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ORIGINAL RESEARCH

On embedding of 4-manifolds

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Abstract We discuss embedding of 4-manifolds in Euclidean spaces of dimension ≤ 8. We give an elementary proof of the fact that every 4-manifold admits a topological embedding in $\mathbb{R}^8$. As an application of our method of proof, we also deduce some embedding results for compact 4-manifolds with nonempty boundary.

Keywords 4-Manifold · Embedding

Mathematics Subject Classification Primary 57 Q35 · Secondary 57 R40

1 Introduction

One of the fundamental problems in geometric topology is to study embeddings of manifolds. By the works of Hirsch [2] and Wall [3], every closed 3-manifold embeds in $\mathbb{R}^5$. For 4-manifolds $\mathbb{R}^8$ is optimal in terms of the codimension of embedding. In this note, we give a new elementary proof of topological embedding of 4-manifolds in $\mathbb{R}^8$.

Theorem 1.1 *Every closed* 4*-manifold admits topological embedding in* $\mathbb{R}^8$.

Note that to prove the smooth version of Theorem 1.1 one requires the famous Whitney trick.

As an application of our method to prove Theorem 1.1, we get the following smooth embedding results for compact 4-manifolds with nonempty boundaries.

Theorem 1.2 *Let* M^3 *be a closed* 3*-manifold and let* $L = \bigsqcup_{i=1}^{n} \gamma_i$ *be a framed link in the boundary of the* 4*-disk* $\mathbb{D}^4$, *where each knot* γ_i *is an unknot and has an even framing.*

(1) There exists a 4-manifold W^4 bounding M^3 such that W^4 admits smooth embedding in $\mathbb{R}^6$.
(2) Let V^4 be a 4-manifold obtained by attaching 2-handles to $\mathbb{D}^4$ along L. Then V^4 smoothly embeds in $\mathbb{R}^5$.

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In [3], Wall showed that every closed 3-manifold bounds a 4-manifold that immerses in $\mathbb{R}^5$. Statement (1) of Theorem 1.2 can be seen as a similar statement for smooth embedding in $\mathbb{R}^6$.

Unless mentioned otherwise, all the embeddings are assumed to be smooth. All the results mentioned in this note are well known classical results in both the topoological and smooth category. In particular, one can use the Smale-Hirsch immersion theory (and Whitney trick for smooth closed 4-manifolds) to deduce most of the results given here. Here, we take a more elementary and visual approach to these results and avoid any use of the Smale-Hirsch theory.

2 Embedding 4-manifolds

Let us begin by recalling some basic definitions.

Definition 2.1 (Meridian/ Longitude) Let $\mathbb{S}^1 \times \mathbb{D}^2$ be a solid torus. We say that a knot γ on $\partial(\mathbb{S}^1 \times \mathbb{D}^2) = \mathbb{S}^1 \times \mathbb{S}^1$ is a meridian(or longitude) provided γ is isotopic to $z_0 \times \mathbb{S}^1$ (or $\mathbb{S}^1 \times w_0$) in $\mathbb{S}^1 \times \mathbb{S}^1$, where $z_0 \in \mathbb{S}^1$ (or $w_0 \in \partial\mathbb{D}^2$). We denote the meridian by $\mathfrak{m}$ and the longitude by $\mathfrak{l}$.

We shall not distinguish between the meridian $\mathfrak{m}$ (or longitude $\mathfrak{l}$) and the homology class $[\mathfrak{m}]$ (or $[\mathfrak{l}]$) representing $\mathfrak{m}$ (or $\mathfrak{l}$).

Definition 2.2 (k-type embedding) We say that an embedding $\phi : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1 \times \mathbb{D}^n$ sending $(z, x, y) \to (z, x, y, \mathbf{0})$ is of k-type provided there exist a family $\phi_t : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1 \times \mathbb{D}^n$ of embeddings that satisfies the following:

(1) $\phi_0 = Id$.
(2) For each t, ϕ_t is identity on the first factor.
(3) $\phi_1(\mathbb{S}^1 \times \mathbb{D}^2) = \phi(\mathbb{S}^1 \times \mathbb{D}^2)$.
(4) $[\phi_1(\mathfrak{l})] = k[\phi(\mathfrak{m})] + [\phi(\mathfrak{l})]$ (see Fig. 1).

