{1}{2}-\frac{1}{2d}+\epsilon}, \qquad \text{otherwise}, \end{cases}$$

$$R_7(y) = E(y) - \int_{E_k} E(\widetilde{y}) dY_k,$$

where $\kappa_0, c_0, c_{lm}, c'_{lm}$ are constants, $M' = Q^2 M$ and

$$\lambda_0 = -\frac{3}{2} + \frac{1}{2d}.$$

1.3 General divisor sums involving Hecke eigenvalues

In this paper, we consider primitive holomorphic cusp forms in the full modular group $SL(2, \mathbb{Z})$, which are eigenfunctions of all Hecke operators T_n. Denote by S_k the set of normalized primitive holomorphic cusp forms of even integral weight $k \geqslant 2$, and $g \in (S_k)^{prim}$ implies π is a cuspidal representation of $GL(2)$, has level one and $\pi^\infty = D_k$, the holomorphic discrete series with lowest k, where k is a positive even integer. It is well known that $g(z)$ has a Fourier expansion at the cusp infinity

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n) n^{(k-1)/2} e(nz), \tag{1.1}$$

where we normalize $\lambda_g(1) = 1$, and $\lambda_g(n) \in \mathbb{R}$ is the nth eigenvalue of the Hecke operator T_n. Deligne [3] proved the Ramanujan-Petersson conjecture, which asserts that

$$|\lambda_g(n)| \leqslant d(n) \ll n^{\epsilon} \tag{1.2}$$

for all integers $n \geqslant 1$, where $d(n)$ denotes the Dirichlet divisor function.

The Hecke L-function associated to $g(z)$ is defined by

$$L(s, g) = \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1} \tag{1.3}$$

for Rs $s > 1$, where local parameters α_p and β_p satisfy

$$\alpha_p + \beta_p = \lambda_g(p) \quad \text{and} \quad \alpha_p \beta_p = 1. \tag{1.4}$$

For any fixed integer $j \geqslant 1$, we define

$$L(s, g)^j := \sum_{n=1}^{\infty} \frac{\lambda_{j,g}(n)}{n^s} \tag{1.5}$$

for Re $s > 1$. It is easy to find that

$$\lambda_{j,g}(n) = \sum_{n_1 n_2 \cdots n_j = n} \lambda_g(n_1) \cdots \lambda_g(n_j) \ll n^{\epsilon}.$$

It is well know that

$$L(s,g) = \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s}$$

is an $L-$function of degree 2, entire on $\mathbb{C}$ and satisfies the functional equation

$$(2\pi)^{-s}\Gamma\left(s + \frac{k-1}{2}\right)L(s,g) = (-1)^{k/2}(2\pi)^{-(1-s)}\Gamma\left(1 - s + \frac{k-1}{2}\right)L(1-s,g).$$

Obviously

$$(2\pi)^{-js}\Gamma^j\left(s + \frac{k-1}{2}\right)L^j(s,g) = (-1)^{jk/2}(2\pi)^{-j(1-s)}\Gamma^j\left(1 - s + \frac{k-1}{2}\right)L^j(1-s,g).$$

From the above equation, we can easily see

$$L^j(s,g) = \sum_{n=1}^{\infty} \frac{\lambda_{j,g}(n)}{n^s}$$

satisfies conditions I, II and III, $\lambda_{j,g}(n) \in \mathbb{R}$, $L^j(s,g)$ has no poles in $\sigma \leqslant \sigma^*$. We can easily get $Q = (2\pi)^{-j}, d = 2j, \omega = (-1)^{\frac{jk}{2}}, h = 2j, a(n) = \lambda_{j,g}(n)$.

