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ORIGINAL RESEARCH

Strict convexity of Orlicz sequence spaces equipped with p-Amemiya norms

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Abstract In this paper, criteria for strict convexity of Orlicz sequence spaces equipped with the p-Amemiya norms ($1 \leq p \leq \infty$) are presented. The obtained results widen the characterization of strict convexity of Orlicz sequence spaces. From our results it follows that there are Orlicz sequence spaces which are strictly convex for p-Amemiya norms with $1 < p < \infty$ only, that is, they are neither strictly convex for Luxemburg norm corresponding to the case $p = \infty$ nor for Orlicz norm corresponding to the case $p = 1$.

Keywords Orlicz sequence space · P-Amemiya norm · Strict convexity

1 Introduction

Let $(X, \|\cdot\|)$ be a real Banach space and let $B(X)$ and $S(X)$ be the closed unit ball and the unit sphere of X, respectively.

A point $x \in S(X)$ is called an extreme point if for every $y,\ z \in S(X)$ with $x = \frac{y+z}{2}$, we have $y = z = x$. The notion of extreme point plays an important role in some branches of mathematics. For example, the Krein-Milman theorem, Choquet integral representation theorem, Rainwater theorem on convergence in the weak topology, Bessaga-Pelczyński theorem and Elton test for unconditional convergence are formulated in terms of extreme points (see [17]).

A point $x \in S(X)$ is called a strong U-point (SU-point for short) if for any $y \in S(X)$ with $\|x + y\| = 2$, we have $x = y$. Recall that the nature of an SU-point is such that a point $x \in S(X)$ is a point of local uniform rotundity if and only if x is a point of compact local uniform rotundity and an SU-point (see [6]).

A Banach space X is called strict convexity if every point of $S(X)$ is an SU-point of $B(X)$. In a Euclidean space, strict convexity corresponds geometrically to the property that the unit sphere $S(X)$ does not contain a segment. We remark that, if a space has the property of uniform convexity, then it possesses that of strict convexity as well; However, the converse implication is generally untrue.

The notion of SU-point is essentially stronger than the notion of extreme point. Namely in l_∞^2 (two-dimensional l_∞ space) the points $x = (1, 1)$ and $y = (1, -1)$ are extreme points of $B(l_\infty^2)$. Since $x \neq y$ and $\|x + y\| = 2$, neither x nor y is a strong U-points of $B(l_\infty^2)$.

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It is well known that strict convexity of a normed space X is important for the uniqueness of the best approximation element in any bounded closed convex and nonempty set $A \subset X$ for any $x \in X\backslash A$. Namely, if X is a strictly convex normed space, A is a bounded closed convex and nonempty set in X and $x \in X\backslash A$ then if $y \in A$ is such that $\|x - y\| = d(x, A) := \inf\{\|x - z\| : z \in A\}$, then for any $z \in A, z \neq y$, we have $d(x, A) < \|x - z\|$. If X is not strictly convex then there is a bounded closed convex and nonempty set $A \subset X$ and $x \in X\backslash A$ such that there is a continuum of points in A that realize the distance $d(x, A)$.

If we want to consider the problem of the uniqueness of the best approximation locally , that is , for some fixed $x \in X\backslash A$ only, then the notion of SU-point can be applied and the notion of extreme point cannot be used in place of the SU-point (see [11]). So the notion of the SU-point is important for the local best approximation problem.

In 1957, Milnes gave criteria for strict convexity of Orlicz function spaces equipped with Orlicz norm in [19]. Cui, Hudzik, Nowak and Pluciennik (see [7]) gave the criteria for strict convexity of Orlicz sequence spaces endowed with Orlicz norm. Criteria for strict convexity of more general Musielak-Orlicz sequence spaces equipped with Luxemburg and Orlicz norm were given in [14,15].

In the sequel N and R denote the set of natural numbers and the set of real numbers, respectively.

The triple $(N, 2^N, \mu)$ stands for the counting measure space and $l^0(\mu) = l^0$ denotes the space of all real sequence $x = (x(i))$. A Banach space X is called a Köthe sequence space, if it is a subspace of l^0 such that

(i) if $x \in l^0$, $y \in X$ and $|x(i)| \leq |y(i)|$ for $i \in N$, then $x \in X$ and $\|x\| \leq \|y\|$,
(ii) there exists $x \in X$ such that $supp\ x = N$, where $supp\ x = \{i \in N : x(i) \neq 0\}$.

A Köthe sequence space X is said to have the Fatou property if for any (x_n) in X with $0 \leq x_n \uparrow x$ and $\sup_n \|x_n\| < \infty$, $\sup_n x_n$ exists in X, then $\sup_n x_n = x$ and $\|x_n\| \uparrow \|x\|$.

A map Φ is said to be an Orlicz function if it is even, convex, left continuous on $[0, +\infty)$, $\Phi(0) = 0$ and there exists $u > 0$ such that $0 < \Phi(u) < \infty$.

For every Orlicz function Φ, we define the complementary function Ψ, in sense of Young, by the formula

$$\Psi(v) = \sup\{u|v| - \Phi(u) : u \geq 0\}.$$

Define

$$a_\Phi = \sup\{u \geq 0 : \Phi(u) = 0\}, \quad b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\}.$$

Let $p_+(u)$ and $p_-(u)$ ($q_+(v)$ and $q_-(v)$) stand for the right and left derivatives of $\Phi(\Psi)$ at $u \in R$ with $0 \leq u < b_\Phi$ (at $v \in R$ with $0 \leq v < b_\Psi$), respectively. Here we define $p_+(b_\Phi) = \infty$, and $p_+(u) = p_-(u) = \infty$ for all $u > b_\Phi$, $q_+(b_\Psi) = \infty$, and $q_+(v) = q_-(v) = \infty$ for all $v > b_\Psi$.

Let us emphasize that p_-, p_+, q_-, q_+ always mean functions, while letters p, q always denote numbers from the interval $[1, \infty)$.

Moreover, for every $u, v \in R$, we have the following Young Inequality:

$$|uv| \leq \Phi(u) + \Psi(v).$$

Further, $|uv| = \Phi(u)+\Psi(v)$ if and only if $p_-(|u|) \leq |v| \leq p_+(|u|)$, when u is fixed, or $q_-(|v|) \leq |u| \leq q_+(|v|)$, when v is fixed.