Lemma 2.3 *A natural embedding $\phi : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1 \times \mathbb{D}^4$ sending $(z, x, y) \to (z, x, y, 0, 0)$ is of k-type.*

Proof Let $\tau_{\mathbb{C}P^1} = \{([l], z_0, z_1) \;/\; (z_0, z_1) \in l\} \subset \mathbb{C}P^1 \times \mathbb{C}^2$ be the tautological line bundle over $\mathbb{C}P^1$. Consider the quotient map $q : \mathbb{S}^1 \times [0, 1] \to \mathbb{S}^2 = \mathbb{C}P^1$ collapsing $\mathbb{S}^1 \times \{0\}$ to south pole and $\mathbb{S}^1 \times \{1\}$ to north pole and the map $a_t : \mathbb{S}^1 \to \mathbb{S}^1 \times [0, 1]$ defined by $a_t(z) = (z, t)$ for each $t \in [0, 1]$.

Next consider the family of solid tori obtained as a disc bundle of $(a_t \circ q)^*(\tau_{\mathbb{C}P^1})$, where $(a_t \circ q)^*(\tau_{\mathbb{C}P^1})$ denotes the pull back bundle of $\tau_{\mathbb{C}P^1}$ via $a_t \circ q$. It is easy to see that each solid tori in the family is contained in $\mathbb{S}^1 \times \mathbb{R}^4$. This is because, $\tau_{\mathbb{C}P^1}$ is contained in $\mathbb{C}P^1 \times \mathbb{C}^2$. Therefore $(a_t \circ q)^*(\tau_{\mathbb{C}P^1})$ is also contained in $\mathbb{S}^1 \times \mathbb{C}^2$. Notice that, since a_0 is constant, we can assume that $(a_0 \circ q)^*$ is an embedding ϕ. Next, observe that when we trivialize $\tau_{\mathbb{C}P^1}$ over upper hemisphere as $U_1 \times \mathbb{C}$ and lower hemisphere $U_2 \times \mathbb{C}$ of $\mathbb{S}^2 = \mathbb{C}P^1$ the transition map from $\mathbb{S}^1 \times \mathbb{C} \to \mathbb{S}^1 \times \mathbb{C}$ sends longitude $\mathfrak{l}$ of $\mathbb{S}^1 \times \mathbb{D}^2$ to the curve γ such that $[\gamma] =$

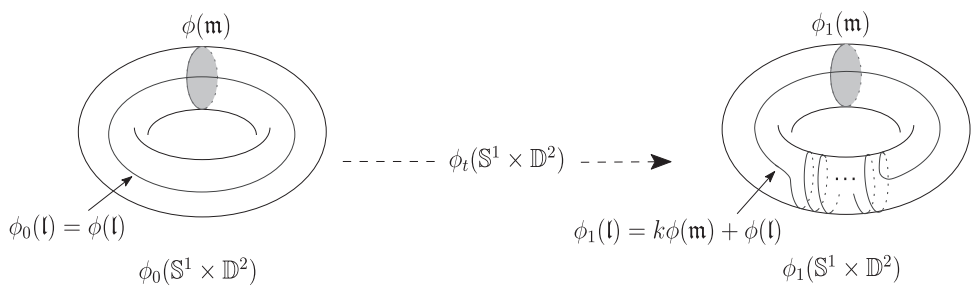

Fig. 1 For a k-type embedding, the longitude $[\mathfrak{l}]$ can be sent to the curve $[\mathfrak{l}] + k[\mathfrak{m}]$ via an isotopy of $\mathbb{S}^1 \times \mathbb{D}^n$ satisfying the conditions in Definition 2.2

$1[\mathfrak{m}] + 1[\mathfrak{l}]$. Thus we conclude that the embedding ϕ is of $+1$-type. By repeated and reverse application of this we can ensure that the embedding ϕ is of k-type. □

Note that Lemma 2.3 also implies that the natural embedding of $\mathbb{S}^1 \times \mathbb{D}^2$ in $\mathbb{S}^1 \times \mathbb{D}^m$ is of k-type for all $m \geq 4$.