We define

$$S_{j,g}(x) := \sum_{n \leqslant x} \lambda_{j,g}(n),$$

using the mean square in short intervals of $S_{j,g}(x)$

$$\int_T^{T+H} \left| S_{j,g}(x+U) - S_{j,g}(x) \right|^2 dx,$$

we can study the shifted convolution sum of

$$\sum_{h \leqslant H} \sum_{N < n \leqslant 2N} \lambda_{j,g}(n)\lambda_{j,g}(n+h), \quad j \geqslant 1.$$

Lü and Wang [9] have studied the averages of shifted convolutions of general divisor sums involving Hecke eigenvalues of classical holomorphic cusp forms for the full modular group $SL(2,\mathbb{Z})$, namely

$$\sum_{h \leqslant H} \sum_{N < n \leqslant 2N} \lambda_{j,g}(n)\lambda_{j,g}(n+h), \qquad \text{for} \quad j \geqslant 2.$$

In [9], the authors mainly use the mean value of Hecke L- function, the subconvexity bounds for Hecke L-function, double perron formula, partial summation, and Hölder inequality.

The goal of this paper is to study the averages of shifted convolutions of general divisor sums involving Hecke eigenvalues of classical holomorphic cusp forms for the full modular group $SL(2,\mathbb{Z})$ under a certain assumption by the Tong-type truncated formula of $\sum_{N < n \leqslant 2N} \lambda_{j,g}(n)$. In other words, we reduce the mean square of $\sum_{N < n \leqslant 2N} \lambda_{j,g}(n)$ to the problem of finding a suitable constant σ^* which is related to the mean square estimate of $L(s,g)^j$.

1.4 Statement of results

Let $1/2 \leqslant (\sigma_j)^* < 1$ denote a number such that the estimate

$$(\sigma_j)^* := \inf\left\{\sigma \Big| \int_0^T |L(\sigma + it, g)|^{2j} dt \ll T^{1+\epsilon}\right\}$$

holds for any $\epsilon > 0$. For the number $(\sigma_j)^*$, we assume $(\sigma_j)^* \leqslant \frac{2j+1}{4j}$.

Theorem 1 *Assume σ^* satisfy the condition above, and $1 \leqslant H \leqslant N$. Then for any $\epsilon > 0$, we have*

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{1,g}(n)\lambda_{1,g}(n+h) \ll N^{1+\epsilon}H^{\frac{1}{2}}.$$

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{2,g}(n)\lambda_{2,g}(n+h) \ll \begin{cases} N^{1+\epsilon}H^{1/2}, & \text{for } H \geqslant N^{3/4}, \\ N^{11/8+\epsilon}, & \text{for } N^{7/16} \leqslant H \leqslant N^{3/4}, H \leqslant N^{1/3}, \\ N^{\frac{6}{5}+\epsilon}H^{\frac{2}{5}}, & \text{for } N^{1/3} \leqslant H \leqslant N^{7/16}. \end{cases}$$

For $j \geqslant 3$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\lambda_{j,g}(n+h) \ll \begin{cases} N^{1+\epsilon}H^{1/2}, & \text{for } N^{1-\frac{1}{2j}} \leqslant H \leqslant N, \\ N^{\frac{3}{2}-\frac{1}{4j}+\epsilon}, & \text{for } H \leqslant N^{1-\frac{1}{2j}}. \end{cases}$$

Corollary 1 *Assume that $\sigma_j^* = 1/2$, Then for $1 \leqslant H \leqslant N$, we have*

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\lambda_{j,g}(n+h) \ll \begin{cases} N^{1+\epsilon}H^{1/2}, & \text{for } N^{1-\frac{1}{2j-1}} \leqslant H \leqslant N, \\ N^{\frac{3}{2}-\frac{1}{4j-2}+\epsilon}, & \text{for } H \leqslant N^{1-\frac{1}{2j-1}}. \end{cases}$$

2 Some lemmas

To prove Theorem 1, we need the following lemmas.

