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ORIGINAL RESEARCH

The closure operator and flats of Γ-extension matroids

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Abstract Slater introduced the point-addition operation on graphs to characterize 4-connected graphs. The Γ-extension operation on binary matroids is a generalization of the point-addition operation. In this paper, for a given binary matroid M we characterize the closure operator and the flats of the Γ-extension binary matroid M^X in terms of the closure operator of original binary matroid M.

Keywords binary matroid · closure operator · flats of matroids · Γ-extension

Mathematics Subject Classification 05B35

1 Introduction

We refer to Oxley [3] and West [8] for standard terminology in matroids and graphs, respectively. Slater [6] introduced the point-addition operation on graphs and used it to classify 4-connected graphs. Azanchilar [1] extended this operation to binary matroids as follows.

Definition 1 [1]. Let M be a binary matroid with ground set E and matrix representation A over the field $GF(2)$. Let $X = \{x_1, x_2, ..., x_m\} \subseteq E$ be an independent set in M and let $\Gamma^X = \{\gamma_1, \gamma_2, ..., \gamma_m\}$ be a set such that $E \cap \Gamma^X = \emptyset$. Suppose A' is the matrix obtained from the matrix A by adjoining m columns labeled by $\gamma_1, \gamma_2, ..., \gamma_m$ such that the column labeled by γ_i is same as the column labeled by x_i, for $i = 1, 2, ..., m$. Let A^X be the matrix obtained by adjoining one extra row to A' which has entry 1 in the column labeled by γ_i for $i = 1, 2, ..., m$ and zero elsewhere. The vector matroid of the matrix A^X, denoted by M^X, is called as the Γ-extension of M with respect to X and the transition from M to M^X is called as the Γ-extension operation on M.

Not that the ground set of the matroid M^X is $E \cup \Gamma^X$ and $M^X \setminus \Gamma^X = M$. Therefore M^X is an extension of M. Some basic properties of M^X are studied in [1] and [2].

The Γ-extension operation is related to the splitting operation on binary matroid which is defined by shikare et. al. [7] as follows.

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Definition 2 [7]. Let M be a binary matroid with standard matrix representation A over the field $GF(2)$ and let X be a set of elements of M. Let A_X be the matrix obtained by adjoining an extra row to the matrix A whose entries are 1 in the columns labeled by the elements of the set X and zero otherwise. The vector matroid of the matrix A_X, denoted by M_X, is called as the splitting matroid of M with respect to X, and the transition from M to M_X is called as the splitting operation.

Let M be a binary matroid with ground set E and let $X = \{x_1, x_2, ..., x_m\}$ be an independent set in M. Obtain the extension M' of M with ground set $E \cup \Gamma^X$, where $\Gamma^X = \{\gamma_1, \gamma_2, ..., \gamma_m\}$ is disjoint from E, such that $\{x_i, \gamma_i\}$ is a 2-circuit in M' for each i. The matroid M'_Γ obtained from M' by splitting with respect to the set Γ^X is the Γ-extension matroid M^X.

2 Properties of Γ-extension matroids

The following propositions characterize the circuits of Γ-extension matroids in terms of the circuits of the matroid M.

Proposition 1 [1]. *Let M be a binary matroid and $X \subseteq E(M)$. Then*

(1) Every circuit of M is a circuit of M^X;
(2) Every circuit of M^X contains at least one element of E; and
(3) Every circuit of M^X contains an even number of elements of Γ^X.

Proposition 2 [1]. *Let M be a binary matroid on E with representation matrix $A = [I_r|Y]$ over $GF(2)$ and X be a subset of a base of M. Then a subset Z of $E \cup \Gamma^X$ is a circuit of M^X if and only if one of the following conditions hold:*

(1) $Z = \{e_i, e_j, \gamma_i, \gamma_j\}$ where $i \neq j$, $\gamma_i, \gamma_j \in \Gamma^X$ and $e_i, e_j \in X$ for $1 \leq i, j \leq r$;
(2) $Z = J \cup D$ where $J \subseteq \Gamma^X$, $|J|$ is an even integer and $\emptyset \neq D \subseteq E$ is such that $D \cup X_J$ is a circuit in M where $X_J = \{e_i \in X | \gamma_i \in J\}$.

Let M be a matroid on E with rank function r. Then the function cl from 2^E into 2^E defined, for all $A \subseteq E$, by $cl(A) = \{x \in E | r(A \cup x) = r(A)\}$, where 2^E denote the power set of E, is called the closure operator of M.

