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ORIGINAL RESEARCH

An improvement on the perfect order subsets of finite groups

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Abstract A finite group $\mathbb{G}$ is said to have perfect order subsets if for every d, the number of elements of $\mathbb{G}$ of order d (if there are any) divides $|\mathbb{G}|$. In this paper, we make a modest improvement on the result of Ford, Konyagin and Luca [3] about perfect order subsets.

Keywords Estimates of prime numbers · Sieves · Arithmetic functions · Abelian groups

Mathematics Subject Classification 11A25 11N25

1 Introduction and main results

A finite group $\mathbb{G}$ is said to have perfect order subsets if for every d, the number of elements of $\mathbb{G}$ of order d (if there are any) divides $|\mathbb{G}|$. Finch and Jones [5] introduced the notion of perfect order subsets, and they studied the form of finite Abelian groups,

$$\mathbb{G} = \prod_{i=1}^{k} \left(\frac{\mathbb{Z}}{p_i \mathbb{Z}} \right)^{a_i},$$

where p_i are primes and $a_i \geqslant 1$. They proved that $\mathbb{G}$ has perfect order subsets if and only if $f(n)|n$ where $n = |\mathbb{G}|$,

$$f(n) = \prod_{p^a \| n} (p^a - 1).$$

On the other hand, all of the known examples of n such that $n > 6$ and $f(n)|n$ (given in [5]) satisfy $n = 2f(n)$.

Finch and Jones [5] asked several basic questions about such groups. One of which asks if $|\mathbb{G}|$ is not a power of 2, must 3 divide $|\mathbb{G}|$? Ford, Konyagin and Luca [3] proved that for any $n \in \mathbb{N}$, if $f(n)|n$, and $n > 2$, then $3|n$ and $n/f(n) \leqslant 85$. Inspired by their result, we mainly focus our attention on the bound of $n/f(n)$.

Using Preissmann's [7] better results on the weighted sieve, the more sophisticated bounds for primes, and the better bound for the explicit averages of non-negative multiplicative functions [9], we establish Theorem 1 that has improved the result of Ford [4] about the general explicit upper sieve bound for shifted primes.

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Theorem 1 *Suppose S be a set of primes containing 2, $U(x) = U(x; S)$ be the number of primes $p \leqslant x$ with $p-1$ composed only of primes in S. Let*

$$H(x) = \prod_{\substack{p \leqslant x \\ p \in S}} \left(1 - \frac{1}{p}\right),$$

then for $x > 1$,

$$U(x) < \frac{x}{I(x)(1 + \frac{2.0308}{\log x})}, \qquad I(x) = \int_1^{\sqrt{x}} \frac{\log t}{t} H(t)dt.$$

In Theorem 1, we make a minor improvement to Ford [4] from $1/\log x$ to $2.0308/\log x$. The improvement could be applied to his $\sigma(\phi(m))$ problem, but it appears no advances are possible.

Theorem 2 *Let S be any set of primes with the property that for all $p \in S$ and prime $q|(p-1)$, $q \in S$. For any $x \geqslant 2$,*

$$\prod_{p \in S} \left(1 - \frac{1}{p}\right) \geqslant \prod_{\substack{p \leqslant x \\ p \in S}} \left(1 - \frac{1}{p}\right) - \frac{7.5306}{\log x}.$$

By the parameter optimization method, we make a minor improvement to Ford, Konyagin and Luca [3] from $8/\log x$ to $7.5306/\log x$.

Theorem 3 *Suppose $f(n) = \prod_{p^a \| n}(p^a - 1)$, $f(n)|n$ and $Q = \{p : p|n, p^{90} \nmid n\}$, then*

$$\prod_{q \in Q} \left(1 - \frac{1}{q}\right)^{-1} \leqslant \frac{1}{83.26}.$$

Using the parameter optimization method and the sharper bound for the Mertens products [1], [8], we make a minor improvement to Ford, Konyagin and Luca [3] from 85.32 to 83.26.

Theorem 4 *Let $f(n) = \prod_{p^a \| n}(p^a - 1)$. For any $n \in \mathbb{N}$, if $f(n)|n$, then $n/f(n) \leqslant 83$.*

Using Lemma 2.6 and Theorem 3, we improve the result of Ford, Konyagin and Luca [3] from 85 to 83.

2 Some lemmas

Let $S(x)$ be a trigonometric polynomial,

$$S(x) = \sum_{M+1}^{M+N} a_n e(nx),$$

where $N > 0$ and M are integers, a_n are arbitrary complex numbers, and $e(x) = e^{2\pi i x}$. Let x_r be arbitrary real numbers which are distinct modulo 1. δ_r is a measure of their spacing,

$$\delta_r = \min_{\substack{s \\ s \neq r}} \|x_r - x_s\|,$$

where the minimum is taken over all s with $r \neq s$, and $\|x\|$ denotes the distance from x to the nearest integer. Montgomery and Vaughan [6] established that

$$\sum_{r=1}^{R} \left(N + \frac{3}{2}\delta_r^{-1}\right)^{-1} |S(x_r)|^2 \leqslant \sum_{M+1}^{M+N} |a_n|^2.$$

Later Preissmann [7] improved the result of Montgomery and Vaughan [6], he improved $\frac{3}{2}$ to 1.3155. In this paper, we will use the following corresponding result.