Lemma 2.1 *Assume that $\sigma^* \leqslant \frac{2j+1}{4j}$. Let δ be a real number such that $0 < \delta < (5/4)^j - 1$. Then for U and H such that $U \gg 1, H \gg 1$ and $U + H < \delta T$, we have*

$$\int_T^{T+H}\Big(S_{j,g}(x+U) - S_{j,g}(x)\Big)^2 dx \ll T^{\epsilon}\Big\{HU + HT^{1-1/j}U^{1/2j} + T^{2-\frac{3-4\sigma^*}{4j(1-\sigma^*)-1}}\Big\}.$$

Proof See Theorem 1 of Cao, Tanigawa and Zhai [2]. □

Lemma 2.2 *Assume that $\sigma^* \leqslant \frac{2j+1}{4j}$. Let δ be a real number such that $0 < \delta < (5/4)^j - 1$. Then for $1 \ll U \ll T^{1-1/2j-\epsilon}$ and $1 \ll H < \delta T$, we have*

$$\int_T^{T+H}\Big(S_{j,g}(x+U) - S_{j,g}(x)\Big)^2 dx \ll HU^2T^{-1+1/2j+\epsilon} + T^{1+\epsilon}U(T^{2j-1}U^{-2j})^{2\sigma^*-1} + HT^{1-1/2j+\epsilon}.$$

Proof See Theorem 2 of Cao, Tanigawa and Zhai [2]. □

Lemma 2.3 *For any $\epsilon > 0$, we have*

$$\int_1^T\left|L\left(\frac{1}{2}+it,g\right)\right|^2 dt \sim CT\log T, \tag{2.1}$$

$$\int_1^T\left|L\left(\frac{5}{8}+it,g\right)\right|^4 dt \ll T^{1+\epsilon}, \tag{2.2}$$

uniformly for $T \geqslant 1$.

Proof The result (2.1) was established by Good [4]; (2.2) was proved by Ivić [5]. □

Lemma 2.4 *Let $S_{j,g}(x) = \sum_{n\leqslant x}\lambda_{j,g}(n)$, then for any ϵ, we have*

$$S_{j,g}(x) \ll \begin{cases} \frac{x^{1/3}}{\log^A x}, & \text{for } j=1, \\ x^{1/2+\epsilon}, & \text{for } j=2, \\ x^{3/5+\epsilon}, & \text{for } j=3, \\ x^{1-3/2j+\epsilon}, & \text{for } j \geqslant 4, \end{cases}$$

where $A = 0.118\cdots$.

Proof See Theorem 2 of Wu [11] and Theorem 1 of Lü [7]. □

3 Proof of Theorem 1

We define

$$G_j(N,H) := \sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\lambda_{j,g}(n+h).$$

By switching the order of summations, we get

$$\begin{aligned} G_j(N,H) &= \sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\sum_{h\leqslant H}\lambda_{j,g}(n+h) \\ &= \sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\left(\sum_{m\leqslant n+H}\lambda_{j,g}(m)-\sum_{m\leqslant n}\lambda_{j,g}(m)\right). \end{aligned} \tag{3.1}$$

By Cauchy's inequality, we have

$$\begin{aligned} G_j(N,H) &\ll \left(\sum_{N<n\leqslant 2N}\lambda_{j,g}^2(n)\right)^{1/2}\left(\sum_{N<n\leqslant 2N}\left(\sum_{m\leqslant n+H}\lambda_{j,g}(m)-\sum_{m\leqslant n}\lambda_{j,g}(m)\right)^2\right)^{1/2} \\ &\ll N^{1/2+\epsilon}\times\left(\sum_{N<n\leqslant 2N}\Big(S_{j,g}(n+H)-S_{j,g}(n)\Big)^2\right)^{1/2}, \end{aligned} \tag{3.2}$$

where we have used the trivial bound

$$\sum_{N<n\leqslant 2N}\lambda_{j,g}^2(n) \ll N^{1+\epsilon}.$$

Next we mainly study the sum

$$W = \sum_{N<n\leqslant 2N}\Big(S_{j,g}(n+H)-S_{j,g}(n)\Big)^2.$$

Write

$$W = \sum_{n=N+1}^{2N}\int_n^{n+1}\Big(S_{j,g}(n+H)-S_{j,g}(n)\Big)^2 dy = W_1+W_2,$$

where

$$W_1 = \sum_{n=N+1}^{2N}\int_n^{n+1}\Big(S_{j,g}(y+H)-S_{j,g}(y)\Big)^2 dy = \int_{N+1}^{2N+1}\Big(S_{j,g}(y+H)-S_{j,g}(y)\Big)^2 dy$$

and

$$W_2 = \sum_{n=N+1}^{2N} \int_n^{n+1} \left(\left(S_{j,g}(n+H) - S_{j,g}(n) \right)^2 - \left(S_{j,g}(y+H) - S_{j,g}(y) \right)^2 \right) dy.$$