Lemma 1 [3]. *Let M be a matroid on E and $A \subseteq E$. Then $cl(A) = A \cup \{x \in E | M$ has a circuit C such that $x \in C \subseteq A \cup x\}$*

Lemma 2 [3]. *Suppose M is a matroid on E with the rank function r, $A \subseteq E$ and $x \in cl(A)$. Then $cl(A \cup x) = cl(A)$.*

Let cl and cl' be the closure operators of M and M^X, respectively. The following result characterizes the rank function of the matroid M^X in terms of the rank function of the matroid M.

Proposition 3 [1]. *Let r and r' be the rank functions of the matroids M and M^X, respectively. Suppose that $Y_1, Y_2 \subseteq X$ and $A \subseteq E(M) - X$. Then*

(1) $r'(Y_1 \cup A \cup \Gamma^{Y_2}) = r(Y_1 \cup A) + 1$, if $Y_2 \subseteq Y_1$;
(2) $r'(Y_1 \cup A \cup \Gamma^{Y_2}) = r(Y_2 \cup A) + 1$, if $Y_1 \subseteq Y_2$;
(3) $r'(Y_1 \cup A) = r(Y_1 \cup A)$;
(4) $r'(A \cup \Gamma^{Y_2}) = r(A \cup Y_2)$, if $A \subseteq E(M) - X$; and
(5) $r'(\Gamma^{Y_2}) = |\Gamma^{Y_2}|$.

The following results follow immediately from Proposition 3.

Corollary 1 *Let r and r' be the rank functions of the matroids M and M^X, respectively. Then $r'(M^X) = r(M)+1$.*

Corollary 2 *A subset B' of $E \cup \Gamma^X$ is a base for M^X if and only if $B' = (B \setminus D) \cup J$, where $\emptyset \neq J$, $D \subset B$, $|D| = |J| - 1$ and $(B \setminus D) \cup X_j$ contains no circuit of M.* □

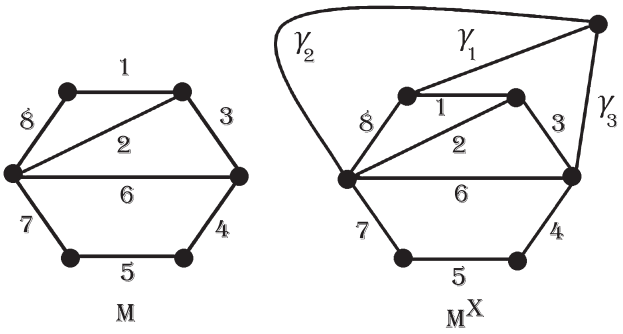

Fig. 1 Graphic representation of M and M^X

3 The main theorem

Throughout this section, we assume that M is a binary matroid on a set E and M^X is the Γ-extension matroid of M with respect to the subset X of E. Further, $A' \subseteq E \cup \Gamma^X = E \cup \{\gamma_1, \gamma_2, ..., \gamma_m\}$ and $A = A' \backslash \Gamma^X$.

We shall use a sequence of four elementary lemmas to find the closure operator of the Γ-extension matroid M^X in terms of the closure operator of the matroid M.

Lemma 3 *Suppose $A' = Y_1 \cup A$ and $Y_1 \cup A \subseteq E(M)$. Then $cl'(A') = cl(Y_1 \cup A)$.*

Proof Suppose, $A' = Y_1 \cup A$ and $Y_1 \cup A \subseteq E(M)$. Let $x \in cl'(A')$. Then $x \in A' = Y_1 \cup A$ or $x \in cl'(A')$-A'. If $x \in Y_1 \cup A$, then we are through. Now if $x \in cl'(A') - A'$, then there is a circuit C' of M^X such that $x \in C'$ and $C' \subseteq A' \cup x = Y_1 \cup A \cup x$.

If $x \in E$, then by Proposition 3, C' is a circuit of M. Therefore, $x \in cl(Y_1 \cup A)$ and $cl'(A') \subseteq cl(Y_1 \cup A)$. If $x = \gamma_i$, then $\gamma_i \in C'$ and C' is a circuit of M^X contained in $A \cup \gamma_i$. This implies that $\gamma_i \in C' \subseteq A \cup \gamma_i$. Since by Proposition 3, $|\Gamma|$ is even, $\exists \gamma_i \in A$ and this is a contradiction to the fact that A contains no member of Γ^X.