Lemma 2.4 *Let M be a compact 4-manifold consisting of 0-handle, 1-handles and 2-handles. Then there exists an embedding of M into $\mathbb{R}^6$.*

Proof Let $H_0 \cup H_1^1 \cup \ldots \cup H_1^{r_1} \cup H_2^1 \cup \ldots \cup H_2^{r_2}$ be a handlebody decomposition of M, where H_k^r denotes the k-handle. The handlebody consisting of 0-handles and 1-handles is diffeomorphic to $M_1 = \mathbb{S}^1 \times \mathbb{D}^3 \# \mathbb{S}^1 \times \mathbb{D}^3 \# \ldots \# \mathbb{S}^1 \times \mathbb{D}^3 \ \ (s_1 \ \ times) \# \mathcal{M}_b \times \mathbb{D}^2 \# \cdots \# \mathcal{M}_b \times \mathbb{D}^2 \ \ (s_2 \ \ times)$, where $s_1 + s_2 = r_1$ and $\mathcal{M}_b$ is the compact Mobius band. Note that, if M is orientable then $s_2 = 0$. We decompose M as $M = M_1 \cup H_2^1 \cup \ldots \cup H_2^{r_2}$. Let $\gamma_i \subset \partial M_1$ be an attaching circle with framing coefficient n_i for each 2-handle H_2^i. We denote the tubular neighborhoods of γ_i in ∂M_1 by $\mathcal{N}_{\partial M_1}(\gamma_i)$. Let $\alpha_i, \beta_i \subset \partial \mathcal{N}_{\partial M_1}(\gamma_i)$ be the knots corresponding to framing coefficients $n_i, 0$ respectively. We then observe the following.

(1) M is diffeomorphic to $M_1 \bigsqcup_{Id,i=1}^{n} \mathcal{N}_{\partial M_1}(\gamma_i) \times [-1,1] \bigsqcup_{f_i,i=1}^{n} (\mathbb{D}^2 \times \mathbb{D}^2)_i$, where Id is the identity map between $\mathcal{N}_{M_1}(\gamma_i)$ and $\mathcal{N}_{M_1}(\gamma_i) \times \{-1\}$ and f_i identifies $\mathcal{N}_{M_1}(\gamma_i) \times \{1\}$ and $(\mathbb{S}^1 \times \mathbb{D}^2)_i = \partial(\mathbb{D}^2 \times \mathbb{D}^2)_i$ so that the 2–handle $(\mathbb{D}^2 \times \mathbb{D}^2)_i$ is attached with framing n_i.
(2) $\mathbb{D}^6$ is diffeomorphic to $\mathbb{D}^6 \bigsqcup_h \partial\mathbb{D}^6 \times [-1,1]$, where h identifies $\partial\mathbb{D}^6$ and $\partial\mathbb{D}^6 \times \{-1\}$ and h is isotopic to the identity map.

Let g be a proper embedding of M_1 in $\mathbb{D}^6$. Since M_1 actually embeds in $\mathbb{S}^5$, such a g always exists. Note that, g embeds the tubular neighborhoods $\bigsqcup_{i=1}^{n} \mathcal{N}_{M_1}(\gamma_i)$ in $\mathbb{S}^5 = \partial\mathbb{D}^6$ as $\bigsqcup_{i=1}^{n} g(\mathcal{N}_{M_1}(\gamma_i))$. Let $\mathbb{S}^2 \subset \mathbb{S}^3 \subset \mathbb{S}^4 \subset \mathbb{S}^5$ be fixed inclusion of the standard spheres. Now by the single parameter version of the Thom transversality theorem (see Lemma 7.8 in [1] and), these embeded tubular neighborhoods can be isotoped by an isotopy ρ_t of $\mathbb{S}^5$ such that the followings are satisfied.