By Lemma 2.1, we have

$$W_1 \ll N^{1+\epsilon} H + N^{2-\frac{1}{j}+\epsilon} H^{\frac{1}{2j}} + N^{2-\frac{3-4\sigma^*}{4j(1-\sigma^*)-1}+\epsilon}. \tag{3.3}$$

By Lemma 2.2, we have

$$W_1 \ll N^{\frac{1}{2j}+\epsilon} H^2 + N^{2-\frac{1}{2j}+\epsilon} + N^{1+\epsilon} H (N^{2j-1} H^{-2j})^{2\sigma^*-1}. \tag{3.4}$$

Next we consider S_2. The integrand in S_2 is

$$\begin{aligned} & \left(S_{j,g}(n+H) - S_{j,g}(n) \right)^2 - \left(S_{j,g}(y+H) - S_{j,g}(y) \right)^2 \\ = \; & \left(S_{j,g}(n+H) - S_{j,g}(n) + S_{j,g}(y+H) - S_{j,g}(y) \right) \\ \times \; & \left(S_{j,g}(n+H) - S_{j,g}(n) - S_{j,g}(y+H) + S_{j,g}(y) \right) \\ \ll \; & S_{j,g}(N) \left| S_{j,g}(n+H) - S_{j,g}(n) - S_{j,g}(y+H) + S_{j,g}(y) \right|, \end{aligned} \tag{3.5}$$

here we use the mean value theorem, for $n \leqslant y < n+1$,

$$\begin{aligned} S_{j,g}(n+H) - S_{j,g}(n) &= S'_{j,g}(n_1) H, \\ S_{j,g}(y+H) - S_{j,g}(y) &= S'_{j,g}(y_1) H, \\ S'_{j,g}(n_1) H - S'_{j,g}(y_1) H &= S''_{j,g}(z_1)(n_1 - y_1) H. \end{aligned}$$

where n_1, y_1, z_1 are real numbers such that $n < n_1, y_1 \leqslant n+H$. Inserting the above estimates into the formula (3.5), we have

$$W_2 \ll NH |S_{j,g}(N)| |S''_{j,g}(N)|. \tag{3.6}$$

From (3.3) and (3.6), we get

$$W \ll N^{1+\epsilon} H + N^{2-\frac{1}{j}+\epsilon} H^{\frac{1}{2j}} + N^{2-\frac{3-4\sigma^*}{4j(1-\sigma^*)-1}+\epsilon} + N^{1+\epsilon} H |S_{j,g}(N)| |S''_{j,g}(N)|,$$

which together with (3.2) gives

$$G_j(N,H) \ll \; N^{1+\epsilon} H^{1/2} + N^{\frac{3}{2}-\frac{1}{2j}+\epsilon} H^{\frac{1}{4j}} + N^{\frac{3}{2}-\frac{3-4\sigma^*}{8j(1-\sigma^*)-2}+\epsilon} + N^{1+\epsilon} H^{1/2} |S_{j,g}(N)|^{1/2} |S''_{j,g}(N)|^{1/2}. \tag{3.7}$$

From (3.4) and (3.6), we get for $H \leqslant N^{1-1/2j-\epsilon}$,

$$W \ll N^{\frac{1}{2j}+\epsilon} H^2 + N^{2-\frac{1}{2j}+\epsilon} + N^{1+\epsilon} H (N^{2j-1} H^{-2j})^{2\sigma^*-1} + N^{1+\epsilon} H |S_{j,g}(N)| |S''_{j,g}(N)|,$$

which together with (3.2) gives for $H \leqslant N^{1-1/2j-\epsilon}$,

$$\begin{aligned} G_j(N,H) \ll \; & N^{\frac{1}{2}+\frac{1}{4j}} H + N^{\frac{3}{2}-\frac{1}{4j}+\epsilon} + N^{1+\epsilon} H^{1/2} (N^{2j-1} H^{-2j})^{\sigma^*-1/2} \\ + \; & N^{1+\epsilon} H^{1/2} |S_{j,g}(N)|^{1/2} |S''_{j,g}(N)|^{1/2}. \end{aligned} \tag{3.8}$$

