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ORIGINAL RESEARCH

A comparison of generalized opers and (G, P)-opers

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Abstract The goal of this paper is to describe the relationship between generalized B-opers, generalized $\mathrm{SO}(2n, \mathbb{C})$-opers and (G, P)-opers. In particular, we show that to each generalized B-oper there is a naturally associated (G, P)-oper, but there are some (G, P)-opers that do not arise as generalized B-opers or $\mathrm{SO}(2n, \mathbb{C})$-opers.

Keywords Oper · Parabolic subgroup · Differential operator · Connection

Mathematics Subject Classification 14H60 · 31A35 · 33C80 · 53C07

Introduction

In a recent paper [4], Collier and Sanders defined an (G, P)-oper, which generalizes Beilinson and Drinfeld's definition of an G-oper to allow any parabolic subgroup $P < G$ as opposed to a Borel subgroup as part of the data. For a special class of parabolic subgroups called even Jacobson–Morozov parabolics, Collier and Sanders gave a parametrization of (G, P)-opers in higher Teichmuller space. In a similar direction but with a different approach, [2] and [3] introduced generalized B-opers and generalized $\mathrm{SO}(2n, \mathbb{C})$-opers which allow partial flags of vector bundles where the successive quotients have a constant rank $r \geq 1$. As two generalizations of opers, (G, P)-opers and generalized B-opers are closely related to each other in the way that principal bundles are related to their associated vector bundles. In the first two sections, we recall from [2] and [3] generalized B-opers and generalized $\mathrm{SO}(2n, \mathbb{C})$-opers and give an exposition of known results. In section 3 we define the (G, P)-opers, as introduced in [4]. Then, we dedicate Sections 4 and 5 to prove the following:

Theorem (4.1) *To each generalized B-oper* (Definition 1.3)*, there is a naturally associated (G, P)-oper* (Definition 3.2)*.*

Finally we show that (G, P)-opers are more general class of opers.

Theorem (5.3) *There exist some (G, P)-opers which do not arise as the associated oper of a generalized B-oper or a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper.*

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1 Generalized B-opers

In this section we review the results of [2], where we show some properties of generalized B-opers, then outline the steps necessary for finding the parameter space of generalized B-opers.

Let X be a compact Riemann surface of genus $g \geq 2$. Let E be a holomorphic vector bundle over X.

Let B be a holomorphic nondegenerate bilinear form on E. It is a holomorphic map $B : E \otimes E \to \mathcal{O}_X$ such that at each $x \in X$, the fiberwise map $B_x : E_x \otimes E_x \to \mathbb{C}$ is a nondegenerate bilinear form on the complex vector space E_x. Nondegeneracy means for each nonzero vector $v \in E_x$, there is some $w \in E_x$ such that $B_x(v, w) \neq 0$.

A **B-connection** on E is a holomorphic connection D on E such that

$$\partial(B(s,\ t)) \ = \ B(D(s),\ t) + B(s,\ D(t))$$

for all smooth local sections s and t of E. Note that $\partial(B(s,\ t)) \ = \ d(B(s,\ t))$ when s, t are holomorphic sections, because B is compatible with the holomorphic structure.

Now suppose the holomorphic vector bundle E is equipped with a B-connection D. For any holomorphic subbundle $F \subset E$, the **orthogonal complement** of F with respect to the bilinear form B is

$$F^{\perp} := \{w \in E \mid B_{p(w)}(w, v) = 0 \,\forall\, v \in F_{p(w)}\},$$

where $p : E \to X$ is the natural projection of the vector bundle.

Proposition 1.1 *The orthogonal complement $F^{\perp}$ with respect to B satisfies the following properties.*

(1) *$F^{\perp} \subset E$ is a holomorphic subbundle.*
(2) *$F^{\perp}$ is canonically isomorphic to the dual bundle $(E/F)^*$.*
(3) *$(F^{\perp})^{\perp} = F$.*

Proof Let $\overline{\partial} : E \to E \otimes \Omega^{0,1}_X$ be the Dolbeault operator that defines the holomorphic structure on E. For any locally smooth sections s_1 of F and s_2 of $F^{\perp}$, we have the following equation from the holomorphicity of B:

$$\overline{\partial}(B(s_1, s_2)) = B(\overline{\partial} s_1, s_2) + B(s_1, \overline{\partial} s_2).$$

Since F is assumed to be a holomorphic subbundle, we have $\overline{\partial} s_1$ is a smooth section of $F \otimes \Omega^{0,1}_X$. Both s_1 and $\overline{\partial} s_1$ are orthogonal to s_2 with respect to B implies $B(s_1, \overline{\partial} s_2) = 0$, hence $\overline{\partial} s_2$ must be a smooth section of $F^{\perp} \otimes \Omega^{0,1}_X$, which implies that $F^{\perp}$ is a holomorphic subbundle of E as required.

The nondegeneracy of B gives a canonical identification $B^{\flat} : E \simeq E^*$. The composition $F^{\perp} \hookrightarrow E \xrightarrow{B^{\flat}} E^* \to (E/F)^*$ gives an isomorphism.

The third statement follows from definition. □

Definition 1.2 Suppose B is either a symplectic or a symmetric form. A **B-filtration** of E is a filtration of holomorphic subbundles

$$0 = F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = E$$

such that

(1) the filtration length n is odd when B is symplectic and even when B is symmetric;
(2) F_i/F_{i-1} are holomorphic vector bundles;
(3) and $F_i^{\perp} = F_{n-i}$ for all $1 \leq i \leq n-1$.

In particular the first half of a B-filtration is fiberwise an isotropic flag of vector spaces (Definition 5.1). When B is symmetric and n is odd, the middle subbundle is Lagrangian.

Definition 1.3 A **generalized B-oper** is a triple $(E, \{F_i\}, D)$, where $\{F_i\}$ is a B-filtration and D is a B-connection on E, such that

(1) (Griffith's transversality) $D(F_i) \subset F_{i+1} \otimes K_X$ for all $1 \leq i \leq n-1$;
(2) (Nondegeneracy) for every $1 \leq i \leq n-1$, the second fundamental form of D with respect to F_i induces an isomorphism

$$\begin{aligned} S_D : F_i/F_{i-1} &\xrightarrow{\sim} (F_{i+1}/F_i) \otimes K_X \\ [s] &\mapsto [D(s)]. \end{aligned}$$

1.1 Taking apart a B-oper

In this section we show that a generalized B-oper can be naturally rewritten as a product of an rank r flat orthogonal bundle and a classical $\mathrm{SL}(n, \mathbb{C})$-oper.

Let K be the canonical bundle on X and fix a theta characteristic $\theta^2 = K$. Let $J^m(W)$ denote the m'th order jet bundle of a holomorphic vector bundle W on X.

Theorem 1.4 ([2]) *Let $(E, \{F_i\}, D)$ be a generalized B-oper on X. Let n be the filtration length of $\{F_i\}$ and r be the rank of associated graded bundles. Then E can be canonically identified with the holomorphic vector bundle $(F_n/F_{n-1} \otimes \theta^{n-1}) \otimes J^{n-1}(\theta^{-n+1})$. Furthermore $F_n/F_{n-1} \otimes \theta^{n-1}$ is a rank r flat orthongal bundle and $J^{n-1}(\theta^{-n+1})$ is a classical $\mathrm{Sp}(n, \mathbb{C})$-oper when n is even and a classical $\mathrm{SO}(n, \mathbb{C})$-oper when n is odd.*

Proof For ease of notation, write $Q = F_n/F_{n-1}$, $\mathcal{Q} = F_n/F_{n-1} \otimes \theta^{n-1}$, and $\mathcal{J}_n = J^{n-1}(\theta^{-n+1})$. Here is an outline of the proof given in [2].