(1) $\rho_1(g(\mathcal{N}_{M_1}(\gamma_i))) \subset \mathbb{S}^3 \subset \mathbb{S}^5$ and $\rho_1(g(\gamma_i))$'s bounds mutually disjoint discs $\mathcal{D}_i^2$ in $\mathbb{S}^2 \subset \mathbb{S}^3$.
(2) Tubular neighborhoods $\mathcal{N}_{\mathbb{S}^4}(\mathcal{D}_i^2)$ of $\mathcal{D}_i^2$'s in $\mathbb{S}^4 \subset \mathbb{S}^5$ are mutually disjoint. For each i, $\mathcal{N}_{\mathbb{S}^4}(\mathcal{D}_i^2)$ is identified with $\mathcal{D}_i^2 \times \mathbb{D}^2$ via a diffeomorphism $w_i : \mathcal{D}_i^2 \times \mathbb{D}^2 \to \mathcal{N}_{\mathbb{S}^4}(\mathcal{D}_i^2) \subset \mathbb{S}^5$.
(3) $\rho_1(g(\mathcal{N}_{M_1}(\gamma_i))) = w_i(\partial\mathcal{D}_i^2 \times \mathbb{D}^2)$.

Using ρ_t, we define an embedding $\psi' : M_1 \bigsqcup_{Id,i=1}^{n} \mathcal{N}_{M_1}(\gamma_i) \times [-1,0] \to \mathbb{D}^6 \bigsqcup_h (\partial\mathbb{D}^6 \times [-1,0])$ as follows.

(1) $\psi' : M_1 \to \mathbb{D}^6$ is given by $\psi'(x) = g(x)$.
(2) $\psi' : \mathcal{N}_{M_1}(\gamma_i) \times [-1,0] \to \partial\mathbb{D}^6 \times [-1,0]$ is given by $\psi'(x,t) = (\rho_{t+1}(g(x)), t)$.

Let k_i be the linking number of $\psi'(\gamma_i \times \{0\})$ and $\psi'(\alpha_i \times \{0\})$ for each i. Next, we use $-k_i$-type embeddings of $\mathbb{S}^1 \times \mathbb{D}^2$ in $\mathbb{S}^1 \times \mathbb{D}^4 \subset \mathbb{S}^5 \times \{0\}$ for embedding the remaining part of M. To do this, we fix an identification $\xi_i : \mathcal{N}_{\partial M_1}(\gamma_1) \to (\mathbb{S}^1 \times \mathbb{D}^2)_i$ such that $\xi_i(\beta_i)$ is the longitude $\mathfrak{l}$ of $(\mathbb{S}^1 \times \mathbb{D}^2)_i$. We also fix an identification η_i from the tubular neighborhoods $\mathcal{N}_{\partial\mathbb{D}^6 \times \{0\}}(\psi'(\gamma_i \times \{0\}))$ of $\psi'(\gamma_i \times \{0\})$ in $\partial\mathbb{D}^6 \times \{0\}$ to $(\mathbb{S}^1 \times \mathbb{D}^4)_i$. We make this identification such that the maps $\eta_i \circ \psi' \circ \xi_i^{-1} : (\mathbb{S}^1 \times \mathbb{D}^2)_i \to (\mathbb{S}^1 \times \mathbb{D}^4)_i$ sends (z, x, y) to $(z, x, y, 0, 0)$. By lemma 2.3, the maps $\phi^i = \eta_i \circ \psi' \circ \xi_i^{-1}$ are $-k_i$–type embeddings. Thus, for each i there exists a family $\phi_t^i : (\mathbb{S}^1 \times \mathbb{D}^2)_i \to (\mathbb{S}^1 \times \mathbb{D}^4)_i$ of embeddings which satisfies Definition 2.2. In particular, ϕ_1^i sends the curve $\mathfrak{l} + k_i[\mathfrak{m}]$ to $\mathfrak{l} - k_i[\mathfrak{m}] + k_i[\mathfrak{m}] = \mathfrak{l}$. This allows us to extend the embedding ψ' in the following way.