Given any Orlicz function Φ, we define on l^0 the convex modular of x by

$$I_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x(i)), \quad \text{for any} \quad x = (x(i)) \in l^0.$$

The Orlicz sequence space l_Φ generated by an Orlicz function Φ is a linear space of real sequences defined by

$$l_\Phi = \{x \in l^0 : I_\Phi(cx) < \infty, \text{ for some } c > 0 \text{ depending on } x\}.$$

The space h_Φ is the subspace of all order continuous elements of l_Φ defined by

$$h_\Phi = \{x \in l^0 : I_\Phi(cx) < \infty \text{ for any } c > 0\}.$$

The Orlicz sequence space l_Φ is a Köthe sequence space when it equipped with the Orlicz norm

$$\|x\|_\Phi^o = \sup_{I_\Psi(v) \leq 1} \sum_{i=1}^{\infty} x(i)v(i),$$

or the Luxemburg norm

$$\|x\|_\Phi = \inf\{\varepsilon > 0 : I_\Phi(\frac{x}{\varepsilon}) \le 1\}.$$

At first sight the Orlicz norm and Luxemburg norm seem far from similar. In fact, in many cases the geometric properties of Orlicz space under each of these norms differ from each other. But in the fifties, I. Amemiya (see [21]) considered a norm defined by the formula

$$\|x\|_\Phi^A = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx))$$

Krasnoslskii and Rutickii ([16]), Nakano ([21]), Luxemburg and Zaanen ([18]) proved, under additional assumptions on the function Φ, that the Orlicz norm can be expressed exactly by the Amemiya formula, i.e., $\|x\|_\Phi^o = \|x\|_\Phi^A$. In the most general case of Orlicz function Φ, the similar result was obtained by Hudzik and Maligranda ([12]). Moreover, it is not difficult to verify that Luxemburg norm can also be expressed by an Amemiya-like formula (see [4,22]), namely

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k} \max\{1, I_\Phi(kx)\}.$$

The only difference between the two Amemiya formulas is the function under the infimum operation (we will call it the outer function): for all $u \ge 0$, $s(u) = 1+u$ (for the Orlicz norm) and $s(u) = \max\{1, u\}$ (for the Luxemburg norm). In the paper [12], Hudzik and Maligranda proposed to investigate the Amemiya formula-norms generated by the outer functions of the type

$$s_p(u) = \begin{cases} (1 + u^p)^{\frac{1}{p}}, & \text{for } 1 \le p < \infty, \\ \max\{1, u\}, & \text{for } p = \infty, \end{cases}$$

and

$$\|x\|_{\Phi,p} = \inf_{k>0} \frac{1}{k} s_p(I_\Phi(kx)) \quad (1 \le p \le \infty).$$

In that case we obtain a family of topologically equivalent norms (called p-Amemiya norms),

$$\|x\|_\Phi = \|x\|_{\Phi,\infty} \le \|x\|_{\Phi,p} \le \|x\|_{\Phi,q} \le \|x\|_{\Phi,1} = \|x\|_\Phi^o \le 2\|x\|_\Phi$$

for all $1 \le q \le p \le \infty$.

The purpose in investigating the p-Amemiya norms is to look at the theory of Orlicz spaces from a common point of view. In recent years, an intensive development of research connected with Orlicz spaces equipped with the p-Amemiya norm have taken place, many important results broaden the knowledge about the geometry of these spaces (see [2,4,5,8,9,13]) and some open questions were put [24]. Recently, Wisla presented a general and universal method of introducing norms in Orlicz spaces that cover all the cases of p-Amemiya norms. Namely, using the concept of outer function, s-norms $\|x\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_\Phi(kx))$ are introduced. Some basic and crucial results of Orlicz spaces equipped with the s-norms have been presented in [25].

However, because of the complication of Orlicz sequence spaces equipped with the p-Amemiya norm, at present, the criteria for strict convexity has not been discussed yet. The aim of this paper is to give criteria for strict convexity of Orlicz sequence spaces equipped with the p-Amemiya norm ($1 \le p \le \infty$).

In order to simplify notation, the Orlicz space equipped with the p-Amemiya norm is denoted by $l_{\Phi,p} = (l_\Phi, \|\cdot\|_{\Phi,p})$.

We say an Orlicz function Φ satisfies the δ_2-condition ($\Phi \in \delta_2$, for short) if there are positive constants k and x_0 with $0 < \Phi(x_0) < \infty$ such that $\Phi(2x) \le k\Phi(x)$ for $|x| \le x_0$.

Set $\sigma_p = \max\{u \ge 0 : s_p(u) = 1\}$, $\varepsilon_p = 1 - [\frac{1}{p}]$ where $[\frac{1}{p}]$ is the largest integer not exceeding $\frac{1}{p}$. Obviously,

$$\sigma_p = \begin{cases} 0, & \text{for } 1 \le p < \infty, \\ 1, & \text{for } p = \infty, \end{cases} \qquad \varepsilon_p = \begin{cases} 0, & \text{for } p = 1, \\ 1, & \text{for } 1 < p \le \infty. \end{cases}$$

For any $i \in N$, we say that a point $w \in R$ is a strict convexity point of Φ, if $\Phi(\frac{u+v}{2}) < \frac{1}{2}(\Phi(u) + \Phi(v))$ whenever $w = \frac{u+v}{2}$ and $u \ne v$. We write $w \in SC_\Phi$.

Let (e_i) be the sequence of unit vectors, i.e. $e_i = (0, \cdots, 0, 1, 0, \cdots)$.

Further details about Orlicz spaces equipped with the Luxemburg or the Orlicz norm, can be found in [3,20,22,23]. Basic results on the Orlicz spaces equipped with the p-Amemiya norms have been presented in [4].

2 Auxiliary result

In this section we will present a few auxiliary of results that will be used in the main part of the paper.

Define the function $\alpha_p : l_{\Phi,p} \to [-1, \infty]$ by

$$\alpha_p(x) = \begin{cases} I_\Phi^{p-1}(x) I_\Psi(p_+(|x|)) - 1, & \text{for } 1 \le p < \infty, \\ -1, & \text{for } p = \infty,\ I_\Phi(x) \le 1, \\ I_\Psi(p_+(|x|)), & \text{for } p = \infty,\ I_\Phi(x) > 1. \end{cases}$$

and the functions $k_p^* : l_{\Phi,p} \to [0, \infty)$, $k_p^{**} : l_{\Phi,p} \to (0, \infty]$ by

$$k_p^*(x) = \inf\{k \ge 0 : \alpha_p(kx) \ge 0\} \ (\text{with} \inf \emptyset = \infty),$$

$$k_p^{**}(x) = \sup\{k \ge 0 : \alpha_p(kx) \le 0\}.$$

It is evident that $k_p^*(x) \le k_p^{**}(x)$ for every $1 \le p \le \infty$ and $x \in l_{\Phi,p} \backslash \{0\}$.