(1) The bilinear form B induces a symmetric bilinear form $\mathbb{S}$ on $\mathcal{Q}$.
(2) There exists a holomorphic coupled connection ∇ on $\mathcal{Q}$ which is an $\mathbb{S}$-connection [1]. In other words, $\mathbb{S}$ is covariant constant with respect to ∇.
(3) The generalized B-oper induces a naturally self-adjoint order n differential operator $\mathcal{D} \in H^0(\mathrm{Diff}^n(Q, Q \otimes K^n))$.
(4) Take trace to get $\tau(\mathcal{D}) \in H^0(\mathrm{Diff}^n(\theta^{-n+1}, \theta^{n+1}))$ and view it as a holomorphic $\mathrm{SL}(n, \mathbb{C})$-connection $\widehat{D}$ on the jet bundle $J^{n-1}(\theta^{-n+1})$.
(5) $J^{n-1}(Q) \simeq \mathcal{Q} \otimes J^{n-1}(\theta^{-n+1})$
(6) Establish an isomorphism $\mathcal{Q} \otimes J^{n-1}(\theta^{-n+1}) \simeq E$ that preserves the connections $\nabla \otimes 1 + 1 \otimes \widehat{D}$ on $\mathcal{Q} \otimes J^{n-1}(\theta^{-n+1})$ and D on E.
(7) Induce a bilinear form $\widehat{B}$ on the jet bundle $J^{n-1}(\theta^{-n+1})$ from the isomorphism $\mathcal{Q} \otimes J^{n-1}(\theta^{-n+1}) \simeq E$. In particular, $\widehat{B}$ is covariant constant with respect to $\widehat{D}$ and thus $\widehat{D}$ is a $\widehat{B}$-connection.

□

1.2 Reconstructing a B-oper

Previously we have seen that a B-oper contains the data of a flat orthogonal bundle and a classical $\mathrm{SL}(n, \mathbb{C})$-oper. In this section we reconstruct a B-oper given a flat orthogonal bundle and a classical $\mathrm{SL}(n, \mathbb{C})$-oper.

Theorem 1.5 ([2]) *Fix integers $n \geq 2$, $n \neq 3$ and $r \geq 1$. There is a bijective correspondence between the space of of all generalized B-opers of filtration length n and associated graded rank $\mathrm{rk}(F_i/F_{i-1}) = r$ and*

$$\mathcal{C}(X) \times \mathcal{P}(X) \times \bigoplus_{j=2}^{\lceil n/2 \rceil} H^0(X, K^{2j})$$

where $\mathcal{C}(X) = \mathrm{Hom}(\pi_1(X), \mathrm{SO}(n, \mathbb{C}))/\mathrm{SO}(n, \mathbb{C})$ is the character variety that parameterizes flat orthogonal bundles on X and $\mathcal{P}(X)$ is the space of projective structures on X up to marked isomorphism.

Proof Let W be any rank r holomorphic vector bundle on X with a nondegenerate symmetric bilinear form $\mathbb{S}_W$. Again we summarize the proof in [2] as follows.

(1) Similar to the existence of a Chern connection, there is a unique holomorphic connection that preserves $\mathbb{S}_W$, which we denote by ∇^W.
(2) If n is even let ω denote an $\mathrm{Sp}(n, \mathbb{C})$-oper and if n is odd, let ω denote an $\mathrm{SO}(n, \mathbb{C})$-oper.
(3) ω defines a holomorphic connection $\widehat{D}$ on $\mathcal{J}_n = J^{n-1}(\theta^{-n+1})$ and a nondegenerate bilinear form $\widehat{B}$.
(4) Fact: there is a classical correspondence between the space of $\mathrm{SL}(n, \mathbb{C})$-opers on X with the space

$$\mathcal{P}(X) \times \bigoplus_{i=3}^{n} H^0(X, K^i).$$

The subspace

$$\mathcal{P}(X) \times \bigoplus_{j=2}^{\lceil n/2 \rceil} H^0(X, K^{2j})$$

corresponds to $\mathrm{Sp}(n, \mathbb{C})$-opers when n is even and $\mathrm{SO}(n, \mathbb{C})$-opers when n is odd.

(5) Set $E = W \otimes \mathcal{J}_n$, $B = \mathbb{S}_W \otimes \widehat{B}$, and $D = \nabla^W \otimes 1 + 1 \otimes \widehat{D}$. Let $F_i = J_i \otimes W$, for J_i the filtration of $\mathcal{J}_n$. Then $(E, \{F_i\}, D)$ is a generalized B-oper.

□

2 Generalized $\mathrm{SO}(2n, \mathbb{C})$-opers

On some even rank orthogonal vector bundles we could not define generalized B-opers for certain associated graded ranks r. To fill in this gap, we study the generalized $\mathrm{SO}(2n, \mathbb{C})$-opers and describe their parameter space. The detailed results are in [3].

Let X be a compact Riemann surface of genus $g \geq 2$, and E be a holomorphic vector bundle on X. An **orthogonal structure** on E is a holomorphic nondegenerate symmetric bilinear form B on the vector bundle. Equivalently we think of an orthogonal structure as a reduction of structure group of E from $\mathrm{GL}(n, \mathbb{C})$ to $\mathrm{SO}(n, \mathbb{C})$ for $n = \mathrm{rk}(E)$.

Let E be a holomorphic vector bundle of rank $2n$. Fix an integer $r \geq 1$ and let $l = 2n/r$. When l is odd, there is a generalized B-filtration (Definition 1.2) of E of length l and associated graded rank r, with B being a symmetric form. However if l is even, then we may not find a corresponding B-filtration. This motivated us to study generalized $\mathrm{SO}(2n, \mathbb{C})$-opers to cover the case of an even l.

Definition 2.1 Let (E, B) be a holomorphic vector bundle on X with an orthogonal structure B, and suppose $\mathrm{rk}(E) = 2n$. A **generalized filtration** $\{F_i\}$ of (E, B) is a sequence of nested holomorphic subbundles

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{2m+1} = E$$

that satisfies the following conditions:

(1) the associated graded sheaves F_{i+1}/F_i are holomorphic vector bundles for all $i = 0, 1, 2, \ldots, 2m$;
(2) there is a positive integer r such that $\mathrm{rk}(F_{i+1}/F_m) = r$ for all $i \neq m$ and $\mathrm{rk}(F_{m+1}/F_m) = 2r$;
(3) and $F_i^{\perp} = F_{2m+1-i}$ for all $i = 0, 1, 2, ..., m$.

We call $(E, B, \{F_i\})$ a **filtered** $\mathrm{SO}(2n, \mathbb{C})$**-bundle**.

Remark 2.2 The filtration in a generalized B-oper has a fixed associated graded rank, but the the middle associated graded bundles of a filtered $\mathrm{SO}(2n, \mathbb{C})$-bundle has a jump in rank. For this reason we refer to r as the **generic associated graded rank** to mean that all but the middle associated graded bundle has rank r.

Lemma 2.3 *Let* $(E, B, \{F_i\})$ *be a filtered* $\mathrm{SO}(2n, \mathbb{C})$*-bundle of length* $2m + 1$ *with generic associated graded rank* r*, then*

$$r = \frac{n}{m+1}.$$

In particular, we must have $l = 2n/r = 2(m + 1)$, which is even.

2.1 The naive definition

We hope to define a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper as follows, reminiscent of the classical opers. However more structures are needed to parametrize these objects, hence we refer to them as quasiopers for now.

Definition 2.4 Let $(E, B, \{F_i\})$ be a filtered $\mathrm{SO}(2n, \mathbb{C})$-bundle of length $2m + 1$. Let D be a holomorphic B-connection on E. The tuple $(E, B, \{F_i\}, D)$ is called a **generalized** $\mathrm{SO}(2n, \mathbb{C})$**-quasioper** on X if the connection satisfies the following properties.