We define $\psi' : \bigsqcup_{Id,i=1}^{n} \mathcal{N}_{M_1}(\gamma_i) \times [0,1] \to \partial\mathbb{D}^6 \times [0,1]$ as $\psi'(x,t) = (\eta_i^{-1} \circ \phi_t^i \circ \xi_i(x), t)$. Now observe that $\psi'(\mathcal{N}_{M_1}(\gamma_i) \times \{1\})$ belongs to the boundary of a 2–handle $w_i(\mathcal{D}_i^2 \times \mathbb{D}^2) \times \{1\}$, embedded in $\mathbb{S}^4 \subset \mathbb{S}^5 \times \{1\}$. Note that $w_i(\mathcal{D}_i^2 \times \{0\}) \cup (\psi'(\gamma_i \times [-1,1])$ gives the core disk of the 2-handle that is attached along $\gamma_i \subset \mathbb{S}^5 \times \{-1\}$ with framing n_i. Moreover, if $\hat{\mathcal{D}}_i^2$ denotes a parallel disk bounding the longitude $\psi'(\alpha_i \times \{1\})$ of $\psi'(\mathcal{N}_{M_1}(\gamma_i) \times \{1\})$, then $\hat{\mathcal{D}}_i^2 \cup (\psi'(\alpha_i \times [-1,1])$ gives a parallel disk bounding the