Conversely, let $x \in cl(Y_1 \cup A)$. If $x \in Y_1 \cup A$, then there is nothing to prove. If $x \in cl(Y_1 \cup A) - \{Y_1 \cup A\}$, then there exist circuit C of M such that $x \in C$ and $C \subseteq Y_1 \cup A \cup x$. Now $C' = C$ is a circuit of M^X and $x \in C' \subseteq Y_1 \cup A \cup x$. This implies that $x \in cl'(A')$. Therefore $cl(Y_1 \cup A) \subseteq cl'(A')$ and we conclude that $cl'(A') = cl(Y_1 \cup A)$. □

We illustrate the above Lemma with the help of the following example (Fig. 1).

Example 1 Consider the cycle matroid $M(G)$ corresponding to the graph shown in the Figure 1. Let $X = \{1, 2, 3\}$. Then M^X is the corresponding Γ-extension matroid.

Let $A' = \{2, 6\}$ and $Y_1 = \{2\}$. Then $A' = Y_1 \cup A$, $cl(Y_1 \cup A) = \{2, 6, 3\} = cl'(A')$.

Lemma 4 *Let $A' = Y_1 \cup A \cup \Gamma^{Y_2}$ and $Y_2 \subseteq Y_1 \subseteq X$. Then $cl'(A') = cl(Y_1 \cup A) \cup \Gamma^C$ where $C \in cl(Y_1 \cup A)$.*

Proof Suppose $A' = Y_1 \cup A \cup \Gamma^{Y_2}$ and $Y_2 \subseteq Y_1$. If $x \in A' = Y_1 \cup A \cup \Gamma^{Y_2}$, then $x \in cl(Y_1 \cup A)$. If $x \in cl'(A') - A'$, then there is a circuit C' of M^X such that $x \in C'$ and $C' \subseteq A' \cup x$. If $\Gamma^{Y_2} \nsubseteq C'$, then $x \in C' \subseteq Y_1 \cup A \cup x$. This implies that $x \in cl(Y_1 \cup A)$ in M. If $\Gamma^{Y_2} \subseteq C'$ and $x \notin Y_1$ then C' has one of the following form.

i) $C' = Y_1 \cup \Gamma^{Y_1}$. Then $x \in C' = Y_1 \cup \Gamma^{Y_1} \subseteq Y_1 \cup A \cup \Gamma^{Y_2} \cup x$. This implies that $\Gamma^{Y_1} \subseteq \Gamma^{Y_2}$ and $Y_1 \subseteq Y_2$; a contradiction.

ii) $C' = D \cup \Gamma^{Y_1}$ and $|\Gamma^{Y_1}|$ is even.

Then $x \in C' = D \cup \Gamma^{Y_1} \subseteq Y_1 \cup A \cup \Gamma^{Y_2} \cup x$. We conclude that $x \in \Gamma^C$ where $C \subseteq cl(Y_1 \cup A)$ and $cl'(A') \subseteq cl(Y_1 \cup A) \cup \Gamma^C$.

If $\gamma_x \in C'$ and $x \in Y_1$ then $C' = Y_1 \cup \Gamma^{Y_1}$, thus $x \in C' = Y_1 \cup \Gamma^{Y_1} \subseteq Y_1 \cup A \cup \Gamma^{Y_2} \cup x$. This implies that $\Gamma^{Y_1} = \Gamma^{Y_2} \cup x$ and $x \in \Gamma^{Y_1} \subseteq \Gamma^X$.

Conversely, let $x \in cl(Y_1 \cup A) \cup \Gamma^C$ where $C \subseteq cl(Y_1 \cup A)$. If $x \in A$ or $x \in \Gamma^{Y_2}$, then $x \in cl'(A')$. If $x \in cl(Y_1 \cup A) - \{Y_1 \cup A\}$, then there is a circuit C of M such that $x \in C \subseteq Y_1 \cup A \cup x$. As $Y_2 \subseteq Y_1$, thus C is also a circuit of M^X and $x \in cl'(A')$. In the case $x \in \Gamma^C$, there is a circuit C of M^X such that $x \in C \subseteq Y_1 \cup A \cup \Gamma^{Y_2}$. We conclude that $x \in cl'(Y_1 \cup A \cup \Gamma^{Y_2}) = cl'(A')$. □

To examplify the above Lemma, let $Y_1 = \{1, 2\}$, $A = \{6\}$, $\Gamma^{Y_2} = \{\gamma_1\}$ in Example 1. Then $cl'(A') = \{1, 2, 3, 6, 8, \gamma_1, \gamma_2, \gamma_3\} = cl(Y_1 \cup A) \cup \Gamma^C$ where $C \subseteq cl(Y_1 \cup A)$.