(1) (Griffith's transversality) For all $i = 1, 2, \ldots, 2m$

$$D(F_i) \subset F_{i+1} \otimes K.$$

(2) (Nondegeneracy) Let S_D be the second fundamental form of D with respect to the filtered subbundles F_i. For all $i \neq m, m+1$, the second fundamental form induces isomorphisms

$$S_D : F_i/F_{i-1} \xrightarrow{\sim} F_{i+1}/F_i \otimes K.$$

Furthermore S_D also induces an isomorphism via self composition

$$S_D^2 : F_m/F_{m-1} \xrightarrow{\sim} F_{m+2}/F_{m+1} \otimes K^2.$$

We say two generalized $\mathrm{SO}(2n, \mathbb{C})$-quasiopers $(E, B, \{F_i\}, D)$ and $(E', B', \{F_i'\}, D')$ are isomorphic if there is a holomorphic isomorphism $\Phi : E \xrightarrow{\sim} E'$ that takes B to B', $\{F_i\}$ to $\{F_i'\}$ and D to D'.

Example 2.5 Any classical $\mathrm{SO}(2n, \mathbb{C})$-oper is a generalized $\mathrm{SO}(2n, \mathbb{C})$-quasioper with generic associated graded rank $r = 1$ and filtration length $2m + 1 = n - 1$.

Proposition 2.6 ([3]) *Let $(E, B, \{F_i\}, D)$ be a generalized $\mathrm{SO}(2n, \mathbb{C})$-quasioper. There exists a natural dual generalized $\mathrm{SO}(2n, \mathbb{C})$-quasioper isomorphic to $(E, B, \{F_i\}, D)$.*

Proof Let B^* and D^* be the natural dual bilinear form and dual connection on E^*. Let $G_{2m+1-i} = (E/F_i)^*$ be the kernel of $E^* \to F_i^*$, which is a holomorphic vector bundle by Proposition 1.1. Let $B^\flat : E \simeq E^*$ be the isomorphism induced by B. Then $F_i = F_{2m+1-i}^{\perp} \xrightarrow{B^\flat} G_i$ implies $B^\flat$ is an isomorphism of generalized $\mathrm{SO}(2n, \mathbb{C})$-quasiopers $(E, B, \{F_i\}, D)$ and $(E^*, B^*, \{G_i\}, D^*)$. □

We give a slightly modified but equivalent proof to the following proposition, which is first presented in [3].

Proposition 2.7 ([3]) *Let $(E, B, \{F_i\}, D)$ be a generalized $\mathrm{SO}(2n, \mathbb{C})$-quasioper. Let $2m + 1$ be the length of $\{F_i\}$ and r be the generic associated graded rank of $\{F_i\}$. There is a natural B-orthogonal decomposition of $E = \mathbb{F} \oplus \mathbb{F}^\perp$ given by the connection D with $\mathrm{rk}(\mathbb{F}) = (2m + 1)r$.*

Proof We pass to the category of coherent sheaves to get the holomorphic subbundle we want because it is an abelian category, and thus closed under taking image and cokernel. In the following proof we use the calligraphic font to denote the coherent sheaf associated to a holomorphic vector bundle.

We use a standard procedure for obtaining a vector subbundle generically generated by a subsheaf. Let $G \subset E$ be any holomorphic subbundle. Consider the second fundamental form as a morphism of sheaves $S_D : \mathcal{G} \to \mathcal{E}/\mathcal{G}\otimes\mathcal{K}$. Let $\mathcal{T}$ be the torsion subsheaf of $(\mathcal{E}/\mathcal{G}\otimes\mathcal{K})/S_D(\mathcal{G})$ and let $p : \mathcal{E}/\mathcal{G}\otimes\mathcal{K} \to (\mathcal{E}/\mathcal{G}\otimes\mathcal{K})/S_D(\mathcal{G})$ be the natural projection map. Then $p^{-1}(\mathcal{T})$ is a locally free sheaf of the same rank as $S_D(\mathcal{G})$ and possibly higher degree.

Since we actually want a subbundle of E, we take preimage again. Let $q : \mathcal{E} \to \mathcal{E}/\mathcal{G}$ be the natural projection map and identify $(\mathcal{E}/\mathcal{G} \otimes \mathcal{K}) \otimes \mathcal{K}^{-1} \simeq \mathcal{E}/\mathcal{G}$. We get a holomorphic vector bundle associated to the locally free sheaf $q^{-1}(p^{-1}(\mathcal{T}) \otimes \mathcal{K}^{-1})$, which we denote by $\widehat{S_D}(G) \subset E$.

Note that $\mathrm{rk}(\widehat{S_D}(G)) = \mathrm{rk}(S_D(\mathcal{G})) + \mathrm{rk}(G)$. Moreover G is preserved by the connection D if and only if $\widehat{S_D}(G) = G$. Let $\widehat{S_D}^{k+1}(G) := \widehat{S_D}(\widehat{S_D}^k(G)) \subset E$ denote the vector bundle obtained by applying the above procedure iteratively $k + 1$ times. Let $F_1 \subset E$ be the first holomorphic subbundle in the filtration $\{F_i\}$ and set $\mathbb{F} = \widehat{S_D}^{2m}(F_1)$.

By the nondegeneracy property of generalized $\mathrm{SO}(2n, \mathbb{C})$-quasiopers, each iteration of applying the second fundamental form increases the rank by r. As a result $\mathbb{F}$ is a rank $(2m + 1)r$ holomorphic subbundle of E.

Furthermore in the $2m$'th iteration the second fundamental form must take on a nontrivial value in F_{2m+1}/F_{2m} by nondegeneracy. This combined with the fact that $F_i^\perp \subset F_{2m} \subsetneq F_{2m+1}$ for all $i = 1, 2, \ldots, 2m$ implies B is nondegenerate on $\mathbb{F}$. Then $\mathbb{F}$ is transverse to $\mathbb{F}^\perp$, so we have the decomposition $E = \mathbb{F} \oplus \mathbb{F}^\perp$. □

2.2 A better definition

Now that we have seen some properties of generalized $\mathrm{SO}(2n, \mathbb{C})$-quasiopers, we impose some natural restrictions that will allow us to parametrize generalized $\mathrm{SO}(2n, \mathbb{C})$-opers in a similar way to the generalized B-opers.

Definition 2.8 Let $(E, B, \{F_i\}, D)$ be a generalized $\mathrm{SO}(2n, \mathbb{C})$-quasioper. Let $E = \mathbb{F} \oplus \mathbb{F}^\perp$ be the decomposition given in Proposition 2.7. Suppose in addition

(1) $\mathbb{F}^\perp \simeq F_1 \otimes K^m$ for F_1 being the first filtered holomorphic subbundle in $\{F_i\}$;
(2) $\widehat{S_D}(\mathbb{F}^\perp) \simeq \mathbb{F}^\perp \oplus (\mathbb{F}^\perp \otimes K^m)$;
(3) and there exists a holomorphic section $\phi \in H^0(X, K^{m+1})$ such that the second fundamental form of D with respect to $\mathbb{F}^\perp$ viewed as a map

$$S_D : \mathbb{F}^\perp \to (\mathbb{F}^\perp \otimes K^m) \otimes K$$

is trivial in the sense $S_D = \mathrm{Id}_{\mathbb{F}^\perp} \otimes \phi$.

Then we call $(E, B, \{F_i\}, D)$ a **generalized** $\mathrm{SO}(2n, \mathbb{C})$**-oper**.

Two generalized $\mathrm{SO}(2n, \mathbb{C})$-opers are isomorphic if the underlying quasiopers are.