Set $K_p(x) = \{0 < k < \infty \ : \ k_p^*(x) \le k \le k_p^{**}(x)\}$.

Lemma 1 *[4] For every $1 \le p \le \infty$ and $x \in l_{\Phi,p} \backslash \{0\}$, the following conditions hold:*

(i) If $k_p^*(x) = k_p^{**}(x) = \infty$, then $\|x\|_{\Phi,p} = \lim_{k\to\infty} \frac{1}{k}(1 + I_\Phi^p(kx))^{\frac{1}{p}}$.

(ii) If $k_p^*(x) < k_p^{**}(x) = \infty$, then the p-Amemiya norm $\|x\|_{\Phi,p}$ is attained at every $k \in K_p(x) = [k_p^*(x), \infty)$.

(iii) If $k_p^{**}(x) < \infty$, then the p-Amemiya norm $\|x\|_{\Phi,p}$ is attained at every $k \in K_p(x) = [k_p^*(x), k_p^{**}(x)]$.

Lemma 2 *[1,4] (a) $p = 1$.*

(i) If $\lim_{x\to\infty} \frac{\Phi(x)}{x} = A < \infty$, and $card(supp(x)) \cdot \Psi(A) \ge 1$, then $K_1(x) \ne \emptyset$.

(ii) If $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$, then $K_1(x) \ne \emptyset$.

(b) $1 < p < \infty$. If Φ is not linear on $[0, \infty)$, then $K_p(x) \ne \emptyset$. (c) $p = \infty$. For every Orlicz function Φ, $K_\infty(x) \ne \emptyset$.

Lemma 3 *[10] A point $x = (x(i)) \in S(l_{\Phi,p})$ $(1 \le p \le \infty)$ is an extreme point of $S(l_{\Phi,p})$ if and only if:*

(i) If $x \in S(l_{\Phi,p})$ with $card(supp\ x) \ge 2$, then $K_p(x) \ne \emptyset$,

(ii) $card(supp\ x) = 1$ or for any $k \in K_p(x)$, $card(\{i \in N : kx(i) \notin SC_\Phi\}) \le \varepsilon_p$,

(iii) for any $k \in K_p(x)$, $I_\Phi(kx) \ge \sigma_p$ or $|x(i)| = b_\Phi < \infty$, $i \in N$.

Lemma 4 *[15] Let $d = \sup\{0 \le v \le b_\Psi : \Psi(v) < 1\}$. The Orlicz sequence space l_Φ is strictly convex with respect to the Orlicz norm if and only if the following conditions are satisfied:*

(1) $\Psi(d) > \frac{1}{2}$,

(2) q_- is continuous on the interval $[0, d)$,

(3) if $\Psi(d) < 1$ then $q_-(d) = \lim_{v\to d^-} q_-(v) = \infty$.

Lemma 5 *For any Orlicz function Φ, any $c > 0$ and any $1 \le p \le \infty$, we have*

$$\sup\left\{0 \le x < b_\Phi : \ \Phi^{\frac{1}{q}}(x)\Psi^{\frac{1}{p}}(p_+(x)) < c\right\} > 0$$

(with the convention $\frac{1}{\infty} = 0$).

Proof On the country, suppose that $\Phi^{\frac{1}{q}}(x)\Psi^{\frac{1}{p}}(p_+(x)) \ge c$ for all $0 < x < b_\Phi$. Since the function $\Phi^{\frac{1}{q}}(\cdot)\Psi^{\frac{1}{p}}(p_+(\cdot))$ is right-continuous at 0, we have

$$0 = \Phi^{\frac{1}{q}}(0)\Psi^{\frac{1}{p}}(p_+(0)) = \lim_{x\to 0^+} \Phi^{\frac{1}{q}}(x)\Psi^{\frac{1}{p}}(p_+(x)) \ge c > 0,$$

a contradiction. □

3 Main Results

Theorem 1 *If Φ is an Orlicz function which vanishes outside zero, then $l_{\Phi,p}$ $(1 \leq p < \infty)$ contains an order isometric copy of l_∞.*

Proof To build an isometry P of l_∞ onto a closed subspace of $l_{\Phi,p}$ $(1 \leq p < \infty)$ preserving the order it is enough to find a sequence $\{x_n\} \subset S(l_{\Phi,p})$ with pairwise disjoint supports and such that $\| \sum_{i=1}^{\infty} x_n \|_{\Phi,p} = 1$. Then the operator $P : l_\infty \to l_{\Phi,p}$ defined by

$$Py = \sum_{n=1}^{\infty} y_n x_n$$

for each $y = (y_n) \in l_\infty$ is the desired isometry.

Assume that

$$a = a_\Phi = \sup\{x > 0 : \Phi(x) = 0\} > 0.$$

Divide the set N of natural numbers into a sequence $(N_n)_{n=1}^{\infty}$ of pairwise disjoint and infinite sets. Define $x_n = (x_n(i))_{i=1}^{\infty}$, where

$$x_n(i) = \begin{cases} a, & i \in N_n, \\ 0, & \text{otherwise}, \end{cases}$$

for any $n \in N$. Then $(1 + I_\Phi^p(x_n))^{\frac{1}{p}} = 1$ for each $n \in N$. Moreover,

$$\frac{1}{k}(1 + I_\Phi^p(kx_n))^{\frac{1}{p}} = \frac{1}{k} > 1$$

for each $k \in (0, 1)$ and

$$\frac{1}{k}(1 + I_\Phi^p(kx_n))^{\frac{1}{p}} = \infty$$

for each $k > 1$, because the sets N_n are infinite and $\Phi(ka) > 0$ for $k > 1$. This shows that

$$\|x_n\|_{\Phi,p} = \inf_{k>0} \frac{1}{k}(1 + I_\Phi^p(kx_n))^{\frac{1}{p}} = (1 + I_\Phi^p(x_n))^{\frac{1}{p}} = 1$$

for each $n \in N$. In the same way it can be proved that $\| \sum_{n=1}^{\infty} x_n \|_{\Phi,p} = 1$. This finishes the proof. □

Criteria for strict convexity of Orlicz sequence spaces equipped with the Orlicz norm was first given in [15]. The next theorem solves an analogous problem for the Orlicz sequence spaces generated by the arbitrary Orlicz function and equipped with the Orlicz norm, the proof of which use some new techniques in comparison with the ones used in [15].