Lemma 5 *Let $A' = Y_1 \cup A \cup \Gamma^{Y_2}$ and $Y_1 \subseteq Y_2$. Then $cl'(A') = cl(Y_1 \cup A \cup Y_2) \cup \Gamma^C$ where $C \subseteq cl(Y_1 \cup A \cup Y_2)$.*

Proof The proof is similar to the proof of the Lemma 4. □

Lemma 6 *Let $A' = \Gamma^{Y_1}$. Then $cl'(A') = cl(Y_1) \setminus Y_1 \cup \Gamma^{Y_1}$.*

Proof Suppose $A' = \Gamma^{Y_1}$. Let $x \in cl'(A')$. If $x \in A'$, then $x \in \Gamma^{Y_1}$ and we are through. Suppose that $x \in cl'(A') - A'$. Then there is a circuit C' of M^X such that $x \in C' \subseteq A' \cup x$. If $x \in E(M)$, then by Proposition 3, C' is a circuit of M and $x \in C' \subseteq A' \cup x = \Gamma^{Y_1} \cup x$, Thus $C' \subseteq \Gamma^{Y_1} \cup x$, a contradiction.

If $x \in E(M)$, then there is a circuit C' of M^X such that $x \in C' \subseteq A' \cup x = \Gamma^{Y_1} \cup x$.

i) $C' = Y_2 \cup \Gamma^{Y_2}$ and $|\Gamma^{Y_2}|$ is even. Then $x \in C' = Y_2 \cup \Gamma^{Y_2} \subseteq \Gamma^{Y_1} \cup x$, thus $Y_2 \subseteq Y_1$ a contradiction.
ii) $C' = D \cup \Gamma^{Y_2}$ and $|\Gamma^{Y_2}|$ is even.

Then $x \in C' = D \cup \Gamma^{Y_2} \subseteq \Gamma^{Y_1} \cup x$, Thus $x \in cl(Y_1) \setminus Y_1$.

Conversely, let $x \in cl(Y_1) \setminus Y_1 \cup \Gamma^{Y_1}$. If $x \in \Gamma^{Y_1}$,then $x \in A'$ and $x \in cl'(A')$. When $x \in cl(Y_1) \setminus Y_1$, there is a circuit C of M such that $x \in C \subseteq \Gamma^{Y_1} \cup x = A' \cup x$, thus $x \in cl'(A')$. We conclude that $cl(Y_1) \setminus Y_1 \cup \Gamma^{Y_1} \subseteq cl'(A')$. □

Alternatively, one can deduce the following theorem by combining lemmas 3-6.

Theorem 1 *Let M be a binary matroid on a set E and M^X be the Γ-extension matroid of M with respect to a subset X of E and $Y_1, Y_2 \subseteq X$. Suppose $A' \subseteq E \cup \Gamma^X$ and $A \subseteq E(M) - X$. Then $cl'(A')$ is given by one of the following sets.*

(i) $cl(Y_1 \cup A)$;
(ii) $cl(Y_1 \cup A) \cup \Gamma^C$ *where* $C \subseteq cl(Y_1 \cup A)$;
(iii) $cl(Y_1 \cup A \cup Y_2) \cup \Gamma^C$ *where* $C \subseteq cl(Y_1 \cup A \cup Y_2)$; *or*
(iv) $cl(Y_1) \setminus Y_1 \cup \Gamma^{Y_1}$.

4 Flats of Γ-extension matroids

In the following Theorem, we characterize the flats of the Γ-extension matroid M^X in term of the flats of the matroid M.
We denote by C_{EY} the set of all circuits of the matroid M each of which contains an even number of elements of the set $Y \subseteq E(M)$.