Proposition 2.9 *Let $\rho : L \otimes L \to \mathcal{O}$ be a holomorphic isomorphism and let $(E, B, \{F_i\}, D)$ be a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper. Twisting by the line bundle L gives us a new generalized $\mathrm{SO}(2n, \mathbb{C})$-oper.*

Proof Let $E' = E \otimes L$. Note that $\wedge^{2n} E' = \mathcal{O}$. Identify $E' \otimes E' \simeq (E \otimes E) \otimes (L \otimes L)$ so that $B' = B \otimes \rho$ defines an orthogonal structure on E'. Let $F_i' = F_i \otimes L$. Finally set $D' = D \otimes \mathrm{Id}_L + \mathrm{Id}_E \otimes D^L$, where D^L is the unique holomorphic connection on L such that $D^L \otimes \mathrm{Id}_L + \mathrm{Id}_L \otimes D^L = \rho^* d$ is the pullback of the trivial connection on $\mathcal{O}$. We get a generalized $\mathrm{SO}(2n, \mathbb{C})$-quasioper $(E', B', \{F_i'\}, D')$ and need to check that it satisfies the additional conditions for a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper.

The connection D^L has trivial second fundamental form on a line bundle, so

$$\mathbb{F}' = \widehat{S_{D'}}(F_1') \simeq \widehat{S_D}(F_1) \otimes L = \mathbb{F} \otimes L.$$

Since ρ is nondegenerate $L^\perp = 0$ and $\mathbb{F}'^\perp \simeq \mathbb{F}^\perp \otimes L$. By the first property of a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper, we have

$$\mathbb{F}'^\perp \simeq \mathbb{F}^\perp \otimes L \simeq F_1' \otimes K.$$

Next we check the second condition

$$\widehat{S_{D'}}(\mathbb{F}') \simeq \widehat{S_D}(\mathbb{F}) \otimes L \simeq \mathbb{F}'^\perp \oplus (\mathbb{F}'^\perp \otimes K).$$

Finally it's clear that $S_{D'} = S_D \otimes \mathrm{Id}_L = \mathrm{Id}_{\mathbb{F}'} \otimes \phi$ for the same $\phi \in H^0(X, K^{m+1})$ given by the definition of $(E, B, \{F_i\}, D)$ as a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper. □

Let $\mathbb{O}_{n,r}(X)$ be the space of all generalized $\mathrm{SO}(2n, \mathbb{C})$-opers on X of filtration length $2m + 1$ and generic associated graded rank r up to isomorphism. By Lemma 2.3 the filtration length is fixed to be

$$2m + 1 = \frac{2n}{r} - 1.$$

Let $J(X)_2$ be the group of holomorphic line bundles on X of order 2. With the additional constraints of a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper, we can establish the following correspondence.

Theorem 2.10 ([3]) *When r is even $\mathbb{O}_{n,r}(X)$ is in bijective correspondence with*

$$\mathcal{C}(X) \times \mathcal{P}(X) \times H^0(X, K^{m+1}) \times \bigoplus_{j=2}^{m} H^0(X, K^{2j});$$

and when r is odd $\mathbb{O}_{n,r}(X)$ is in bijective correspondence with

$$\mathcal{C}(X) \times \mathcal{P}(X) \times H^0(X, K^{m+1}) \times \bigoplus_{j=2}^{m} H^0(X, K^{2j}) \times J(X)_2.$$

Proof An $\mathrm{SO}(2n, \mathbb{C})$-oper $(E, B, \{F_i\}, D)$ has a decomposition $E = \mathbb{F} \oplus \mathbb{F}^\perp$ by Proposition 2.7. Let $L = \det(\mathbb{F}) \in J(X)_2$, which is trivial if r is even. Take $\mathbb{F}' = \mathbb{F} \otimes L$, then we have a generalized $B_{\mathbb{F}'}$-oper $(\mathbb{F}', B_{\mathbb{F}'}, \{\widehat{S_D}^i(F_1 \otimes L)\}, D_{\mathbb{F}'})$. Note that the filtration length is odd, so the $B_{\mathbb{F}'}$-oper is orthogonal. By the parametrization of B-opers in Theorem 1.5, this corresonds to an element $(\alpha, \beta, \delta) \in \mathcal{C}(X) \times \mathcal{P}(X) \times \bigoplus_{j=2}^m H^0(X, K^{2j})$. Finally the B-connection D can be written as

$$D = \begin{pmatrix} D_{\mathbb{F}} & \mathrm{Id}_{\mathbb{F}^\perp} \otimes \phi \\ -\mathrm{Id}_{\mathbb{F}^\perp} \otimes \phi & D_{\mathbb{F}^\perp} \end{pmatrix}$$

for $\phi \in H^0(X, K^{m+1})$ carrying the second fundamental form data from the definition $S_D = \mathrm{Id}_{\mathbb{F}^\perp} \otimes \phi$. Hence the $\mathrm{SO}(2n, \mathbb{C})$-oper corresponds to a tuple $(\alpha, \beta, \phi, \delta, L)$ as required. □

3 (G, P)-opers

In this section we lay out the Lie algebra foundation to define a (G, P)-oper, and state a parametrization theorem for a class of (G, P)-opers where P is restricted to be even Jacobson–Morozov parabolics.

Let G be a connected complex semisimple Lie group, $P < G$ a parabolic subgroup containing a fixed Borel subgroup B and a fixed maximal torus H. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ be the Lie algebras of $H < B < P < G$ respectively.

Let $R \subset \mathfrak{h}^*$ be the root system of the semisimple Lie algebra $\mathfrak{g}$. Fix a set of positive roots R_+ such that the Borel subalgebra is a direct sum of the Cartan subalgebra and positive root spaces: $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$. Let $\Gamma \subset R_+$ be the set of positive simple roots. The parabolic subgroup P corresponds to a unique subset $I \subset \Gamma$ in the sense that the parabolic subalgebra can be decomposed as a sum of the Cartan subalgebra, the root spaces of all the roots R_I generated by I (positive or negative), and the root spaces of the rest of the positive roots not generated by I:

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_I} \mathfrak{g}_\alpha \oplus \bigoplus_{R_+ \setminus R_I} \mathfrak{g}_\alpha.$$

Then $\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_I} \mathfrak{g}_\alpha$ is the Levi subalgebra of the parabolic, and $\mathfrak{u} = \bigoplus_{R_+ \setminus R_I} \mathfrak{g}_\alpha$ is the nilpotent subalgebra corresponding to the unipotent radical of the parabolic subgroup.

For any root $\alpha = \sum_{\alpha_i \in \Gamma} n_i \alpha_i$, we call $S(\alpha) = \sum_{\alpha_i \in \Gamma \setminus I} n_i \alpha_i$ the **shape** of α, and $l(\alpha) = \sum_{\alpha_i \in \Gamma \setminus I} n_i$ the **level** of α, which is also the height of the shape $\mathrm{ht}(S(\alpha))$. Note that the shape and level are defined with respect to the parabolic.

Definition 3.1 (Grading relative to $\mathfrak{p}$) The root level with respect to $\mathfrak{p}$ defines a natural Lie algebra grading

$$\mathfrak{g}^k = \bigoplus_{l(\alpha)=k} \mathfrak{g}_\alpha$$

such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$.

In particular $\mathfrak{g}^0 = \mathfrak{l}$. Corresponding to this grading is a $\mathfrak{p}$-invariant filtration

$$\mathfrak{g} = \mathfrak{g}_{-m} \supset \cdots \supset \mathfrak{g}_0 \supset \cdots \supset \mathfrak{g}_m = 0$$

for $\mathfrak{g}_k = \bigoplus_{l(\alpha) \geq k} \mathfrak{g}_\alpha$. We have $\mathfrak{g}_1 = \mathfrak{u}$ and $\mathfrak{g}_0 = \mathfrak{p}$ by construction.