Lemma 2.1 *Let $\mathcal{N}$ be a set of Z integers in an interval [M+1, M+N]. For each prime p, let $\omega(p)$ denote the number of residue classes modulo p which contain no elements of $\mathcal{N}$. Then*

$$Z \leqslant L^{-1},$$

where

$$L = \sum_{q \leqslant z} (N + 1.3155qz)^{-1} \mu(q)^2 \prod_{p|q} \frac{\omega(p)}{p - \omega(p)},$$

and z is an arbitrary positive number.

Proof See Preissmann [7] Theorem 1. □

Lemma 2.2 *Let $\pi(x)$ denote the function counting the number of primes less than or equal to some real number x, then*

$$\pi(x) \leqslant 1.2551 \frac{x}{\log x}, \quad x > 1, \qquad \pi(x) \leqslant 1.2236 \frac{x}{\log x}, \quad x \geqslant 151.$$

$$\pi(x) \leqslant \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right), \quad x > 1, \qquad \pi(x) \leqslant \frac{x}{\log x}\left(1 + \frac{1.2497}{\log x}\right), \quad x \geqslant e^{10}.$$

Proof For $x > 1$, the results were proved by Dusart [2]. For $x \geqslant 151$ and $x \geqslant e^{10}$, by the computer programming in C language, we make a small improvement. □

Lemma 2.3 *Let $\pi(x; k, l)$ denote the number of primes $p \leqslant x$ such that $p \equiv l (\mathrm{mod}\ k)$, then*

$$\pi(x; 3, l) > \frac{x}{2 \log x}, \qquad x \geqslant 151,$$

where $l = 1$ or $l = 2$.

Proof See Dusart [2]. □

Lemma 2.4 *For $x > 1$, we have*

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \leqslant \frac{e^{-\gamma}}{\log x}\left(1 + \frac{1}{\log^2 x}\right)$$

and

$$\prod_{p \leqslant x} \frac{p}{p-1} \leqslant e^{\gamma} \log x \left(1 + \frac{1}{\log^2 x}\right).$$

For $x \geqslant 286$, we have

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \leqslant \frac{e^{-\gamma}}{\log x}\left(1 + \frac{0.5}{\log^2 x}\right)$$

and

$$\prod_{p \leqslant x} \frac{p}{p-1} \leqslant e^{\gamma} \log x \left(1 + \frac{0.5}{\log^2 x}\right).$$

For $x \geqslant 2278382$, we have

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \leqslant \frac{e^{-\gamma}}{\log x}\left(1 + \frac{0.2}{\log^3 x}\right)$$

and

$$\prod_{p \leqslant x} \frac{p}{p-1} \leqslant e^{\gamma} \log x \left(1 + \frac{0.2}{\log^3 x}\right),$$

where $\gamma = 0.57721566...$ is the Euler-Mascheroni constant.

Proof The first case and the second case were proved by Rosser and Schoenfeld [8], the third case was proved by Dusart [1]. □

Note that in the proof of Theorem 1, for different ranges of x, we need the different bounds in Lemma 2.4.

Lemma 2.5 *Let $s(n)$ denote the product of the distinct primes dividing n. We have*

$$\sum_{s(n)\leqslant N}\frac{1}{n}=\sum_{n\leqslant N}\frac{\mu^2(n)}{\varphi(n)}=\log N+c_0+O^*(3.95/\sqrt{N}):=\log N+c(N),$$

$$\sum_{s(d)\leqslant y}(1+s(d)/y)^{-1}d^{-1}=\sum_{n\leqslant y}(1+n/y)^{-1}\frac{\mu^2(n)}{\varphi(n)}>\log y,$$

where $c(N)\geqslant 1$ for $1\leqslant N\leqslant 66$, $c(N)\geqslant 1.3$ for $N\geqslant 66$, $\mu(n)$ is the Möbius function, $\varphi(n)$ is the Euler's totient function, $g=O^*(f)$ means that $|f|\leqslant g$, $c_0=1.332582275\cdots$.

Proof The first formula was studied by Ramaré and Akhilesh [9]. For $c(N)$, when $1\leqslant N\leqslant 20000$, this follows the computation of Mathematica. For $N\geqslant 20000$, using the result of Ramaré and Akhilesh [9]

$$c(N)=c_0+O^*(3.95/\sqrt{N}),$$

we can easily get the $c(N)\geqslant 1.3$. For the second formula, see Ford [4]. □

Note that we improve the result of Ford [4] about the bound of $c(N)$, and in the proof of Theorem 1, we will use the exact value of $c(N)$.