Theorem 2 *The Orlicz spaces $l_{\Phi,1}$ is strictly convex with respect to the Orlicz norm (i.e., 1-Amemiya norm) if and only if the following conditions are satisfied:*

(i) $a_\Phi = 0$,
(ii) *there exists a* $x_0 > 0$ *such that* $\Psi(p_+(x_0)) > \frac{1}{2}$,
(iii) Φ *is strictly convex on the interval* $[0, \pi_{\Phi,1}]$ *where*

$$\pi_{\Phi,1} = \sup\{0 \leq t < b_\Phi : \Psi(p_+(t)) \leq 1\}.$$

Proof Let us note that, by Lemma 5, $\pi_{\Phi,1} > 0$.

Sufficiency. Let $x \in S(l_{\Phi,1})$. We will show that the conditions (i), (ii) and (iii) of Lemma 3 are satisfied, i.e., that x is an extreme point of $B(l_{\Phi,1})$.

Suppose that $card(supp\ x) > 1$ and let the indices i, j be such that $i \neq j$ and $0 < |x(i)| < |x(j)|$. Let $0 < x_0 < \infty$ be such that $\Psi(p_+(x_0)) > \frac{1}{2}$. since $x_0 < \infty$, so we can find $0 < k < \infty$ such that $k|x(i)| > x_0$. Hence

$$\Psi(p_+(k|x(i)|)) + \Psi(p_+(k|x(j)|)) \geq 2\Psi(p_+(x_0)) > 1.$$

Therefore $k_1^*(x) < \infty$ and the condition (i) of Lemma 3 is satisfied.

Let $k_0 \in K_1(x)$, we have

$$\Psi(p_+(k_0|x(i)|)) + \Psi(p_+(k_0|x(j)|)) \le I_\Psi(p_+(k_0|x|)) \le 1.$$

By (iii), Φ is strictly convex on a right neighborhood of 0, so $\Psi(p_+(x)) > 0$ for all $x > 0$. Thus

$$\Psi(p_+(k_0|x(i)|)) \le 1 - \Psi(p_+(k_0|x(j)|)) < 1.$$

Therefore, $k_0|x(i)| \le \pi_{\Phi,1}$, whence $k_0 x(i)$ is a point of strict convexity of Φ. By arbitrariness of i, $k_0 x(i) \in SC_\Phi$ for all $i \ge 1$ with $x(i) \ne 0$. Moreover, since $a_\Phi = 0$, we have $0 \in SC_\Phi$, so $k_0 x(i) \in SC_\Phi$ for all $i \ge 1$. Thus the condition (ii) of Lemma 3 is satisfied.

Since $a_\Phi = 0$, we have $I_\Phi(k_0 x) \ge \Phi(k_0 x(i)) > 0$, so the condition (iii) of Lemma 3 holds true as well. Therefore x is an extreme point of $B(l_{\Phi,1})$. By the arbitrariness of x, the Orlicz space $l_{\Phi,1}$ is strictly convex.

Necessity. Assume that $l_{\Phi,1}$ is strictly convex.

(i) If $a_\Phi > 0$, $l_{\Phi,1}$ is contains a linearly isometric copy of l_∞ according to Theorem 1, so it is not strictly convex. This contradiction proves that (i) holds.

(ii) On the contrary, suppose that (ii) does not hold, i.e., $\Psi(p_+(x)) \le \frac{1}{2}$ for all $x > 0$. Then $c = \lim_{x\to\infty} p_+(x)$ is positive (because $\Phi \not\equiv 0$) and finite. Take any $0 < a < \frac{1}{c}$ and let $b = \frac{1}{c} - a$. Evidently $0 < b < \infty$. Put $x = (a, b, 0, \cdots)$. Then $I_\Psi(p_+(kx)) = \Psi(p_+(ka)) + \Psi(p_+(kb)) \le 1$ for all $k > 0$, whence

$$\|x\|_{\Phi,1} = \lim_{k\to\infty} \frac{1}{k}(1 + \Phi(ka) + \Phi(kb)) = \lim_{k\to\infty} (\frac{\Phi(ka)}{ka}a + \frac{\Phi(kb)}{kb}b) = c(a+b) = 1.$$

Suppose that $k \in K_1(x)$ for some $0 < k < \infty$. Let $0 < d < \infty$ be such a number that $u \to cu + d$ is an oblique asymptote of Φ at infinity. Then

$$1 = \frac{1}{k}(1 + \Phi(ka) + \Phi(kb)) \ge \frac{1}{k}(1 + cka + d + ckb + d) = \frac{1}{k}(1 + 2d) + c(a+b) > 1,$$

a contradiction proves that (ii) is satisfied.

(iii) Firstly, we claim that we can find $0 < x_0 < b_\Phi$ such that $0 < \Psi(p_+(x_0)) < \infty$. Indeed, otherwise (since $\Psi \not\equiv 0$) $\Psi(p_+(x)) = 0$ for all $0 \le x < a_\Phi = b_\Phi < \infty$. But this implies that $\Psi(v) = 0$ for all $0 < v \le a_\Psi = b_\Psi < \infty$. Then Φ is linear on R (in fact $\Phi(u) = b_\Psi u$), so $l_{\Phi,1}$ is linearly isometric to l_1. Thus $l_{\Phi,1}$ is not strictly convex - a contradiction that proves our claim. Note that $\Phi(x_0) \le x_0 p_+(x_0) < \infty$.

Suppose that Φ is not strictly convex on $[0, \pi_{\Phi,1}]$, i.e., Φ is affine on some interval $(a-\varepsilon, a+\varepsilon) \subset [0, \pi_{\Phi,1}]$. Then $\Psi(p_+(a)) \le \Psi(p_+(a+\varepsilon)) < 1$.

Let $0 < u_0 < b_\Phi$ be such a number that $0 < \Psi(p_+(u_0)) < \infty$ and let

$$m = \max\{n \in N \cup \{0\} : \Psi(p_+(a)) + n\Psi(p_+(u_0)) < 1\},$$
$$b = \sup\{0 \le u < u_0 : \Psi(p_+(a)) + m\Psi(p_+(u_0)) + \Psi(p_+(u)) \le 1\}.$$