Let X be a a compact Riemann surface of genus $g \geq 2$. Suppose we have a holomorphic principal G-bundle E_G on X with a holomorphic connection ω, and a holomorphic reduction to a principal P-bundle E_P.

Let $\mathcal{O} \subset \mathfrak{g}_{-1}/\mathfrak{p}$ be the unique open P-orbit on $\mathfrak{g}_{-1}/\mathfrak{p}$. Let $\Psi_\omega \in H^0(X, K_X \otimes E_P(\mathfrak{g}/\mathfrak{p}))$ be the second fundamental form of ω with respect to the subbundle E_P. It is determined by the composition

$$TE_P \to TE_G \xrightarrow{\omega} \mathfrak{g} \to \mathfrak{g}/\mathfrak{p}.$$

We say the **position** of ω relative to E_P is equal to $\mathcal{O}$ if $\Psi(v) \in E_P(\mathcal{O})$ for all nonzero $v \in TX$, and denote this by $\mathrm{pos}_{E_P}(\omega) = \mathcal{O}$. This condition corresponds to the Griffith's transversality and nondegeneracy conditions of classical opers.

Definition 3.2 ([4]) A (G, P)**-oper** on X is a triple (E_G, E_P, ω) where E_G is a holomorphic principal G-bundle on X, E_P is a holomorphic reduction to the parabolic $P < G$, and ω is the holomorphic connection on E_G that satisfies $\mathrm{pos}_{E_P}(\omega) = \mathcal{O}$.

A morphism between two (G, P)-opers (E_G, E_P, ω) and (F_G, F_P, η) is an isomorphism of holomorphic principal G-bundles $\Phi : E_G \to F_G$ such that $\Phi|_{E_P} : E_P \simeq F_P$ and $\Phi^*\eta = \omega$.

Let $\mathbb{O}_{G,P}(X)$ denote the category of (G, P)-opers over X.

3.1 Parametrization theorem for even JM-parabolics

Let G be a complex semisimple Lie group, and fix an $\mathfrak{sl}_2$-triple $\mathfrak{s} = \langle f, h, e\rangle$ in the Lie algebra of G. Let

$$I = \{\alpha \in \Gamma \mid \alpha(h) = 0\}$$

be the subset of positive simple roots that interact with $\mathfrak{s}$ trivially. Then we call the parabolic subalgebra corresponding to I a **Jacobson–Morozov parabolic (JM-parabolic)** arising from $\mathfrak{s}$. In the following section, fix a JM-parabolic $\mathfrak{p}$ arising from an **even** $\mathfrak{sl}_2$-triple, which means $\alpha(h) \neq 1$ for all positive simple roots $\alpha \in \Gamma$.

Let $C < G$ be the centralizer subgroup of $\mathfrak{s}$ and $\mathfrak{c}$ be the centralizer subalgebra.

By Weyl's theorem, we may find an irreducible $\mathfrak{s}$-module decomposition of $\mathfrak{g}$. Let W_j denote the direct sum of all $(j+1)$-dimensional irreducible $\mathfrak{s}$-modules contained in $\mathfrak{g}$. In particular $W_0 = \mathfrak{c}$. Since $\mathfrak{s}$ is an even $\mathfrak{sl}_2$-triple, all the nontrivial $\mathfrak{s}$-modules are odd dimensional. Let $1 = m_1 < m_2 < \cdots < m_N$ be all the integers such that $W_{2m_j} \neq 0$. Therefore we have

$$\mathfrak{g} = \mathfrak{c} \oplus \bigoplus_{j=1}^{N} W_{2m_j}$$

as $\mathfrak{s}$-modules.

Let $\mathfrak{l}$ denote the Levi subalgebra of $\mathfrak{p}$, and let $Z_{2m_j} = \mathfrak{l} \cap W_{2m_j}$.

Definition 3.3 The $\mathfrak{s}$**-Slodowy category** $\mathcal{B}_{\mathfrak{s}}(G)$ is the set of objects $(E_C, \psi_0, \psi_1, \ldots, \psi_N)$ where (E_C, ψ_0) is a principal C-bundle on X with a holomorphic connection ψ_0, and $\psi_j \in H^0(X, E_C(Z_{2m_j}) \otimes K^{m_j+1})$. The morphisms are isomorphisms of holomorphic C-bundles that identify holomorphic sections ψ_j for all $1 \leq j \leq N$, where $N = \mathrm{rk}(G)$.

Definition 3.4 The **traceless quadratic** $\mathfrak{s}$**-Slodowy category** is the subcategory $\widehat{\mathcal{B}}_{\mathfrak{s}}(G)$ of $\mathcal{B}_{\mathfrak{s}}(G)$ whose objects are $(E_C, \psi_0, \psi_1, \ldots, \psi_N)$ with $\psi_1 = \widehat{\psi_1}$ under the decomposition $Z_2 = \langle h\rangle \oplus \widehat{Z}_2$.

Theorem 3.5 (cf [4, Theorem 5.8]) *Let G be a complex connected semisimple Lie group. Let $\mathfrak{s} \subset \mathfrak{g}$ be an even $\mathfrak{sl}_2$-triple and let $S < G$ be the connected subgroup with Lie algebra $\mathfrak{s}$. Let $P < G$ be the even Jacobson–Morozov parabolic associated to $\mathfrak{s}$. There exists a functor $F : \mathcal{P}(X) \times \widehat{\mathcal{B}}_{\mathfrak{s}}(G) \to \mathbb{O}_{G,P}(X)$. If $S \simeq \mathrm{PSL}(2, \mathbb{C})$, then F is an equivalence of categories; if $S \simeq \mathrm{SL}(2, \mathbb{C})$, then F is essentially surjective and full.*

There is some similarity between this parametrization theorem and the parametrization theorem for generalized B-opers (Theorem 1.5). In both cases, we extracted the data of a projective structure and used it to decompose the opers into a central component and a Hitchin base component. For (G, P)-opers, these are encapsulated in the definition of an $\mathfrak{s}$-Slodowy category, whereas for generalized B-opers, these components are more explicit spaces: the character variety $\mathcal{C}(X)$ for orthogonal bundles and half of the Hitchin base $\oplus_j H^0(X, K^{2j})$.

4 From B-opers to (G, P)-opers

Theorem 4.1 *To each generalized B-oper* (Definition 1.3)*, there is a naturally associated (G, P)-oper* (Definition 3.2)*.*

Proof Let $(E, \{F_i\}, D)$ be a generalized B-oper on the Riemann surface X. At each $x \in X$, let $(E_P)_x$ be the stabilizer of the flag $\{(F_i)_x\}$, which must be a parabolic subgroup P of either the symplectic group or the orthogonal group G. The 1-cocycles of E define holomorphic transition maps for E_P, so it is a holomorphic principal P-bundle. Let $E_G = E_P \times_P G$ be the associated principal G-bundle.

Since D is assumed to be a B-connection, locally it takes values in the Lie algebra $\mathfrak{g}$ of G as a $\mathbb{C}$-linear map. The second fundamental form S_D also takes values in $\mathfrak{g}$ as it is just a projection of D. Recall from Definition 3.1 that the parabolic subgroup P induces a Lie algebra grading on $\mathfrak{g}$ by the levels of root spaces, where $\mathfrak{g}^k$ is the direct sum of all root spaces of level k relative to $\mathfrak{p}$. We claim that D and S_D are both of degree -1 with respect to this grading.