Ford, Konyagin and Luca [3] proved that if $f(n)=\prod_{p^a\|n}(p^a-1)$ and $f(n)|n, n/f(n)\geqslant 5$, then $\omega(n)\geqslant 46$ and $2^{45}|n$, where $\omega(n)$ is the number of distinct primes factors of n. Next, we mainly study the case $n/f(n)\geqslant 6$.

Lemma 2.6 *Suppose $f(n)=\prod_{p^a\|n}(p^a-1)$, $f(n)|n$ and $n/f(n)\geqslant 6$, then $\omega(n)\geqslant 95$ and $2^{94}|n$, $\omega(n)$ is the number of distinct primes factors of n.*

Proof If $2\parallel n$ and $n>2$, then $n=2p^b$ for a prime p, so $(p^b-1)|2p^b$ and hence $(p^b-1)|2$. This implies $p=3$ and $n=6$. If $2^2|n$ and $2^{12}\nmid n$, then n has at most 12 odd prime factors and

$$\frac{n}{f(n)}\leqslant\frac{4}{3}\prod_{3\leqslant p\leqslant 40}\frac{p}{p-1}<5.$$

Now assume $2^{12}|n$, if $\omega(n)\leqslant 94$, then

$$\frac{f(n)}{n}\geqslant\frac{2^{12}-1}{2^{12}}\prod_{3\leqslant p\leqslant 500}\left(1-\frac{1}{p}\right)>\frac{1}{6}.$$

Hence, $\omega(n)\geqslant 95$, and n has at least 94 odd prime factors. This implies that $2^{94}|f(n)|n$.

Note that we may suppose $n/f(n)$ bigger in Lemma 2.6, but at last we find it meaninglessness for the conjecture $n/f(n)=2$.

□

3 Proof of Theorem 1

For $1<x\leqslant 9$, $U(x)\leqslant 4$, it is trivial. Suppose $x>9$ and all S, if $3\in S$, we have the upper bound

$$\begin{aligned}I(x)=\int_1^{\sqrt{x}}\frac{\log t}{t}H(t)dt&\leqslant\int_1^2\frac{\log t}{t}dt+\frac{1}{2}\int_2^3\frac{\log t}{t}dt+\frac{1}{3}\int_3^{\sqrt{x}}\frac{\log t}{t}dt\\&\leqslant\frac{\log^2 x+5.2967}{24}.\end{aligned}\tag{3.1}$$

For $9 < x \leqslant e^{20}$, using the computation of Matlab to $\pi(x)$, we obtain

$$U(x) \leqslant \pi(x) \leqslant \frac{24x}{(1+2.0308/\log x)(\log^2 x + 5.2967)}.$$

By (3.1), we have

$$U(x) \leqslant \frac{x}{I(x)(1+2.0308/\log x)}.$$

For $x > e^{21}$, by the direct computation of Matlab to $\pi(x)$, we find the inequality is not true.

If $3 \notin S$,

$$\begin{aligned} I(x) = \int_1^{\sqrt{x}} \frac{\log t}{t} H(t)dt &\leqslant \int_1^2 \frac{\log t}{t}dt + \frac{1}{2}\int_2^{\sqrt{x}} \frac{\log t}{t}dt \\ &\leqslant \frac{\log^2 x + 1.922}{16}. \end{aligned} \tag{3.2}$$

For $x > 9$,

$$U(x,S) = \#\left\{p \leqslant x,\ p \in S, 3 \nmid (p-1)\right\} \leqslant \pi(x) - \pi(x;3,1).$$

By Lemma 2.2 and Lemma 2.3, for $x \geqslant 151$, we get

$$U(x,S) \leqslant \pi(x) - \frac{x}{2\log x} \leqslant \frac{7236\pi(x)}{12236}.$$

For $151 \leqslant x \leqslant e^{20}$, using a direct computation of Matlab to $\pi(x)$, we deduce that

$$\pi(x) \leqslant \frac{16x}{(1+2.0308/\log x)(\log^2 x + 1.922)} \times \frac{12236}{7236}.$$

For $2 \leqslant x \leqslant 151$, using a simple computation of Matlab to $\pi(x)$, we have

$$\pi(x) \leqslant \frac{16x}{(1+2.0308/\log x)(\log^2 x + 1.922)} \times \frac{12236}{7236}.$$

Therefore,

$$U(x) \leqslant \frac{x}{I(x)(1+2.0308/\log x)}.$$

Note that by starting with the discussion according to $3 \in S$ or $3 \notin S$, we can extend the range of x to e^{20}, it is different from Ford [4].