3.1 Case $j = 1$

By Lemma 2.3 and Lemma 2.4, we have $\sigma^* = \frac{1}{2} < \frac{3}{4}, d = 2$.

$$S_{1,g}(x) \ll x^{\frac{1}{3}}.$$

By (3.7), we can easily get for $H \leqslant N$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{1,g}(n)\lambda_{1,g}(n+h)\ll N^{1+\epsilon}H^{\frac{1}{2}}. \tag{3.9}$$

By (3.8), for $H\leqslant N^{1/2}$, we can get

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{1,g}(n)\lambda_{1,g}(n+h)\ll N^{5/4+\epsilon}. \tag{3.10}$$

By (3.9) and (3.10), we obtain for $H\leqslant N$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{1,g}(n)\lambda_{1,g}(n+h)\ll N^{1+\epsilon}H^{\frac{1}{2}}.$$

Remark Lü [8] investigated the average of one general $GL(n)\times GL(2k)$ shifted convolution sum, he proved that for any $\epsilon>0$, $X^{2\vartheta_k-1+\epsilon}\leqslant H\leqslant X^{\frac{1}{2\vartheta_k-1}-\epsilon}$,

$$\sum_{h=1}^{H}\sum_{n=1}^{X}a(n)\lambda_{\pi}(n+h)\ll H^{\frac{1}{2}}(H+X)^{\vartheta_k+\epsilon}\|a\|_2,$$

where $a(n)$ is an arbitrary arithmetic function, $\|a\|_2=\left(\sum_{n=1}^{X}|a(n)|^2\right)^{\frac{1}{2}}$, $\pi=\pi_1\boxplus\pi_2\boxplus\cdots\boxplus\pi_k$ is an isobaric representation of $GL(2k)$, and $\pi_1,\pi_2,\cdots,\pi_k$ are cuspidal automorphic representations of $GL(2)$ with trivial conductor and trivial central character. Here $\vartheta_1=\frac{1}{2},\vartheta_2=\frac{3}{4}$, and for $k\geqslant 3,\vartheta_k=\frac{1-\vartheta_{k_1}\vartheta_{k_2}}{2-\vartheta_{k_1}-\vartheta_{k_2}}$ with k_1,k_2 being positive integers satisfying $k=k_1+k_2$.

When we take $k=1$,

$$\sum_{h=1}^{H}\sum_{n=1}^{X}\lambda_g(n)\lambda_g(n+h)\ll H^{\frac{1}{2}}X^{1+\epsilon},$$

we have the same result.

3.2 Cases $j=2$

By Lemma 2.3 and Lemma 2.4, we have $\sigma^*\leqslant\frac{5}{8},d=4$.

$$S_{2,g}(x)\ll x^{\frac{1}{2}+\epsilon}.$$

For

$$f(\sigma^*):=\frac{3}{2}-\frac{3-4\sigma^*}{4d(1-\sigma^*)-2}$$

is a monotonic increasing function, so we can take $\sigma^*=\frac{5}{8}$.

By (3.7), we can easily get for $H\leqslant N$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{2,g}(n)\lambda_{2,g}(n+h)\ll N^{1+\epsilon}H^{\frac{1}{2}}+N^{\frac{11}{8}}. \tag{3.11}$$

By (3.8), we can get for $H\leqslant N^{3/4}$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{2,g}(n)\lambda_{2,g}(n+h)\ll N^{\frac{11}{8}}. \tag{3.12}$$

Combining (3.11) and (3.12), we have for $H\leqslant N$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{2,g}(n)\lambda_{2,g}(n+h)\ll N^{1+\epsilon}H^{\frac{1}{2}}+N^{\frac{11}{8}}.$$

Lü and Wang [9] studied

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{2,g}(n)\lambda_{2,g}(n+h)\ll N^{\frac{6}{5}+\epsilon}H^{\frac{2}{5}}, N^{1/3}\leqslant H\leqslant N^{1-\epsilon}.$$