Let $0 \subset V_1 \subset \cdots \subset V_n = V$ be a flag of complex vector spaces. Suppose B is a nondegenerate symplectic form when n is even and a nondegenerate symmetric form when n is odd and let $G = \mathrm{Sp}(V, B)$ and $G = \mathrm{SO}(V, B)$ respectively. Let $P < G$ be the stabilizer of the flag, which is the unique parabolic subgroup associated to the flag. Denote the vector space of degree k by $V^k = V_k/V_{k-1}$ and let $\mathrm{gr}(V) = \oplus_{k=1}^n V^k$ be a graded vector space. Let $\phi : \mathfrak{g} \to \mathrm{End}(V)$ be the representation of $\mathfrak{g}$ on V induced by the G-action.

Lemma 4.2 *If $\phi(X)$ is an endomorphism of degree 1 on* $\mathrm{gr}(V)$*, then $X \in \mathfrak{g}^{-1}$.*

By the definition of a generalized B-oper, $S_D(x)$ is a degree 1 endomorphism of the graded fiber space $\mathrm{gr}(E_x) = \oplus_{k=1}^n (E_k/E_{k-1})_x$. Lemma 4.2 tells us that it must take value in the unique open P-orbit in $\mathfrak{g}^{-1}$. Hence the connection D induces a principal connection ω that satisfies $\mathrm{pos}_{E_P}(\omega) = \mathcal{O}$. Furthermore the connection D from a generalized B-oper is holomorphic and flat, so (E_G, ω) is a holomorphic flat principal G-bundle. We have thus obtained a (G, P)-oper (E_G, E_P, ω) from the data of a generalized B-oper. □

Proof of Lemma 4.2 We first define a new flag for V using the $\mathfrak{p}$-invariant filtration of $\mathfrak{g}$. Let $\{\mathfrak{g}_k\}$ be the $\mathfrak{p}$-invariant filtration given in Section 3.1. For $1 \le k \le n$, let

$$W_k = \bigcap_{X \in \mathfrak{g}_k} \ker(\phi(X)).$$

We check that this is $\mathfrak{p}$-invariant. For any $X \in \mathfrak{g}_k$, $Y \in \mathfrak{p}$, and $v \in W_k$, we have

$$X \cdot (Y \cdot v) = [X, Y] \cdot v + Y \cdot (X \cdot v) = 0$$

since $[\mathfrak{g}_k, \mathfrak{p}] = [\mathfrak{g}_k, \mathfrak{g}_0] \subset \mathfrak{g}_k$. Hence $Y \cdot v \in W_k$. This flag is maximally $\mathfrak{p}$-invariant by construction. By the bijective correspondence between flags and parabolics in Proposition 5.1 we must have $\{V_k\} = \{W_k\}$ is the same flag. Therefore we may assume that for a nonzero vector $v \in V^k = V_k/V_{k-1}$, we have $X \cdot v = 0$ for all $X \in \mathfrak{g}^k$, and there exists $Y \in \mathfrak{g}^{k-1}$ such that $Y \cdot v \neq 0$.

Suppose $X \in \mathfrak{g}$ acts on $\mathrm{gr}(V)$ by degree 1, that is $X \cdot V^k \subseteq V^{k+1}$ for all $k = 1, 2, \ldots, n-1$. Let $v \in V^k$ be a nonzero vector. By assumption we have $X \cdot v$ is degree $k+1$, which means for all $Y \in \mathfrak{g}^{k+1}$ we have

$$0 = Y \cdot (X \cdot v) = [Y, X] \cdot v + X \cdot (Y \cdot v) = [Y, X] \cdot v.$$

Since v is nonzero degree k, we must have $[Y, X]$ has degree at least k or X has degree at least -1. Moreover there exists $Z \in \mathfrak{g}^k$ such that

$$0 \neq Z \cdot (X \cdot v) = [Z, X] \cdot v + X \cdot (Z \cdot v) = [Z, X] \cdot v.$$

Then $[Z, X]$ has degree at most $k-1$ or X has degree at most -1. Therefore X is of degree -1 and lives in $\mathfrak{g}^{-1}$. □

In the following example we give an explicit construction of a maximally $\mathfrak{p}$-invariant flag as shown in the proof of Lemma 4.2 for the group $G = \mathrm{Sp}(4, \mathbb{C})$ and two of its prabolic subgroups.

Example 4.3 Consider the Lie algebra $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C})$. Let $\varepsilon_i : \mathrm{diag}(h_1, h_2, -h_1, -h_2) \mapsto h_i$ be the weight vectors. The set of roots are $R = \{\pm\varepsilon_1 \pm \varepsilon_2, \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$ and the set of positive roots are $R_+ = \{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\}$. The positive simple roots are $\Gamma = \{\alpha = \varepsilon_1 - \varepsilon_2, \beta = 2\varepsilon_2\}$.

(1) The flag $0 \subset \mathbb{C}^2 \subset \mathbb{C}^4$ corresponds to the parabolic subgroup P defined by $I = \{\alpha = \varepsilon_1 - \varepsilon_2\}$.
The Levi subalgebra is given by the Cartan subalgebra and the root spaces of $R_I = \{\varepsilon_1 - \varepsilon_2, -(\varepsilon_1 - \varepsilon_2)\}$:

$$\mathfrak{l} = \mathfrak{h} \oplus \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array}\right) \oplus \left(\begin{array}{cc|cc} 0 & 0 & & \\ 1 & 0 & & \\ \hline & & 0 & -1 \\ & & 0 & 0 \end{array}\right).$$

The nilpotent subalgebra is given by $R_+ \setminus R_I = \{\varepsilon_1 + \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\}$:

$$\mathfrak{u} = \left(\begin{array}{c|cc} & 0 & 1 \\ & 1 & 0 \\ \hline & & \end{array}\right) \oplus \left(\begin{array}{c|cc} & 2 & 0 \\ & 0 & 0 \\ \hline & & \end{array}\right) \oplus \left(\begin{array}{c|cc} & 0 & 0 \\ & 0 & 2 \\ \hline & & \end{array}\right).$$

For each root in R, its level relative to P is defined to be the sum of coefficients in front of the positive simple roots $\Gamma \setminus I = \{\beta = 2\varepsilon_2\}$. We may define a flag of vector spaces using the level of root spaces relative to P.

Root	*Height*	*Levelrelativeto*P
$\alpha = \varepsilon_1 - \varepsilon_2$	1	0
$\beta = 2\varepsilon_2$	1	1
$-\alpha = -(\varepsilon_1 - \varepsilon_2)$	−1	0
$-\beta = -2\varepsilon_2$	−1	−1
$\alpha + \beta = \varepsilon_1 + \varepsilon_2$	2	1
$-(\alpha + \beta) = -(\varepsilon_1 + \varepsilon_2)$	−2	−1
$2\alpha + \beta = 2\varepsilon_1$	3	1
$-(2\alpha + \beta) = -2\varepsilon_1$	−3	−1

The level 1 roots are $2\varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1$, so the first invariant subspace is

$$\begin{aligned} V_1 &= \bigcap_{A \in \mathfrak{g}_{2\varepsilon_2} \cup \mathfrak{g}_{\varepsilon_1+\varepsilon_2} \cup \mathfrak{g}_{2\varepsilon_1}} \ker(A) \\ &= \ker\left(\begin{array}{c|cc} & 0 & 1 \\ & 1 & 0 \\ \hline & & \end{array}\right) \cap \ker\left(\begin{array}{c|cc} & 2 & 0 \\ & 0 & 0 \\ \hline & & \end{array}\right) \cap \ker\left(\begin{array}{c|cc} & 0 & 0 \\ & 0 & 2 \\ \hline & & \end{array}\right) \\ &= \langle e_1, e_2 \rangle \cap \langle e_1, e_2, e_4 \rangle \cap \langle e_1, e_2, e_3 \rangle \\ &= \langle e_1, e_2 \rangle = \mathbb{C}^2. \end{aligned}$$

There are no roots of level 2, so $V_2 = \ker(0) = \mathbb{C}^4$. Thus $0 \subset V_1 \subset V_2$ recovers the flag of $\mathbb{C}^4$ whose stabilizer is P.