For $x > e^{20}$, let $Q = \sqrt{\frac{x}{1.3155}}$, and hence $Q > 19204$, using Lemma 2.1, we have

$$U(x) \leqslant \pi(Q) + V(x),$$

where $V(x)$ is the number of integers $n \leqslant x$ such that $q \nmid n$ for each prime $q \leqslant Q$, and $q \nmid (n-1)$ for primes $q \leqslant Q, q \notin S$.

By Lemma 2.1, we obtain that

$$U(x) \leqslant \pi(Q) + \frac{x}{L},$$

where

$$L = \sum_{q \leqslant Q} \left(1 + \frac{1.3155qQ}{x}\right)^{-1} \mu^2(q) \prod_{p|q} \frac{\omega(p)}{p - \omega(p)}, \quad \omega(p) = \begin{cases} 1 & p \in S \\ 2 & p \notin S. \end{cases}$$

Let $s(n)$ denote the product of the distinct primes dividing n, $\Omega(n)$ be the number of prime factors of n counted with multiplicity, $\tau(n)$ be the number of positive divisors of n. By the particular choice of Q, i.e $1.3155qQ/x = q/Q$, we obtain

$$\begin{aligned} L &= \sum_{q\leqslant Q}\frac{\mu^2(q)}{1+q/Q}\prod_{\substack{p|q\\ p\in S}}\left(\frac{1}{p}+\frac{1}{p^2}+\cdots\right)\prod_{\substack{p|q\\ p\notin S}}\left(\frac{2}{p}+\frac{4}{p^2}+\cdots\right)\\ &= \sum_{s(d_1d_2)\leqslant Q}\frac{1}{1+s(d_1d_2)/Q}\frac{2^{\Omega(d_2)}}{d_1d_2}\\ &\geqslant \sum_{s(d_1d_2)\leqslant Q}\frac{1}{1+s(d_1d_2)/Q}\frac{\tau(d_2)}{d_1d_2}\\ &\geqslant \sum_{s(d_1d_3d_4)\leqslant Q}\frac{1}{1+s(d_1d_3d_4)/Q}\frac{1}{d_1d_3d_4}. \end{aligned} \tag{3.3}$$

Here d_1 is composed only of primes in S, and $d_2 = d_3d_4$ is composed only of primes not in S.
Let $d_5 = d_1d_3$, so that there are no restrictions on the prime factors of d_5. Since $s(d_4d_5)\leqslant s(d_4)s(d_5)$, we get

$$L \geqslant \sum_{s(d_4)s(d_5)\leqslant Q}\frac{1}{1+s(d_4)s(d_5)/Q}\frac{1}{d_4d_5}. \tag{3.4}$$

Let

$$g(x) = \sum_{s(d)\leqslant x}\frac{1}{d}, \qquad f(x) = \sum_{s(d_4)\leqslant x}\frac{1}{d_4}.$$

Since every $d\leqslant x$ can be written uniquely in the form $d = d_1d_4$, so we have

$$g(x) \leqslant \sum_{s(d_1)\leqslant x}\frac{1}{d_1}\sum_{s(d_4)\leqslant x}\frac{1}{d_4} = H(x)^{-1}f(x). \tag{3.5}$$

By the second formula of Lemma 2.5, partial summation, (3.4) and (3.5), we get

$$\begin{aligned} L &\geqslant \sum_{s(d_4)\leqslant Q/2}\frac{\log(Q/s(d_4))}{d_4}\\ &= \sum_{q_4\leqslant Q/2}\frac{\mu^2(q_4)}{\varphi(q_4)}\log(Q/q_4)\\ &= \int_{1^-}^{Q/2}\log(Q/t)df(t)\\ &= f(Q/2)\log 2+\int_1^{Q/2}\frac{f(t)}{t}dt\\ &\geqslant f(Q/2)\log 2+\int_1^{Q/2}\frac{H(t)g(t)}{t}dt\\ &= f(Q/2)\log 2+\int_1^{Q/2}\frac{H(t)\log t}{t}dt+\int_1^{Q/2}\frac{H(t)c(t)}{t}dt\\ &\geqslant \int_1^{Q/2}\frac{H(t)\log t}{t}dt+H(Q/2)\left(g(Q/2)\log 2+\int_1^{Q/2}\frac{c(t)}{t}dt\right). \end{aligned} \tag{3.6}$$

Supposing $Q\geqslant 19204$, by the first formula of Lemma 2.5 and a short computation of Mathematica, we obtain

$$\int_1^{Q/2}\frac{c(t)}{t}dt \geqslant \int_1^{5000}\frac{c(t)}{t}dt+\int_{5000}^{Q/2}\frac{1.3}{t}dt > 1.3\log Q-1.0458,$$

where we mainly use the exact value of $c(t)$ in the range of $1 \leqslant t \leqslant 5000$.

$$\int_{Q/2}^{\sqrt{x}} \frac{H(t)\log t}{t}dt \leqslant \frac{H(Q/2)}{2}\left(\log^2\sqrt{x} - \log^2(Q/2)\right) \leqslant \log(2\sqrt{1.3155})H(Q/2)\log Q.$$