Theorem 2 *Let M be a binary matroid on a set E and M^X be the Γ-extension matroid of M with respect to a subset X of E. Suppose $A' \subseteq E \cup \Gamma^X$ and $Y \subseteq X$ such that $Y \cup A$ is a flat of M where $A \subseteq E(M) - X$. Then A' is a flat of M^X if one of the following condition holds.*

i) $A' = Y \cup A$.
ii) $A' = Y \cup A \cup \Gamma^Y$.
iii) $A' = A \cup \Gamma^Y$.
iv) $A' = Y \cup A \cup \Gamma^Y \setminus \{\{\gamma_i, \gamma_j\} | \, i \in Y \text{ and } i \neq j, \nexists C \in C_{EY} \text{ such that } C \subseteq Y \cup A\}$.

Proof The proof follows from lemmas 4-6. □

We illustrate the above Theorem with the help of Example 1. The set of all flats of M.

$F = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{2, 4\}, \{2, 5\},$
$\{2, 7\}, \{3, 4\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{4, 8\}, \{5, 6\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{6, 8\},$
$\{7, 8\}, \{1, 2, 8\}, \{2, 3, 6\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 3, 7\}, \{1, 4, 5\}, \{1, 4, 6\}, \{1, 4, 7\}, \{1, 5, 6\}, \{1, 5, 7\},$
$\{1, 6, 7\}, \{4, 5, 6, 7\}, \{2, 3, 4, 6\}, \{2, 3, 6, 5\}, \{2, 3, 6, 7\}, \{1, 2, 4, 8\}, \{1, 2, 5, 8\}, \{1, 2, 7, 8\}, \{1, 3, 5, 7\},$
$\{1, 3, 4, 5\}, \{1, 2, 3, 6, 8\}, \{2, 3, 4, 5, 6, 7\}, \text{E(M)}\}$.

The set of all flats of M^X is

$$F \cup \{\{1, 2, 8, \gamma_1, \gamma_2\}, \{1, 3, \gamma_1, \gamma_3\}, \{2, 3, 6, \gamma_2, \gamma_3\}, \{\gamma_1\},$$
$$\{\gamma_2\}, \{\gamma_3\}, \{8, \gamma_1, \gamma_2\}, \{\gamma_1, \gamma_3\}, \{6, \gamma_2, \gamma_3\}, \{1, 3, 5, \gamma_1, \gamma_3\}, \{5, \gamma_1, \gamma_3\}, \{1, 3, 7, \gamma_1, \gamma_3\},$$
$$\{7, \gamma_1, \gamma_3\}, \{1, 4, 5, \gamma_1\}, \{4, 5, \gamma_1\}, \{1, 4, 6, \gamma_1\}, \{4, 6, \gamma_1\}, \{1, 4, 7, \gamma_1\}, \{4, 7, \gamma_1\},$$
$$\{1, 5, 6, \gamma_1\}, \{5, 6, \gamma_1\}, \{1, 5, 7, \gamma_1\}, \{5, 7, \gamma_1\}, \{1, 6, 7, \gamma_1\}, \{6, 7, \gamma_1\}, \{2, 3, 4, 6, \gamma_2, \gamma_3\},$$
$$\{4, 6, \gamma_2, \gamma_3\}, \{2, 3, 5, 6, \gamma_2, \gamma_3\}, \{5, 6, \gamma_2, \gamma_3\}, \{2, 3, 6, 7, \gamma_2, \gamma_3\}, \{6, 7, \gamma_2, \gamma_3\},$$
$$\{1, 2, 4, 8, \gamma_1, \gamma_2\}, \{4, 8, \gamma_1, \gamma_2\}, \{1, 2, 5, 8, \gamma_1, \gamma_2\}, \{5, 8, \gamma_1, \gamma_2\}, \{1, 2, 7, 8, \gamma_1, \gamma_2\},$$
$$\{7, 8, \gamma_1, \gamma_2\}, \{1, 3, 5, 7, \gamma_1, \gamma_3\}, \{5, 7, \gamma_1, \gamma_3\}, \{1, 3, 4, 5, \gamma_1, \gamma_3\}, \{4, 5, \gamma_1, \gamma_3\},$$
$$\{1, 2, 3, 6, 8, \gamma_1, \gamma_2, \gamma_3\}, \{6, 8, \gamma_1, \gamma_2, \gamma_3\}, \{2, 3, 4, 5, 6, 7, \gamma_2, \gamma_3\}, \{4, 5, 6, 7, \gamma_2, \gamma_3\},$$
$$\{1, 2, 3, 4, 5, 6, 7, \gamma_1, \gamma_2, \gamma_3\}, \{4, 5, 6, 7, \gamma_1, \gamma_2, \gamma_3\}\}.$$

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