(2) The flag $0 \subset \mathbb{C}^1 \subset \mathbb{C}^3 \subset \mathbb{C}^4$ corresponds to the parabolic subgroup Q defined by $I = \{\beta = 2\varepsilon_2\}$.
The Levi subalgebra is given by the Cartan subalgebra and the root spaces of $R_I = \{2\varepsilon_2, -2\varepsilon_2\}$:

$$\mathfrak{l} = \mathfrak{h} \oplus \left(\begin{array}{c|cc} & 0 & 0 \\ & 0 & 2 \\ \hline & & \end{array}\right) \oplus \left(\begin{array}{cc|c} & & \\ \hline 0 & 0 & \\ 0 & 2 & \end{array}\right).$$

The nilpotent subalgebra is given by $R_+ \setminus R_I = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1\}$:

$$\mathfrak{u} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array}\right) \oplus \left(\begin{array}{c|cc} & 0 & 1 \\ & 1 & 0 \\ \hline & & \end{array}\right) \oplus \left(\begin{array}{c|cc} & 2 & 0 \\ & 0 & 0 \\ \hline & & \end{array}\right).$$

Root	*Height*	*Levelrelativeto*Q
$\alpha = \varepsilon_1 - \varepsilon_2$	1	1
$\beta = 2\varepsilon_2$	1	0
$-\alpha = -(\varepsilon_1 - \varepsilon_2)$	−1	−1
$-\beta = -2\varepsilon_2$	−1	0
$\alpha + \beta = \varepsilon_1 + \varepsilon_2$	2	1
$-(\alpha + \beta) = -(\varepsilon_1 + \varepsilon_2)$	−2	−1
$2\alpha + \beta = 2\varepsilon_1$	3	2
$-(2\alpha + \beta) = -2\varepsilon_1$	−3	−2

For each root in R, its level relative to Q is defined to be the sum of coefficients in front of the positive simple roots $\Gamma \setminus I = \{\alpha = \varepsilon_1 - \varepsilon_2\}$. We may define a flag of vector spaces using the level of root spaces relative to Q. The level 1 roots are $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 + \varepsilon_2$, so the first invariant subspace is

$$\begin{aligned} V_1 &= \bigcap_{A \in \mathfrak{g}_{\varepsilon_1-\varepsilon_2} \cup \mathfrak{g}_{\varepsilon_1+\varepsilon_2}} \ker(A) \\ &= \ker \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array} \right) \cap \ker \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & 1 & 0 \\ \hline & & & \\ & & & \end{array} \right) \\ &= \langle e_1, e_4 \rangle \cap \langle e_1, e_2 \rangle \\ &= \langle e_1 \rangle = \mathbb{C}^1. \end{aligned}$$

There is one level 2 root $2\varepsilon_1$, which defines the Q-invariant subspace

$$V_2 = \ker \left(\begin{array}{cc|cc} & & 2 & 0 \\ & & 0 & 0 \\ \hline & & & \\ & & & \end{array} \right) = \langle e_1, e_2, e_4 \rangle = \mathbb{C}^3$$

Finally $V_3 = \ker(0) = \mathbb{C}^4$ since there are no roots of level 3, and $0 \subset V_1 \subset V_2 \subset V_3$ recovers the flag of $\mathbb{C}^4$ whose stabilizer is Q.

5 From (G, P)-opers to generalized opers

We consider two cases of (G, P)-opers, for $G = \mathrm{Sp}(2n, \mathbb{C})$ and $G = \mathrm{SO}(n, \mathbb{C})$, and see if they give rise to generalized B-opers.

Let (E_G, E_P, ω) be a (G, P)-oper. Let E be the associated holomorphic vector bundle of the holomorphic principal G-bundle E_G, which is a frame bundle with the additional structure of a holomorphic nondegenerate bilinear form B. Depending on the structure group, B is either an invariant symplectic form or an invariant symmetric form.

The holomorphic connection form ω on the principal G-bundle determines a holomorphic connection D on E. We may define D locally. Let (U, ϕ) be a trivializing neighbourhood of E_G. Then there is a canonical section $\sigma_e : U \to E_G|_U$, given by $\phi \circ \sigma_e(x) = (x, e)$. Let $A = \sigma_e^* \omega|_U$, which is then a holomorphic connection matrix in $\Omega^{1,0}(U, \mathfrak{g})$. Now $D|_U = d + A$ is a holomorphic connection form on E.

Finally the parabolic P defines a P-invariant filtration $\{F_i\}$ of E. In order for $(E, \{F_i\}, D)$ to be a generalized B-oper, the filtration coming from P must have associated graded bundles of the same rank. Moreover the filtration needs to have even length when $G = \mathrm{Sp}(2n, \mathbb{C})$ and odd length when $G = \mathrm{SO}(n, \mathbb{C})$. This suggests that not all (G, P)-opers correspond to a generalized B-oper.

5.1 Parabolics and isotropic flags

Let (V, B) be a complex vector space with a nondegenerate bilinear form. There are two cases to consider. In the first case $\dim V = 2n$ is even, we assume there is an action from $G = \mathrm{Sp}(2n, \mathbb{C})$ on V, and B is an invariant

symplectic form. In the second case we assume there is an action from $G = \mathrm{SO}(n, \mathbb{C})$, and B is an invariant symmetric form.

An **isotropic flag** of vector spaces in V is a strictly increasing flag

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_l$$

such that each F_i is a nonzero isotropic subspace of V, that is $F_i \subset F_i^{\perp}$ for the orthogonal complement taken with respect to B.

The **signature** (or type) of a flag is a sequence of numbers given by the dimensions of the subspaces in the flag. For a flag $F = \{0 \subset F_1 \subset F_2 \subset \cdots \subset F_l = V\}$ of V with $\dim F_i = d_i$, the signature of F is $(d_1, d_2, \ldots, d_l)$.

Proposition 5.1 ([5]) *Let $n > 1$ be an integer. There is a bijection between the parabolic subgroups of $G = \mathrm{Sp}(2n, \mathbb{C})$ or $G = \mathrm{SO}(n, \mathbb{C})$ and the isotropic flags of V with respect to the bilinear form B.*

Proof Suppose we fix a basis for V and identify it with $\mathbb{C}^m$. Without the structure of a symplectic form or symmetric form, there is a natural action by $\mathrm{GL}(m, \mathbb{C})$, whose parabolic subgroups are the stabilizers of partial flags of $\mathbb{C}^m$. Restricted to the symplectic or orthogonal subgroup, this means their parabolics also arise as the stabilizers of partial flags. We may identify the Borel subgroup as the stabilizer of a maximal isotropic flag of V, so any stabilizer of an isotropic flag contains the Borel subgroup. Next we show that each isotropic flag has a unique stabilizer which is a parabolic subgroup. Suppose F and F' are two isotropic flags with the same signature, and their stabilizer subgroups are $P_F = P_{F'}$. There is $g \in G$ such that $g(F) = F'$, so $g P_F g^{-1} = P_{F'}$ implies $g \in N_G(P_F)$ is in the normalizer. But parabolic subgroups coincide with their normalizers, so $g \in P_F$ implies $F' = g(F) = F$. Furthermore there are exactly 2^n parabolic subgroups, each corresponding to a subset of the positive simple roots of G. Since there are also exactly 2^n signatures for isotropic flags, their stabilizers exhaust all the parabolic subgroups. □

Remark 5.2 Unlike the type A_n case for $\mathfrak{sl}(n, \mathbb{C})$, it's sufficient to specify half of the flag of isotropic subspaces to identify a parabolic subgroup in types C_n, B_n and D_n, because the symplectic or orthogonal form fully determines the other half of nonisotropic subspaces of the flag.