By the first formula of Lemma 2.5 and $Q \geqslant 19204$, $g(Q/2) \geqslant \log(Q/2) + 1.3$, so

$$\begin{aligned} L - I(x) &\geqslant H(Q/2)\left\{\left(1.3 + \log 2 - \log 2\sqrt{1.3155}\right)\log Q - 1.0458 - \log^2 2 + 1.3\log 2\right\} \\ &\geqslant 1.0995H(Q)\log Q. \end{aligned} \tag{3.7}$$

For $e \leqslant t \leqslant u$, by the first formula of Lemma 2.4, we have

$$labeleq3.8H(t) = H(u)\prod_{\substack{p\in S\\ t<p\leqslant u}}\left(1-\frac{1}{p}\right)^{-1} \leqslant H(u)\frac{\log u}{\log t}\left(1+\frac{1}{\log^2 u}\right)\left(1+\frac{1}{\log^2 t}\right). \tag{3.8}$$

Taking $u = \sqrt{x}$ in (3.8) gives

$$\begin{aligned} I(x) &\leqslant H(\sqrt{x})\int_e^{\sqrt{x}} \frac{\log\sqrt{x}}{t}\left(1+\frac{1}{\log^2\sqrt{x}}\right)\left(1+\frac{1}{\log^2 t}\right)dt + \int_1^e \frac{H(t)\log t}{t} \\ &= H(\sqrt{x})\log\sqrt{x}\left(1+\frac{1}{\log^2\sqrt{x}}\right)\left(\log\sqrt{x} - \frac{1}{\log\sqrt{x}}\right) + \frac{1+\log^2 2}{4} \\ &\leqslant H(\sqrt{x})\log^2\sqrt{x} + 0.3702. \end{aligned} \tag{3.9}$$

Since $\sqrt{x} \geqslant e^{10} \geqslant 286$, by the second formula of Lemma 2.4, we get $H(\sqrt{x})\log\sqrt{x} \geqslant 0.55866$, and thus

$$I(x) \leqslant H(\sqrt{x})\log^2\sqrt{x} + 0.3702 \leqslant 1.06627H(\sqrt{x})\log^2\sqrt{x} \leqslant 1.0811H(Q)\log Q\log\sqrt{x}. \tag{3.10}$$

Combining (3.7) and (3.10), it follows that

$$L \geqslant I(x) + \frac{1.0995I(x)}{1.0811\log\sqrt{x}} \geqslant I(x)\left(1+\frac{2.034}{\log x}\right).$$

By (3.1), $I(x) \leqslant 0.04222\log^2 x$ and Lemma 2.2,

$$\pi(Q) \leqslant \pi(\sqrt{x}) \leqslant \frac{\sqrt{x}}{\log\sqrt{x}}\left(1+\frac{1.2497}{\log\sqrt{x}}\right) \leqslant 2.24994\frac{\sqrt{x}}{\log x}.$$

Therefore,

$$U(x) \leqslant \frac{x}{L} + \pi(Q) \leqslant \frac{x}{I(x)}\left(\frac{1}{1+\frac{2.034}{\log x}} + 0.094993\frac{\log x}{\sqrt{x}}\right) < \frac{x}{I(x)\,(1+2.0319/\log x)}.$$

By (3.2), $I(x) \leqslant 0.06281\log^2 x$ and Lemma 2.2,

$$\pi(Q) \leqslant \pi(\sqrt{x}) \leqslant \frac{\sqrt{x}}{\log\sqrt{x}}\left(1+\frac{1.2497}{\log\sqrt{x}}\right) \leqslant 2.24994\frac{\sqrt{x}}{\log x}.$$

Therefore,

$$U(x) \leqslant \frac{x}{L} + \pi(Q) \leqslant \frac{x}{I(x)}\left(\frac{1}{1+\frac{2.034}{\log x}} + 0.14132\frac{\log x}{\sqrt{x}}\right) < \frac{x}{I(x)\,(1+2.0308/\log x)}.$$

This completes the proof of Theorem 1.

4 Proof of Theorem 2

We may assume S is nonempty, so $2 \in S$. Let

$$S(x) = \#\{p \in S : p \leqslant x\}.$$

For $j \geqslant 0$, we define $y_j = x^{2^j}$ and

$$H_j = \prod_{\substack{p \leqslant y_j \\ p \in S}} \left(1 - \frac{1}{p}\right).$$

Without loss of generality, assume

$$H_0 > \frac{8}{A \log x},$$

then in particular $x \geqslant e^{\frac{16}{A}}$, where A is a parameter to be determined later.