Evidently $b < \infty$. We claim that $b > 0$. On the contrary, suppose that $b = 0$, i.e.,

$$\Psi(p_+(a)) + m\Psi(p_+(u_0)) + \Psi(p_+(u)) > 1$$

for all $0 < u < u_0$. Since 0 is a point of right continuity of $\Psi(p_+(\cdot))$ and $\Psi(p_+(0)) = 0$, we have

$$\Psi(p_+(a)) + m\Psi(p_+(u_0)) = \lim_{u\to 0^+} (\Psi(p_+(a)) + m\Psi(p_+(u_0)) + \Psi(p_+(u))) \ge 1,$$

a contradiction. Thus $b > 0$. Define $k_0 = 1 + \Phi(a) + m\Phi(u_0) + \Phi(b)$ and let $x = \frac{1}{k_0}(a, u_0, \cdots, u_0, b, 0, \cdots)$ (u_0 repeated m-times). Then $I_\Psi(p_+(k_0|x|)) \le 1$ and for every $\gamma > 1$,

$$\begin{aligned} I_\Psi(p_+(\gamma k_0|x|)) &= \Psi(p_+(\gamma a)) + m\Psi(p_+(\gamma u_0)) + \Psi(p_+(\gamma b)) \\ &\ge \Psi(p_+(a)) + m\Psi(p_+(u_0)) + \Psi(p_+(\gamma b)) > 1, \end{aligned}$$

whence $k_0 \in K_1(x)$. Thus

$$\|x\|_{\Phi,1} = \frac{1}{k_0}(1 + I_\Phi(k_0 x)) = \frac{1 + \Phi(a) + m\Phi(u_0) + \Phi(b)}{1 + \Phi(a) + m\Phi(u_0) + \Phi(b)} = 1.$$

Since $k_0 x(1) = a \notin SC_\Phi$, x is not an extreme point of $B(l_{\Phi,1})$. Therefore $l_{\Phi,1}$ is not strictly convex and this contradiction ends the proof. □

The criteria of Theorem 2 seem farm from the criteria of Lemma 4. The next proposition compares Theorem 2 and Lemma 4, and it turns out that Theorem 2 and Lemma 4 are equivalent.

Proposition 1 *The Kamińska's conditions (1), (2), (3) of Lemma 4 are equivalent to the conditions (i), (ii), (iii) of Theorem 2.*

Proof (2) $\Rightarrow$ (i) Suppose that $a_\Phi > 0$. Then $\lim_{v\to 0^+} q_-(v) = \lim_{v\to 0^+} q_+(v) \geq a_\Phi > 0 = q_-(0)$, whence 0 is not a point of (right) continuity of q_-, a contradiction. Thus $a_\Phi = 0$.

(1) $\Rightarrow$ (ii) Assume that $\Psi(d) > \frac{1}{2}$. By continuity of Ψ, we can find $\delta > 0$ such that $\Psi(d-\delta) > \frac{1}{2}$ as well. We claim that $p_+(x) \geq d - \delta$ for some $x > 0$. Indeed, otherwise $p_+(x) < d - \delta$ for all $x \geq 0$, whence

$$b_\Psi = \sup_{x\geq 0} p_+(x) \leq d - \delta < d \leq b_\Psi,$$

a contradiction. Thus $\Psi(p_+(x)) \geq \Psi(d-\delta) > \frac{1}{2}$ for some $x > 0$, so (ii) is satisfied.

(2) $\Rightarrow$ (iii) We claim that $p_+(x) \leq d$ for every $0 \leq x < \pi_{\Phi,1}$. On the contrary, suppose that $p_+(x_0) > d$ for some $0 \leq x_0 < \pi_{\Phi,1}$. Then $\Psi(d) \leq \Psi(p_+(x_0)) < 1$ whence $d = b_\Psi$. Thus $b_\Psi = d < p_+(x_0) \leq b_\Psi$, a contradiction that proves the claim. Therefore,

$$v_0 = \sup\{p_+(x) : 0 \leq x \leq \pi_{\Phi,1}\} \leq d.$$

Since q_- is continuous on $[0, d)$, $p_+ = q_-^{-1}$ is strictly increasing on $[0, \pi_{\Phi,1})$, so Φ is strictly convex on $[0, \pi_{\Phi,1}]$.

(ii) $\Rightarrow$ (1) Assume that $\Psi(p_+(x_0)) > \frac{1}{2}$ for some $0 < x_0 < \infty$. If $p_+(x_0) = \infty$ then $b_\Phi < \infty$. Thus $b_\Psi = \infty$, whence $\Psi(d) = 1 > \frac{1}{2}$. Further, if $p_+(x_0) < \infty$ then we can find $\delta > 0$ such that $\Psi(p_+(x_0) - \delta) \geq \frac{1}{2}$. Thus $p_+(x_0) - \delta < b_\Psi$, whence $\Psi(d) > \Psi(p_+(x_0) - \delta) > \frac{1}{2}$.

(iii) $\Rightarrow$ (2) Since by (iii), Φ is strictly convex on $[0, \pi_{\Phi,1}]$, the function $p_+(\cdot)$ is strictly increasing on $(0, \pi_{\Phi,1})$. Hence, $q_- = p_+^{-1}$ is continuous on the interval $(0, v)$, where

$$v = \lim_{x\to \pi_{\Phi,1}^-} p_+(x) = p_-(\pi_{\Phi,1}).$$

Since $\Psi(p_+(x)) \geq 1$ for every $x \geq \pi_{\Phi,1}$, by right-continuity of p_+, $\Psi(p_+(\pi_{\Phi,1})) \geq 1$. Therefore $p_-(\pi_{\Phi,1}) \leq d \leq p_+(\pi_{\Phi,1})$. Since q_- is constant (so continuous) on each nonempty interval $(p_-(u), p_+(u))$, q_- is continuous on $[0, p_+(\pi_{\Phi,1}))$, whence on $[0, d)$ as well.

(iii) $\Rightarrow$ (3) If $\Psi(d) < 1$ then $d = b_\Psi = \lim_{x\to\infty} p_+(x) < \infty$. Thus $b_\Phi = \infty$ and $\Psi(p_+(x)) \leq \Psi(d) < 1$ for all $0 < x < \infty$. Hence $\pi_{\Phi,1} = \infty$. By (iii), Φ is strictly convex on $(0, \infty)$, so $p_+(\cdot)$ is strictly increasing on $(0, \infty)$ to b_Ψ, whence $q_-(d) = q_-(b_\Psi) = \infty$. □

Let us note that, for any $1 < p < \infty$, powering the inequality in Lemma 5 by p, we obtain that $\pi_{\Phi,p} > 0$, where

$$\pi_{\Phi,p} = \sup\{t > 0 : (\Phi(t))^{p-1} \cdot \Psi(p_+(t)) \leq \frac{1}{2^p}\}.$$

Theorem 3 *The Orlicz sequence space $l_{\Phi,p}$ $(1 \leq p < \infty)$ is strictly convex with respect to the p-Amemiya norm if and only if the following conditions are satisfied:*

(i) $a_\Phi = 0$,
(ii) $p = 1$, *there exists a* $x_0 > 0$ *such that* $\Psi(p_+(x_0)) > \frac{1}{2}$,
(iii) Φ *is strictly convex on the interval* $[0, \pi_{\Phi,p}]$.