5.2 Not all (G, P) opers give rise to generalized opers

Theorem 5.3 *There exist some (G, P)-opers which do not arise as the associated oper of a generalized B-oper or a generalized $\mathrm{SO}(2n, \mathbb{C})$-oper.*

Proof From the data of any (G, P)-oper, we may obtain a filtration of P-invariant vector bundles. Fiberwise we have P-invariant flags of vector spaces. It suffices to show that there exist P-invariant flags that do not correspond to any B-filtration (Definition 1.2) or filtered $\mathrm{SO}(2n, \mathbb{C})$-bundle (Definition 2.1). Hence the theorem is proved by the following counterexamples.

Example 5.4 Consider the following examples for the symplectic group $G = \mathrm{Sp}(2n, \mathbb{C})$. Note that a B-filtration must correspond to a parabolic subgroup of signature $(r, 2r, \ldots, lr)$, where $lr = n$.

(1) Let $G = \mathrm{Sp}(4, \mathbb{C})$. Since each conjugacy class of parabolic is determined by an isotropic flag, we know there are distinct parabolics in G that stabilize the following isotropic flags in $\mathbb{C}^4$

$$0 \qquad 0 \subset \mathbb{C}^1 \qquad 0 \subset \mathbb{C}^2 \qquad 0 \subset \mathbb{C}^1 \subset \mathbb{C}^2.$$

Since F is P-invariant implies $F^{\perp}$ is P-invariant, for each of the above isotropic flags, we may add the symplectic complements to obtain a maximally P-invariant flag of $\mathbb{C}^4$

$$0 \subset \mathbb{C}^4 \qquad\qquad 0 \subset \mathbb{C}^1 \subset \mathbb{C}^3 \subset \mathbb{C}^4$$
$$0 \subset \mathbb{C}^2 \subset \mathbb{C}^4 \qquad\qquad 0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3 \subset \mathbb{C}^4.$$

The first two flags do not correspond to a B-filtration because their length is not even, which is required for a symplectic form. On top of that, the second flag does not have a constant dimension of associated graded spaces. Finally, the last two flags do give rise to B-filtrations.

Take P to be the stabilizer of the flag $0 \subset \mathbb{C}^1 \subset \mathbb{C}^3 \subset \mathbb{C}^4$. It is a Klingen parabolic of $\mathrm{Sp}(4, \mathbb{C})$, which is

maximal, proper, and corresponds to the subset of positive simple roots with the unique short root removed. Using the standard basis, suppose P is the stabilizer of

$$0 \subset \mathbb{C}\langle e_1\rangle \subset \mathbb{C}\langle e_1, e_2, e_4\rangle \subset \mathbb{C}\langle e_1, e_2, e_3, e_4\rangle.$$

We can describe the Lie algebra of P explicitly. Recall $\mathfrak{sp}(4,\mathbb{C}) = \{X \in \mathfrak{gl}(4,\mathbb{C}) \mid JX + X^T J = 0\}$ for $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ the standard symplectic form. Its Cartan subalgebra is a set of diagonal matrices $\mathfrak{h} = \{\mathrm{diag}(h_1, h_2, -h_1, -h_2)\}$. For $i = 1, 2$, let $\varepsilon_i \in \mathfrak{h}^*$ be the linear functional that outputs the i'th diagonal entry. So $\{\varepsilon_1, \varepsilon_2\}$ form a basis for $\mathfrak{h}^*$. The standard root system for $\mathfrak{sp}(4,\mathbb{C})$ is $R = \{\pm\varepsilon_1 \pm \varepsilon_2, \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$. The positive roots are $R_+ = \{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2\}$ and the positive simple roots are $\Gamma = \{\varepsilon_1 - \varepsilon_2, 2\varepsilon_2\}$, which are compatible with the choice of P.

For each root α, we describe its root space $\mathfrak{g}_\alpha \subset \mathfrak{sp}(4,\mathbb{C})$. Let $E_{ij} \in \mathfrak{gl}(4,\mathbb{C})$ be the matrix unit with only one nonzero entry 1 in the ij'th position. The $2n$ complex symplectic Lie algebra consists of Hamiltonian matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, B = B^T, C = C^T,$$

where A, B, C are $n \times n$ complex matrices.

- When $\alpha = \varepsilon_i - \varepsilon_j$, $(i \neq j)$, the root space is $\mathfrak{g}_\alpha = \mathbb{C}\langle E_{ij} - E_{n+j,n+i}\rangle$. This corresponds to the subspace of matrices generated by A being the ij'th matrix unit in $\mathfrak{gl}(n,\mathbb{C})$ and $B = C = 0$.
- When $\alpha = \varepsilon_i + \varepsilon_j$, the root space is $\mathfrak{g}_\alpha = \mathbb{C}\langle E_{i,n+j} + E_{j,n+i}\rangle$. This space is generated by B being the symmetric matrix with nonzero ij'th and ji'th entries, and $A = C = 0$.
- When $\alpha = -(\varepsilon_i + \varepsilon_j)$, the root space is $\mathfrak{g}_\alpha = \mathbb{C}\langle E_{n+i,j} + E_{n+j,i}\rangle$. This space is generated by C being the symmetric matrix with nonzero ij'th and ji'th entries, and $A = B = 0$.

The subset $I = \{2\varepsilon_2\} \subset \Gamma$ obtained by removing the short root $\varepsilon_1 - \varepsilon_2$ from the base gives the parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$. As discussed previously, the Levi subalgebra $\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_I} \mathfrak{g}_\alpha$, for $R_I = \{\pm 2\varepsilon_2\}$, consists of matrices of the form

$$\left(\begin{array}{cc|cc} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & a \\ \hline 0 & 0 & -h_1 & 0 \\ 0 & b & 0 & -h_2 \end{array}\right).$$

The nilpotent subalgebra $\mathfrak{u} = \bigoplus_{\alpha \in R_+ \setminus R_I} \mathfrak{g}_\alpha$, for $R_+ \setminus R_I = \{\varepsilon_1 \pm \varepsilon_2, 2\varepsilon_1\}$, consists of matrices of the form

$$\left(\begin{array}{cc|cc} 0 & a & c & b \\ 0 & 0 & b & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \end{array}\right).$$

(2) In general for any $n > 1$ let (V, B) be a symplectic vector space with an action from $G = \mathrm{Sp}(2n,\mathbb{C})$. We may take any one dimensional isotropic subspace $V_1 \subset V$ and let P be the stabilizer of the flag

$$0 \subset V_1 \subset V_{2n-1} \subset V_{2n}.$$

The flag has an odd length and nonconstant associated graded dimensions, so it does not correspond to any B-filtration.

Example 5.5 For $n > 4$ and let (V, B) be an orthogonal vector space with an action from $G = \mathrm{SO}(n,\mathbb{C})$. We may also take a one dimensional isotropic subspace $V_1 \subset V$ and let P be the stabilizer corresponding to the partial flag

$$0 \subset V_1 \subset V_{n-1} \subset V_n.$$

The associated graded spaces have dimensions 1 and $n - 2 > 2$, which do not correspond to any B-filtration. If $n = 2k$, we may ask if there is a generalized $\mathrm{SO}(2k,\mathbb{C})$-oper corresponding to the parabolic. But this is not the case because for $r = 1$, the filtration length of a generalized $\mathrm{SO}(2k,\mathbb{C})$-oper is $n - 1 > 3$ (Example 2.5).

□

References

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