For the lower estimations for H_j, by the method of induction and Theorem 1, when $y_{j-1} \leqslant t \leqslant y_j$, $j \geqslant 1$, we have

$$\begin{aligned} S(t) &\leqslant \frac{t}{I(t)\,(1 + 2.0308/\log t)}, \\ I(t) &= \int_1^{\sqrt{t}} \frac{\log x}{x} H(x) dx \geqslant H_{j-1} \int_1^{\sqrt{t}} \frac{\log x}{x} dx = \frac{H_{j-1} \log^2 t}{8}, \\ S(t) &\leqslant \frac{8t}{(1 + 2.0308/\log t) H_{j-1} \log^2 t}. \end{aligned}$$

By partial summation and the inequality

$$\log\left(1 - \frac{1}{t}\right) \geqslant -\frac{1}{t-1},$$

we deduce that

$$\begin{aligned} &\log\left(\frac{H_j}{H_{j-1}}\right) \\ &= \sum_{\substack{y_{j-1} < p \leqslant y_j \\ p \in S}} \log\left(1 - \frac{1}{p}\right) \\ &= S(y_j) \log\left(1 - \frac{1}{y_j}\right) - S(y_{j-1}) \log\left(1 - \frac{1}{y_{j-1}}\right) - \int_{y_{j-1}}^{y_j} \frac{S(u)}{u^2 - u} du \qquad (4.1) \\ &\geqslant -\frac{S(y_j)}{y_j - 1} - \frac{y_j}{y_j - 1} \int_{y_{j-1}}^{y_j} \frac{S(u)}{u^2} du \\ &\geqslant -\frac{8}{H_{j-1}} \left(\frac{y_j}{y_j - 1}\right) \left(\frac{1}{\log^2 y_j + 2.0308 \log y_j} + \int_{y_{j-1}}^{y_j} \frac{du}{u(\log^2 u + 2.0308 \log u)}\right). \end{aligned}$$

Since $y_j = y_{j-1}^2$, the integral above equals

$$\frac{1}{2.0308} \log\left(1 + \frac{2.0308}{\log y_j + 2.0308}\right).$$

$$\log(1 + \epsilon) \leqslant \epsilon - \frac{9}{20}\epsilon^2, \quad \text{for} \quad \epsilon = \frac{2.0308}{\log y_j + 2.0308} \leqslant \frac{1}{15}.$$

Therefore,

$$\begin{aligned}
&\log\left(\frac{H_j}{H_{j-1}}\right)\\
&\geqslant -\frac{8}{H_{j-1}}\left(\frac{y_j}{y_j-1}\right)\left(\frac{1}{\log^2 y_j+2.0308\log y_j}+\frac{1}{\log y_j+2.0308}-\frac{9}{20}\frac{2.0308}{(\log y_j+2.0308)^2}\right)\\
&\geqslant -\frac{8}{H_{j-1}}\left(\frac{y_j}{y_j-1}\right)\frac{1}{\log y_j}\left(1-\frac{1.0308}{\log y_j+2.0308}-\frac{9}{20}\frac{2.0308\log y_j}{(\log y_j+2.0308)^2}\right)\\
&\geqslant -\frac{8}{H_{j-1}}\left(\frac{y_j}{y_j-1}\right)\frac{1}{\log y_j}\left(1-\frac{1.94466\log y_j+2.09334864}{(\log y_j+2.0308)^2}\right).
\end{aligned}\tag{4.2}$$

Suppose

$$f(x)=\left(\frac{x}{x-1}\right)\left(1-\frac{1.94466\log x+2.09334864}{(\log x+2.0308)^2}\right)\geqslant\frac{1}{A},$$

where $x\geqslant e^{\frac{32}{A}}$. This means that

$$\log\left(\frac{H_j}{H_{j-1}}\right)\geqslant-\frac{8}{H_{j-1}}\frac{1}{\log y_j}\frac{1}{A}.$$

Therefore,

$$H_j\geqslant H_{j-1}\exp\left\{-\frac{8}{AH_{j-1}\log y_j}\right\}\geqslant H_{j-1}-\frac{8}{A\log y_j}>H_{j-1}-\frac{8}{A\log x}.$$

By the computation of Matlab, we get $\frac{1}{A}=0.94131\cdots$, it follows that,

$$H_j\geqslant H_{j-1}-\frac{8}{A\log x}\geqslant H_{j-1}-\frac{7.5306}{\log x}.$$

This completes the proof of Theorem 2.

5 Proof of Theorem 3

Using Lemma 2.6, we assume $2^{94}|n$, then $2\notin Q$. Suppose t_0 is the smallest prime that satisfies

$$\prod_{\substack{p\leqslant t_0\\ p\in Q}}\left(1-\frac{1}{p}\right)^{-1}\geqslant Me^{\gamma},\tag{5.1}$$

where $Me^{\gamma}<80$, and hence $M<45$, and M is a parameter to be determined later.
If no such t_0 exists, Theorem 3 follows, since we assume $Me^{\gamma}<80$.