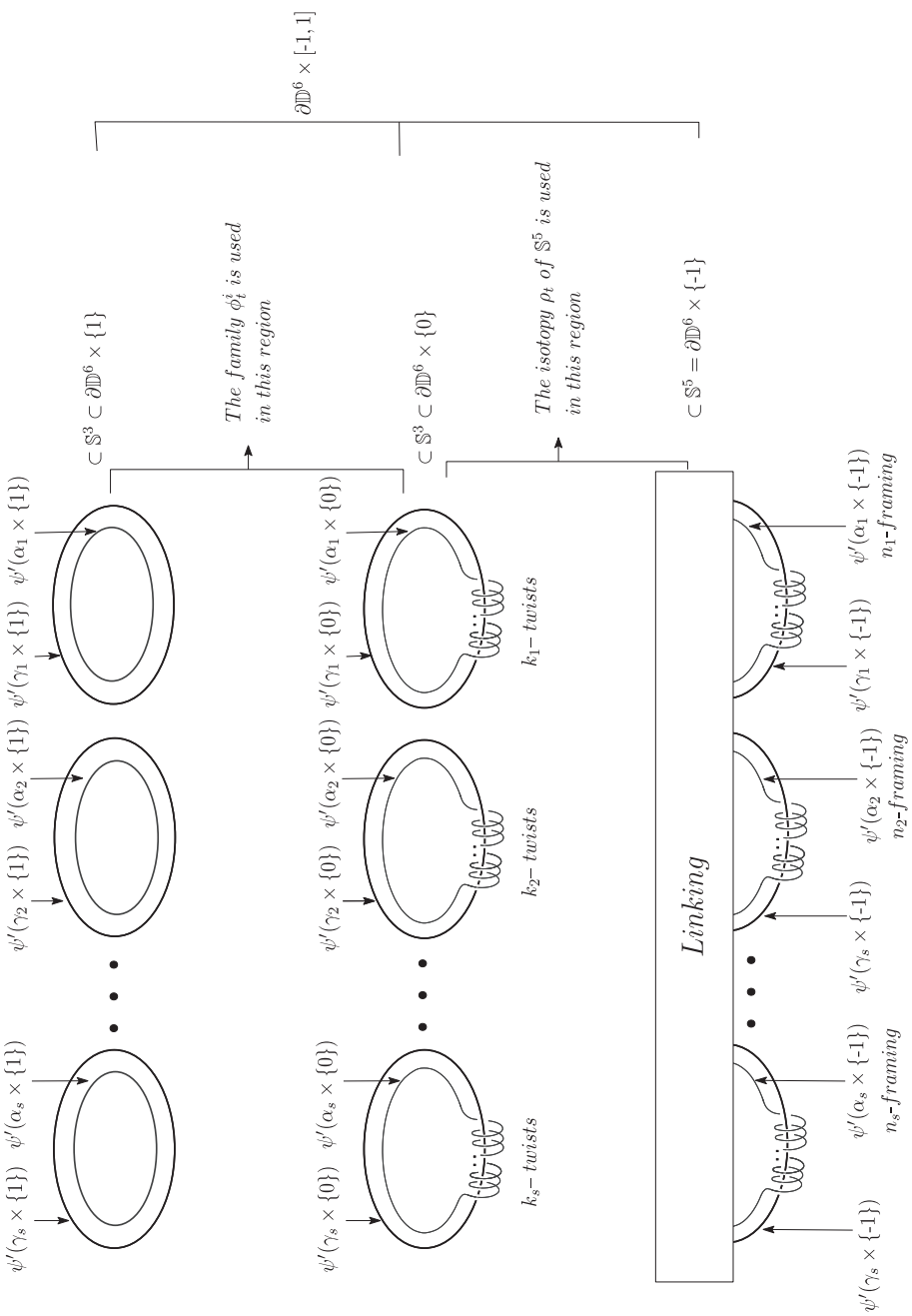

Fig. 2 Figure depicts the pictorial description of the embedding $\psi' : \mathcal{N}_{\partial\mathbb{D}^4}(\gamma) \times [0,1] \to \partial\mathbb{D}^6 \times [-1,1]$

curve with framing n_i. The union of the map ψ' and the maps $w_i \times \{1\}$'s together defines the desired embedding. For a pictorial description of the embedding, see Figure 2. □

Proof (**Proof of Theorem** 1.1) Let M be a closed 4-manifold and let $H_0 \cup H_1^1 \cup \ldots \cup H_1^{r_1} \cup H_2^1 \cup \ldots \cup H_2^{r_2} \cup H_3^1 \cup \ldots \cup H_3^{r_3} \cup H_4$ be its handlebody decomposition, where H_k^r denotes the k-handle. We denote the manifold consisting of all the handles of M upto index k by M_k.

We write $\mathbb{R}^8$ as $\mathbb{R}^6 \times \mathbb{R} \times \mathbb{R}$. Then we use Lemma 2.4 to embed M_2 in $\mathbb{R}^6 \times \{(0,0)\}$. we denote this embedding by the map ψ_1. Now notice that there is only one choice for the framming of attaching 3-handles. Therefore, to extend the embedding ψ_1 to the 3-handles of M, it is enough to show that the attaching spheres of 3-handles embeded in $\mathbb{R}^6 \times \{(0,0)\}$ bounds properly embeded disjoint 3-balls in $\mathbb{R}^7_+ \times \{0\}$, where $\mathbb{R}^7_+ = \{(\mathbf{x}, y) \ / \ \mathbf{x} \in \mathbb{R}^6, y > 0\}$. Let us show that the attaching spheres $\mathcal{S}_i^2$ of 3-handles embeded in $\mathbb{R}^6 \times \{(0,0)\}$ bounds properly embeded disjoint 3-balls in $\mathbb{R}^7_+ \times \{0\}$. Again, by Thom's Transversality theorem, there exists an isotopy $\rho_t^i : \mathbb{S}^2 \to \mathbb{R}^6$ such that $\rho_0^i(\mathbb{S}^2) = \mathcal{S}_i^2$ and $\rho_1^i(\mathbb{S}^2)$ is a standard 2-sphere in $\mathbb{R}^3 \subset \mathbb{R}^6$. For each i the map $\rho^i : \mathbb{S}^2 \times [0,1] \cup \mathbb{D}^3 \times \{1\} \to \mathbb{R}^6 \times \mathbb{R} \times \mathbb{R}$ is defined as follows.