We have improved the result of [14],

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{2,g}(n)\lambda_{2,g}(n+h)\ll\begin{cases} N^{1+\epsilon}H^{1/2}, & \text{for } H\geqslant N^{3/4},\\ N^{11/8+\epsilon}, & \text{for } N^{7/16}\leqslant H\leqslant N^{3/4}, H\leqslant N^{1/3},\\ N^{\frac{6}{5}+\epsilon}H^{\frac{2}{5}}, & \text{for } N^{1/3}\leqslant H\leqslant N^{7/16}.\end{cases}$$

3.3 Cases $j=3$

By Lemma 2.3 and Lemma 2.4, we have $\sigma^*\leqslant\frac{2j+1}{4j}, d=6$.

$$S_{3,g}(x)\ll x^{\frac{3}{5}+\epsilon}.$$

By (3.7), we can easily get for $H\leqslant N$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{3,g}(n)\lambda_{3,g}(n+h)\ll N^{1+\epsilon}H^{\frac{1}{2}}+N^{\frac{4}{3}}H^{\frac{1}{12}}+N^{\frac{3}{2}-\frac{3-4\sigma^*}{24(1-\sigma^*)-2}+\epsilon}. \tag{3.13}$$

By (3.8), we can get for $H\leqslant N^{5/6}$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{3,g}(n)\lambda_{3,g}(n+h)\ll N^{\frac{17}{12}+\epsilon}+NH^{1/2}(N^5H^{-6})^{\sigma^*-1/2}. \tag{3.14}$$

If we take $\sigma^*=7/12$, we have for $H\leqslant N$,

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{3,g}(n)\lambda_{3,g}(n+h)\ll N^{1+\epsilon}H^{\frac{1}{2}}+N^{\frac{17}{12}+\epsilon}.$$

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{3,g}(n)\lambda_{3,g}(n+h)\ll\begin{cases} N^{1+\epsilon}H^{1/2}, & \text{for } N^{5/6}\leqslant H\leqslant N,\\ N^{17/12+\epsilon}, & \text{for } H\leqslant N^{5/6}.\end{cases}$$

Lü and Wang [9] studied

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{3,g}(n)\lambda_{3,g}(n+h)\ll N^{\frac{4}{3}+\epsilon}H^{\frac{1}{3}}, N^{1/2}\leqslant H\leqslant N^{1-\epsilon}.$$

We have improved the result of [9].

3.4 Cases $j\geqslant 4$

We use the same method to study, we have the following result.

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\lambda_{j,g}(n+h)\ll\begin{cases} N^{1+\epsilon}H^{1/2}, & \text{for } N^{1-\frac{1}{2j}}\leqslant H\leqslant N,\\ N^{\frac{3}{2}-\frac{1}{4j}+\epsilon}, & \text{for } H\leqslant N^{1-\frac{1}{2j}}.\end{cases}$$

Lü and Wang [9] studied

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\lambda_{j,g}(n+h)\ll N^{\frac{4j+1}{2j+4}+\epsilon}H^{\frac{2}{j+2}}, N^{(2j-3)/2j}\leqslant H\leqslant N^{1-\epsilon}.$$

We have improved the result of [9].

If we take $\sigma_j^*=1/2$ in Lemma 2.1 and Lemma 2.2, by the above method, we can get

Corollary 1 *Assume that* $\sigma_j^*=1/2$, *Then for* $1\leqslant H\leqslant N$, *we have*

$$\sum_{h\leqslant H}\sum_{N<n\leqslant 2N}\lambda_{j,g}(n)\lambda_{j,g}(n+h)\ll\begin{cases} N^{1+\epsilon}H^{1/2}, & \text{for } N^{1-\frac{1}{2j-1}}\leqslant H\leqslant N,\\ N^{\frac{3}{2}-\frac{1}{4j-2}+\epsilon}, & \text{for } H\leqslant N^{1-\frac{1}{2j-1}}.\end{cases}$$

This completes the proof of Theorem 1.

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