Proof The case of $p = 1$ has been proved by Theorem 2. So we only need to prove the cases of $1 < p < \infty$.

Sufficiency. Let $x \in S(l_{\Phi,p})$. Using Lemma 3, we will show that x is an extreme point of $B(l_{\Phi,p})$. This is clear if $card(supp\ x) = 1$. So, we can assume that $card(supp\ x) > 1$.

Since, by Lemma 5, $\pi_{\Phi,p} > 0$, Φ is strictly convex on an open neighborhood of 0, whence $\Psi(p_+(u)) > \Psi(p_+(0)) = 0$ for all $u > 0$. Thus, Φ cannot be linear on R, by Lemma 2, $K_p(z) \neq \emptyset$ for all $z \in l_{\Phi,p}\backslash\{0\}$. Let $k \in K_p(x)$, we claim that the inequality

$$\Phi^{p-1}(kx(i))\Psi(p_+(k|x(i)|)) > \frac{1}{2^p},$$

can hold true for at most one index $i \geq 1$. Indeed, otherwise we could find $i, j \geq 1, i \neq j$, such that $kx(i) \neq 0$, $kx(j) \neq 0$ and the above inequality holds true for both i and j. Without loss of generality, we can assume that $|x(i)| \leq |x(j)|$. Then

$$\begin{aligned} 1 &< 2[\Phi^{p-1}(kx(i))\Psi(p_+(k|x(i)|))]^{\frac{1}{p}} \\ &= 2^{\frac{1}{q}}\Phi^{\frac{1}{q}}(kx(i)) \cdot 2^{\frac{1}{p}}\Psi^{\frac{1}{p}}(p_+(k|x(i)|)) \\ &\leq [\Phi(kx(i)) + \Phi(kx(j))]^{\frac{1}{q}} \cdot [\Psi(p_+(k|x(i)|)) + \Psi(p_+(k|x(j)|))]^{\frac{1}{p}} \\ &\leq I_\Phi^{\frac{1}{q}}(kx) \cdot I_\Psi^{\frac{1}{p}}(p_+(k|x|)) = [I_\Phi^{p-1}(kx)I_\Psi(p_+(k|x|))]^{\frac{1}{p}} \leq 1 \end{aligned}$$

a contradiction that proves the claim. Therefore, by (iii), Φ is not strictly convex at most at one value $k|x(i)|$. By Lemma 3, x is an extreme point of $B(l_{\Phi,p})$. By the arbitrariness of x, the Orlicz spaces $l_{\Phi,p}$ is strictly convex.

Necessity. Assume that $l_{\Phi,p}$ is strictly convex.

(i) If $a_\Phi > 0$, $l_{\Phi,1}$ is contains a linearly isometric copy of l_∞ according to Theorem 1, so it is not strictly convex. This contradiction proves that (i) holds.

(ii) Suppose that Φ is not strictly convex on $[0, \pi_{\Phi,p}]$, i.e., Φ is affine on some interval $(a-\varepsilon, a+\varepsilon) \subset [0, \pi_{\Phi,p}]$. Then

$$0 \leq \Phi^{p-1}(a)\Psi(p_+(a)) \leq \Phi^{p-1}(a+\varepsilon)\Psi(p_+(a+\varepsilon)) < \frac{1}{2^p}.$$

Let

$$b = \sup\{0 \leq x < b_\Phi : (2\Phi(a) + \Phi(x))^{p-1}(2\Psi(p_+(a)) + \Psi(p_+(x))) \leq 1\}.$$

We claim that $b > 0$. On the country, suppose that $b = 0$, i.e.,

$$(2\Phi(a) + \Phi(x))^{p-1}(2\Psi(p_+(a)) + \Psi(p_+(x))) > 1$$

for all $x > 0$. Since 0 is a point of right continuity of the function $\Psi(p_+(\cdot))$ and $\Psi(p_+(0)) = 0$, we get

$$2^p\Phi^{p-1}(a)\Psi(p_+(a)) = \lim_{x\to 0^+}(2\Phi(a) + \Phi(x))^{p-1}(2\Psi(p_+(a)) + \Psi(p_+(x))) \geq 1$$

a contradiction. Thus $b > 0$.

Moreover, $b < b_\Phi$ as long as $\lim_{x\to b_\Phi^-} \Phi(x) = \Phi(b_\Phi) = \infty$ (then $\lim_{x\to b_\Phi^-} p_+(u) = \infty$) and we have $b \leq b_\Phi < \infty$ whenever $\Phi(b_\Phi) < \infty$. Thus, in all cases of Φ, $0 < b < \infty$ and $0 < \Phi(b) < \infty$.

Define $k_0 = (1 + (2\Phi(a) + \Phi(b))^p)^{\frac{1}{p}}$ and let $x = \frac{1}{k_0}(a, a, b, 0, \cdots)$. Then $I_\Phi^{p-1}(k_0x)I_\Psi(p_+(k_0|x|) \leq 1$ and, for every $\gamma > 1$,

$$\begin{aligned} I_\Phi^{p-1}(\gamma k_0x)I_\Psi(p_+(\gamma k_0|x|) &= (2\Phi(\gamma a) + \Phi(\gamma b))^{p-1}(2\Psi(p_+(\gamma a)) + \Psi(p_+(\gamma b))) \\ &\geq (2\Phi(a) + \Phi(\gamma b))^{p-1}(2\Psi(p_+(a)) + \Psi(p_+(\gamma b))) > 1, \end{aligned}$$

whence $k_0 \in K_p(x)$. Thus

$$\|x\|_{\Phi,p} = \frac{1}{k_0}s_p(I_\Phi(k_0x)) = \frac{(1 + (2\Phi(a) + \Phi(b))^p)^{\frac{1}{p}}}{(1 + (2\Phi(a) + \Phi(b))^p)^{\frac{1}{p}}} = 1.$$

Since $k_0x(1) = k_0x(2) = a \notin SC_\Phi$, x is not an extreme point of $B(l_{\Phi,p})$. Therefore $l_{\Phi,p}$ is not strictly convex and this contradiction ends the proof. □

Criteria for strict convexity of Orlicz sequence space equipped with the Luxemburg norm was given by Kamińska (see [14]). The following theorem solves an analogous problem for the Orlicz sequence spaces generated by the arbitrary Orlicz function and equipped with the Luxemburg norm. Although the criteria is the same as Kamińska's conclusion, our proof is shorter than Kamińska's proof.