By the third formula of Lemma 2.4, we have

$$\frac{1}{2Me^{\gamma}}\geqslant\prod_{p\leqslant t_0}\left(1-\frac{1}{p}\right)\geqslant\frac{e^{-\gamma}}{\log t_0}\left(1+\frac{0.2}{\log^3 t_0}\right)^{-1},$$

where we assume $t_0\geqslant 2278382$.
From the above formula, we get

$$\log t_0\left(1+\frac{0.2}{\log^3 t_0}\right)\geqslant 2M,$$

where we write $t_0\geqslant e^{2L}$, then $L>7$. For convenience, we assume L be the integer and take $M=L(1+\frac{1}{40L^3})$, then we get $L<45$.

Let $S=\{p:p|n\}$ and $S(x)=\#\{p\leqslant x:p\in S\}$. For any prime q with $q^a\parallel n$, there are at most a primes $p|n$ with $p\equiv 1(\mathrm{mod}\, q)$. For $x\geqslant t_0$, using Theorem 1, we have

$$\begin{aligned} S(x) &\leqslant S(\sqrt{x})+\sum_{\substack{q\in Q\\ q\leqslant\sqrt{x}}}\sum_{\substack{p|n\\ p\equiv 1(\mathrm{mod}q)}}1\\ &+\ \#\left\{\sqrt{x}<p\leqslant x:\forall q\leqslant\sqrt{x}\ \ \text{with}\ \ q\in Q, q\nmid(p-1)\right\}\\ &\leqslant\ 90S(\sqrt{x})+\#\left\{\sqrt{x}<p\leqslant x:\forall q\leqslant\sqrt{x}\ \ \text{with}\ \ q\in Q, q\nmid(p-1)\right\}\\ &\leqslant\ 45\sqrt{x}+\frac{x}{I(x)(1+2.0308/\log x)}, \end{aligned} \tag{5.2}$$

where

$$H(t)=\prod_{\substack{p\leqslant t\\ p^{90}|n}}\left(1-\frac{1}{p}\right),\qquad I(x)=\int_1^{\sqrt{x}}\frac{\log t}{t}H(t)dt.$$

By the third formula of Lemma 2.4 and (5.1),

$$\begin{aligned} H(t) &\geqslant \prod_{p\leqslant\max(t_0,t)}\left(1-\frac{1}{p}\right)\prod_{\substack{p\leqslant t_0\\ p\in Q}}\left(1-\frac{1}{p}\right)^{-1}\\ &\geqslant\ \frac{M}{\log\max(t,t_0)}\left(1+\frac{0.2}{\log^3 t_0}\right)^{-1}\\ &\geqslant\ \frac{L}{\log\max(t,t_0)}. \end{aligned} \tag{5.3}$$

Therefore,

$$I(x)\geqslant\begin{cases}\frac{L\log^2 x}{8\log t_0} & (x\leqslant t_0^2),\\ \frac{L}{2}\log\left(\frac{x}{t_0}\right) & (x>t_0^2).\end{cases}$$

For $t_0\leqslant x\leqslant t_0^2$, we have

$$\begin{aligned} S(x) &\leqslant 45\sqrt{x}+\frac{x}{\frac{L\log^2 x}{8\log t_0}(1+2.0308/\log x)}\\ &\leqslant\ \frac{8x\log t_0}{L\log^2 x(1+A/\log x)}, \end{aligned} \tag{5.4}$$

where A is a parameter to be determined by L later.
For $x>t_0^2$, we have

$$\begin{aligned} S(x) &\leqslant 45\sqrt{x}+\frac{2x}{L\log\left(\frac{x}{t_0}\right)(1+2.0308/\log x)}\\ &\leqslant\ \frac{2x}{L\log\left(\frac{x}{t_0}\right)(1+B/\log x)}, \end{aligned} \tag{5.5}$$

where B is a parameter to be determined by L later.
By (5.1), (5.4), (5.5), $S(t_0)\geqslant 1$ and partial summation, supposing

$$t=t_0^{C+1}\geqslant t_0^2,\quad C\geqslant 1,$$

we have

$$\begin{aligned}
&\prod_{\substack{p\in S\\ t_0<p\leqslant t}}\left(1-\frac{1}{p}\right)\\
&\geqslant \exp\left\{-\sum_{\substack{p\in S\\ t_0<p\leqslant t}}\frac{1}{p}-\sum_{p>t_0}\frac{1}{p^2}\right\}\\
&\geqslant \exp\left\{-\frac{1}{t_0}+\frac{S(t_0)}{t_0}-\frac{S(t)}{t}-\int_{t_0}^{t}\frac{S(u)}{u^2}du\right\}\\
&\geqslant \exp\left\{-\frac{2}{LC\log t_0+\frac{LBC}{C+1}}-\frac{8\log t_0}{LA}\log\left(1+\frac{A}{2\log t_0+A}\right)-\frac{2}{L}\log C\right\}.
\end{aligned}\tag{5.6}$$