(1) $\rho^i : \mathbb{S}^2 \times [0,1] \to \mathbb{R}^6 \times \mathbb{R} \times \mathbb{R}$ is defined as $\rho^i(x,t) = (\rho_t^i(x), t, 0)$.
(2) $\rho^i : \mathbb{D}^3 \times \{1\} \to \mathbb{R}^6 \times \mathbb{R} \times \mathbb{R}$ is defined as $\rho^i(y,1) = (y, 0, 0, 0, 1, 0)$.

Thus each $\mathcal{S}_i^2$ bounds a properly embeded 3-ball in $\mathbb{R}^7_+ \times \{0\}$. Moreover, by transversality, we can assume these 3-balls in $\mathbb{R}^7_+ \times \{0\}$ to be mutually disjoint. Thus we get an embedding of M_3 in $\mathbb{R}^7 \times \{0\}$. We denote this embedding by the map ψ_2. Note that ∂M_3 is an embeded $\mathbb{S}^3$ in $\mathbb{R}^7$. We embed the cone $\mathcal{C}(\partial M_3)$ over ∂M_3 in $\mathbb{R}^7 \times [0,1] \subset \mathbb{R}^8$ to obtain a topological embedding of M in $\mathbb{R}^8$. □

Proof (**Proof of Theorem** 1.2)

(1) Due to the works of Lickorish ([4, 5]) we know that every closed, oriented, connected 3-manifold can be seen as the boundary of a 4-manifold obtained from $\mathbb{D}^4$ by attaching 2-handles along a framed link of unknots in $\partial\mathbb{D}^4$. For non-orientable 3-manifolds, one needs to attach a single non-orientable 1-handle ($\mathcal{M}_b \times \mathbb{D}^2$) to $\mathbb{D}^4$ and then attach 2-handles along framed links in $\partial(\mathbb{D}^4 \cup \mathcal{M}_b \times \mathbb{D}^2)$. Thus, the result follows from Lemma 2.4.
(2) We claim that the natural embedding $\Phi : \mathbb{S}^1 \times \mathbb{D}^2 \to \mathbb{S}^1 \times \mathbb{R}^3$ sending $(z,x,y) \to (z,x,y,0)$ is of $2k$–type for any $k \in \mathbb{Z}$. One can see this in the following way. Recall that in the proof of Lemma 2.3, we used the tautological line bundle $\tau_{\mathbb{C}P^1} \subset \mathbb{C}P^1 \times \mathbb{C}^2$ over $\mathbb{C}P^1$ to produce the isotopy reqired to show k–type embedding in $\mathbb{S}^1 \times \mathbb{D}^4$. Let us now consider the tangent bundle $T\mathbb{S}^2 \subset \mathbb{S}^2 \times \mathbb{R}^3$ of $\mathbb{S}^2$. Note that when we trivialize $T\mathbb{S}^2$ by restricting to the upper hemisphere and the lower hemisphere of S^2, the transition map from $\mathbb{S}^1 \times \mathbb{D}^2$ to itself sends the longitude $\mathfrak{l}$ to the curve $[\gamma] = 2[\mathfrak{m}] + 1[\mathfrak{l}]$. Using arguments similar to Lemma 2.3, one can then show that Φ is a $2k$-type embedding. Now consider a proper embedding $g : \mathbb{D}^4 \to \mathbb{D}^5$. Take the even framed link L inside $g(\partial\mathbb{D}^4)$. Let S_0^1 be a component of L. Note that g induces the embedding Φ of $S_0^1 \times \mathbb{D}^2 \subset g(\partial\mathbb{D}^4)$ in $S_0^1 \times \mathbb{D}^3 \subset \partial\mathbb{D}^5$. An argument similar to Lemma 2.4, then shows that V^4 embeds in $\mathbb{D}^5 \cup \partial\mathbb{D}^4 \times [0,1] \cong \mathbb{D}^5$.

□

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