Let $\pi_{\Phi,\infty} = \sup\{0 \leq u < b_\Phi : \Phi(u) < \frac{1}{2}\}$. By Lemma 5, $\pi_{\Phi,\infty} > 0$.

Theorem 4 *The Orlicz sequence space $l_{\Phi,\infty}$ is strictly convex with respect to the Luxemburg (∞-Amemiya) norm if and only if the following conditions are fulfilled:*

(i) There exists a $x_0 \in R$ such that $\Phi(x_0) = 1$,
(ii) the function Φ satisfies the condition δ_2,
(iii) Φ is strictly convex on the interval $[0, \pi_{\Phi,\infty}]$.

Proof Sufficiency. Let $x \in S(l_{\Phi,\infty})$. Using Lemma 3, we will show that x is an extreme point $B(l_{\Phi,p})$. This is clear if $card(supp\ x) = 1$. So, we can assume that $card(supp\ x) > 1$.

Since, by Lemma 5, $\pi_{\Phi,\infty} > 0$, Φ is strictly convex on an open neighborhood of 0. Thus, Φ cannot be linear on R, by Lemma 2, $K_\infty(z) \neq \emptyset$ for all $z \in l_{\Phi,\infty}\backslash\{0\}$. Let $k \in K_\infty(x)$. We claim that the equality

$$\Phi(k|x(i)|) > \frac{1}{2}$$

can be hold true for at most one index $i \geq 1$. Indeed, otherwise we could find $i, j \geq 1, i \neq j$, such that $kx(i) \neq 0$, $kx(j) \neq 0$ and the above inequality holds true for both i and j. Without loss of generality, we can assume that $|x(i)| \leq |x(j)|$. Then

$$1 < 2\Phi(kx(i)) \leq \Phi(kx(i)) + \Phi(kx(j)) \leq I_\Phi(kx) \leq \|x\|_\infty = 1$$

a contradiction that proves the claim. Therefore, by (iii), Φ is not strictly convex at most at one value $k|x(i)|$.

We claim that $I_\Phi(x) = 1$. On the contrary, suppose that $I_\Phi(x) < 1$. By the δ_2 condition, there exists $n \in N$ and a constant $K > 1$ such that

$$\sum_{i=n}^{\infty} \Phi(2x(i)) \leq K \sum_{i=n}^{\infty} \Phi(x(i)).$$

Since $supp\ x \geq 2$ and $\Phi(x_0) = 1$ where x_0 is the number from (i), $|x(i)| < \frac{x_0}{2}$ $(i \in N)$. Hence

$$I_\Phi(2x) = \sum_{i=1}^{\infty} \Phi(2x(i)) \leq K \sum_{i=n}^{\infty} \Phi(x(i)) + (n-1)\Phi(x_0) < \infty,$$

Similarly, $I_\Phi(kx) < \infty$ for all $k \geq 1$. But the function $k \to I_\Phi(kx)$ is continuous on $(1, \infty)$ and $\lim_{k\to\infty} I_\Phi(kx) \geq \Phi(k|x(i)|) = \infty$. Therefore there exists $k > 1$ such that $I_\Phi(kx) = 1$. Hence

$$1 = \|x\|_{\Phi,\infty} = \frac{1}{k} s_\infty(I_\Phi(kx)) = \frac{1}{k} \max\{1,\ I_\Phi(kx)\} = \frac{1}{k} < 1,$$

a contradiction. Thus $I_\Phi(x) = 1$.

By Lemma 3, x is an extreme point of $B(l_{\Phi,\infty})$. By the arbitrariness of x, the Orlicz space $l_{\Phi,\infty}$ is strictly convex.

Necessity. Assume that $l_{\Phi,\infty}$ is strictly convex.

(i) Suppose that, for all $u \geq 0$, either $\Phi(u) < 1$ or $\Phi(u) = \infty$. Then $b_\Phi < \infty$ and, by the left-continuity of Φ, $\Phi(b_\Phi) < 1$ as well. By continuity of Φ, we can find $0 < a < b_\Phi$ such that $\Phi(a) + \Phi(b_\Phi) < 1$. Let $x = (a, b_\Phi, 0, \cdots)$. Then $I_\Phi(x) < 1$ and, for all $\gamma > 1$, $I_\Phi(\gamma x) = \infty$. Therefore $1 \in K_\infty(x)$ and $\|x\|_{\Phi,\infty} = \max\{1, I_\Phi(x)\} = 1$ i.e., $x \in S(l_{\Phi,\infty})$. Since $I_\Phi(x) < 1$ and $x(1) = a < b_\Phi$, x is not an extreme point of $B(l_{\Phi,\infty})$, a contradiction. Thus the condition (i) is satisfied.
(ii) If Φ does not satisfy the δ_2 condition then $l_{\Phi,\infty}$ contains a linearly isometric copy of l_∞ (see [3]), so it is not strictly convex. Thus (ii) holds true.
(iii) Suppose that Φ is not strictly convex on $[0, \pi_{\Phi,\infty}]$, i.e., Φ is affine on some interval $(a - \varepsilon, a + \varepsilon) \subset [0, \pi_{\Phi,\infty}]$. Then

$$0 < \Phi(a) < \Phi(a + \varepsilon) < \frac{1}{2}.$$

Let

$$b = \sup\{0 \leq x \leq x_0 : 2\Phi(a) + \Phi(x) \leq 1\},$$

where the number x_0 is taken from the condition (i). Evidently $0 < b < \infty$ and $0 < \Phi(b) < \infty$. Moreover, by (i) and by continuity of Φ, $2\Phi(a) + \Phi(b) = 1$. Define $x = (a, a, b, 0, \cdots)$. Then $I_\Phi(x) = 1$, so $1 \in K_\infty(x)$, whence

$$\|x\|_{\Phi,\infty} = s_\infty(I_\Phi(x)) = \max\{1, 2\Phi(a) + \Phi(b)\} = 1.$$

Since $k_0x(1) = k_0x(2) = a \notin SC_\Phi$, x is not an extreme point of $B(l_{\Phi,\infty})$. Therefore $l_{\Phi,\infty}$ is not strictly convex and this contradiction ends the proof.

□

We can also looked at Theorem 3 and Theorem 4 from a chain of norms point of view with two boundaries: the Orlicz norm ($p = 1$) and the Luxemburg norm ($p = \infty$). Let us note that strict convexity of $l_{\Phi,p}$ for any ($1 < p < \infty$) does not imply the strict convexity of either of the boundaries norms. The following example shows that there are Orlicz spaces $l_{\Phi,p}$ that are strictly convex for $1 < p < \infty$ only.