By Theorem 2, we obtain

$$\begin{aligned}
&\prod_{p\in S}\left(1-\frac{1}{p}\right)\\
&\geqslant \prod_{\substack{p\leqslant t\\ p\in S}}\left(1-\frac{1}{p}\right)-\frac{7.5306}{\log t}\\
&\geqslant \prod_{\substack{p\leqslant t_0\\ p\in S}}\left(1-\frac{1}{p}\right)\cdot\exp\left\{-\frac{2}{LC\log t_0+\frac{LBC}{C+1}}-\frac{8\log t_0}{LA}\log\left(1+\frac{A}{2\log t_0+A}\right)-\frac{2}{L}\log C\right\}\\
&-\frac{7.5306}{(C+1)\log t_0}.
\end{aligned}\tag{5.7}$$

By the third formula of Lemma 2.4, we get

$$\begin{aligned}
&\prod_{\substack{p>t_0\\ p\in Q}}\left(1-\frac{1}{p}\right)\\
&\geqslant \prod_{\substack{p>t_0\\ p\in S}}\left(1-\frac{1}{p}\right)\\
&\geqslant \exp\left\{-\frac{2}{LC\log t_0+\frac{LBC}{C+1}}-\frac{8\log t_0}{LA}\log\left(1+\frac{A}{2\log t_0+A}\right)-\frac{2}{L}\log C\right\}\\
&-\frac{7.5306}{(C+1)\log t_0}\prod_{p\leqslant t_0}\left(1-\frac{1}{p}\right)^{-1}\\
&\geqslant \exp\left\{-\frac{2}{LC\log t_0+\frac{LBC}{C+1}}-\frac{8\log t_0}{LA}\log\left(1+\frac{A}{2\log t_0+A}\right)-\frac{2}{L}\log C\right\}\\
&-\frac{7.5306e^{\gamma}(1+0.2/\log^3 t_0)}{(C+1)}.
\end{aligned}\tag{5.8}$$

For $t_0 = e^{2L}$,

$$\prod_{\substack{p>e^{2L}\\ p\in Q}}\left(1-\frac{1}{p}\right) \geqslant \exp\left\{-\frac{2}{2L^2C+\frac{LBC}{C+1}} - \frac{16}{A}\log\left(1+\frac{A}{4L+A}\right) - \frac{2}{L}\log C\right\} - \frac{7.5306e^{\gamma}(1+1/40L^3)}{(C+1)}. \tag{5.9}$$

Let

$$F(L,x) = \exp\left\{-\frac{2}{2L^2x+\frac{LBx}{x+1}} - \frac{16}{A}\log\left(1+\frac{A}{4L+A}\right) - \frac{2}{L}\log x\right\} - \frac{7.5306e^{\gamma}(1+1/40L^3)}{(x+1)}, \quad x\geqslant 1, \quad 7<L<45, \tag{5.10}$$

where the parameters A and B depend on L.

By (5.1) and (5.10), we have

$$\begin{aligned}&\prod_{p\in Q}\left(1-\frac{1}{p}\right)\\ &\geqslant \prod_{\substack{p<e^{2L}\\ p\in Q}}\left(1-\frac{1}{p}\right)\prod_{\substack{p>e^{2L}\\ p\in Q}}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{e^{2L}}\right)\\ &\geqslant \frac{1}{L\left(1+\frac{1}{40L^3}\right)e^{\gamma}}\left(1-\frac{1}{e^{2L}}\right)F(L,x).\end{aligned} \tag{5.11}$$

Let

$$G(L,x) = \frac{1}{L\left(1+\frac{1}{40L^3}\right)e^{\gamma}}\left(1-\frac{1}{e^{2L}}\right)F(L,x), \quad x\geqslant 1, \quad 7<L<45,$$

where $G(L,x)$ depends on L and x. By the computation of Matlab, we find that when $L=17$, $A=2.0256$, $B=2.0307$, $M=17.00008651$, we have a better bound for $G(L,x)$. Finally we get

$$\prod_{p\in Q}\left(1-\frac{1}{p}\right) \geqslant \frac{1}{83.26}.$$

This completes the proof of Theorem 3.

6 Proof of Theorem 4

By Theorem 3 and Lemma 2.6,

$$\frac{n}{f(n)} = \prod_{p^{\alpha}\|n}\frac{1}{1-p^{-\alpha}} \leqslant \prod_{p\in Q}\frac{1}{1-p^{-1}}\prod_{p\notin Q}\frac{1}{1-p^{-90}} \leqslant 83.3.$$

This completes the proof of Theorem 4.

Note that the parameters A and B depend on L, the bigger A and B, the better bound we can get for $\frac{n}{f(n)}$. By the computation of Matlab, we find that when $L=17$, we have a better bound for $\frac{n}{f(n)}$.

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