Example 1 Let $\Phi(x) = x \cdot e^{-\frac{1}{3x}}$ for $x > 0$ and $\Phi(0) = 0$. For every $1 < p < \infty$, the Orlicz space $l_{\Phi,p}$ is strictly convex but it is strictly convex neither for $p = 1$ nor for $p = \infty$.

Proof It is easy to check that Φ is strictly convex on $[0, \infty)$. By Lemma 2, $K_p(x) \neq \emptyset$ ($1 < p < \infty$). Thus $l_{\Phi,p}$ is strictly convex for all $1 < p < \infty$. For every $x > 0$,

$$\begin{aligned}\Psi(p_+(x)) &= x \cdot p_+(x) - \Phi(x) = x(1 + \frac{1}{3x}) \cdot e^{-\frac{1}{3x}} - x \cdot e^{-\frac{1}{3x}} \\ &= \frac{1}{3}e^{-\frac{1}{3x}} \nearrow \frac{1}{3} \quad \text{as} \quad x \to \infty.\end{aligned}$$

Take $e_1 = (1, 0, \cdots)$, $e_2 = (0, 1, 0, \cdots)$. Then for every $k > 0$,

$$I_\Psi(p_+(ke_i)) = \Psi(p_+(k)) < \frac{1}{3} < 1 \quad (i = 1, 2).$$

Thus, $k_1^*(e_i) = k_1^{**}(e_i) = \infty$ i.e., $K_1(e_i) = \emptyset$, and

$$\|e_i\|_{\Phi,1} = \lim_{k\to\infty} \frac{1 + (ke^{-\frac{1}{3k}})}{k} = 1 \quad (i = 1, 2).$$

Set $x = e_1 + e_2$, for any $k > 0$,

$$I_\Psi(p_+(kx)) = 2\Psi(p_+(k)) < \frac{2}{3} < 1.$$

We conclude $K_1(x) = \emptyset$,

$$\|e_1 + e_2\|_{\Phi,1} = \|x\|_{\Phi,1} = \lim_{k\to\infty} \frac{1 + 2(ke^{-\frac{1}{3k}})}{k} = 2.$$

Hence, we obtain a contradiction with the strict convexity of $l_{\Phi,1}$.

Further, for every $x > 0$,

$$\frac{\Phi(2x)}{\Phi(x)} = \frac{2x \cdot e^{-\frac{1}{6x}}}{x \cdot e^{-\frac{1}{3x}}} = 2e^{\frac{1}{6x}} \to \infty \quad \text{as} \quad x \to 0.$$

So the function Φ does not satisfy the δ_2 condition. Thus, $l_{\Phi,\infty}$ is not strictly convex either. □

The next example shows that there are Orlicz spaces $l_{\Phi,p}$ ($1 \leq p < \infty$) are strictly convex, but the Orlicz space $l_{\Phi,\infty}$ is not strictly convex.

Example 2 Let

$$\Phi(x) = \begin{cases} 0, & x = 0, \\ xe^{-\frac{1}{x}}, & x \in (0, 1], \\ \frac{1}{e}x^2, & x \in (1, \infty). \end{cases}$$

For every $1 \leq p < \infty$ the Orlicz sequence space $l_{\Phi,p}$ is strictly convex but it is not strictly convex for $p = \infty$.

Proof We get easily

$$p_+(x) = \begin{cases} 0, & x = 0, \\ (1+\frac{1}{x})e^{-\frac{1}{x}}, & x \in (0,1], \\ \frac{2}{e}x, & x \in (1,\infty). \end{cases}$$

Hence $p_+(x)$ is continuous and finite on the whole R^+. By Lemma 2, we have $K_p(x) \neq \emptyset$ and Φ is strictly convex on the interval $[0, \pi_{\Phi,p}]$ for all $1 \leq p < \infty$. Therefore $l_{\Phi,p}$ $(1 \leq p < \infty)$ is strictly convex. However,

$$\lim_{x\to 0^+} \frac{\Phi(2x)}{\Phi(x)} = 2e^{\frac{1}{2x}} \to \infty,$$

so Φ fails to satisfy the condition δ_2. Then $l_{\Phi,\infty}$ is not strictly convex. □

The last example shows that there are Orlicz spaces $l_{\Phi,p}$ $(1 < p \leq \infty)$ are strictly convex, but the Orlicz space $l_{\Phi,1}$ is not strictly convex.

Example 3 Let

$$\Phi(x) = \begin{cases} e^x - 1, & x \in [0, \ln(\frac{3}{2})], \\ \frac{3}{2}x + \frac{1}{2} - \frac{3}{2}\ln(\frac{3}{2}), & x \in (\ln(\frac{3}{2}), \infty). \end{cases}$$

For every $1 < p \leq \infty$, the Orlicz sequence space $l_{\Phi,p}$ is strictly convex but it is not strictly convex for $p = 1$.

Proof It is easy to calculate $p_-(x)$, $q_-(v)$, we get easily

$$\Psi(v) = \begin{cases} 0, & v \in [0,1], \\ v\ln(v) + 1 - v, & v \in (1, \frac{3}{2}], \\ \infty, & v \in (\frac{3}{2}, \infty). \end{cases}$$

Considering the function Ψ we have $b_\Psi = \frac{3}{2}$ and $\Psi(b_\Psi) = \frac{3}{2}\ln\frac{3}{2} - \frac{1}{2} < \frac{1}{2}$. By Theorem 3, the space $l_{\Phi,1}$ is not strictly convex.

Since $b_\Psi = \frac{3}{2}$, we have $b_\Phi = \infty$. For any $1 < p < \infty$, we have

$$\Phi^{p-1}(\ln\frac{3}{2})\Psi(p_+(\ln\frac{3}{2})) < \frac{1}{2^p},$$

and for any $x > \ln\frac{3}{2}$, we get

$$\Phi^{p-1}(x)\Psi(p_+(x)) = \infty.$$

Whence $\pi_{\Phi,p} = \ln\frac{3}{2}$ $(1 < p < \infty)$. Φ is strictly convex on $[0, \pi_{\Phi,p}]$, by Theorem 3, $l_{\Phi,p}$ $(1 < p < \infty)$ are strictly convex.

Since for every $x > 0$, $\lim_{x\to 0^+} \frac{\Phi(2x)}{\Phi(x)} = 2 < \infty$, the function Φ satisfy the δ_2 condition. Moreover, Φ is finite on R^+ and it is strictly convex on the interval $\{x \in R^+ : x \leq \ln(\frac{3}{2})\} = \{x \in R^+ : \Phi(x) \leq \frac{1}{2}\}$. Hence $l_{\Phi,\infty}$ is strictly convex.

□

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