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ORIGINAL RESEARCH

Structure of a quotient ring R/P with generalized derivations acting on the prime ideal P and some applications

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Abstract The main purpose of this paper is to develop a new approach that consists to investigate the structure of a quotient ring R/P via action of generalized derivations on the prime ideal P. In this direction, we initiate new classes of additive mappings extending both of centralizing and commuting mappings. Furthermore, for an arbitrary ring, we will consider algebraic identities based on its prime ideals. As an application, we obtain some results concerning invariance of minimal prime ideals of a semiprime ring under derivations.

Keywords Prime ideal · Generalized derivations · Quotient ring · Commutativity

MSC Classification 16U80 · 16D25

1 Introduction

Throughout, R denotes an associative ring with center $Z(R)$. Recall that an ideal P of R is said to be *prime* if $P \neq R$ and for $x, y \in R$, $xRy \subseteq P$ implies that $x \in P$ or $y \in P$. A prime ideal P of R is minimal if P does not properly include any prime ideals of R. Therefore, R is called a *prime ring* if and only if (0) is the only minimal prime ideal of R. The Lie product of two elements x and y of R is $[x, y] = xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. A map $d : R \to R$ is a *derivation* of a ring R if d is additive and satisfies the Leibnitz'rule; $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

Over the last years, several authors have investigated the relationship between the commutativity, the structure of the ring R and certain special types of maps on R (viz.; [6], [20], [13]). Herstein [14] showed that a prime ring R with nonzero derivation d satisfying $d(x)d(y) = d(y)d(x)$ for all $x, y \in R$, must be a commutative integral domain if its characteristic is not two, and, if the characteristic equals two, must be commutative or an order in a simple algebra which is 4-dimensional over its center. Several authors have proved commutativity theorems for prime rings admitting derivations which are centralizing on R. We first recall that a mapping $f : R \to R$ is called *centralizing* (on R) if $[f(x), x] \in Z(R)$ for all $x \in R$; in the special case where $[f(x), x] = 0$ for all

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$x \in R$, the mapping f is said to be *commuting* on R. In [19], Posner proved that if a prime ring R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. Since then many authors have extended the Posner's result in several directions. A considerable number of researchers have investigated the ideals (Jordan ideal, Lie ideal...) in rings as well as the commutativity of rings that consider derivations and generalized derivations. For example, in [15] Lanski generalizes the result of Posner by considering a derivation d such that $[d(x), x] \in Z(R)$ for all x in a nonzero Lie ideal U of R.

One may observe that the concept of generalized derivations includes the concept of derivations. Hence it should be interesting to extend some results concerning these notions to generalized derivations. More specifically an additive map $F : R \longrightarrow R$ is said to be a *generalized derivation* if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Generalized derivations have been primarily studied on operator algebras. Thus any investigation from the algebraic point of view might be interesting (see for example [9], [12], [16] and [18]).

In the present paper we will adopt a new approach by considering algebraic identities involving a prime ideal without the primeness (semiprimeness) assumption on the considered ring. Furthermore, as an application of our results, we will study the problem of invariance of minimal prime ideals of a semiprime ring under derivations.

P-Centralizing and P-Commuting maps

The motivation for this section lies in an attempt to extend in some way the well known first Posner's Theorem contained in [19]: there Posner proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Moreover, this result of Posner has been extended by several authors in various directions.

Our purpose here is to continue this line of investigation by considering a class of additive mappings whose concept covers the ones of centralizing and commuting mappings. More precisely, motivated by the notion of centralizing and commuting mappings, we initiate the concepts of I-centralizing and I-commuting mappings where I is an ideal of the considered ring as follows.

Definition 1 Let I be an ideal of a ring R.

1. A map $f : R \to R$ is I-centralizing if $\overline{[f(x), x]} \in Z(R/I)$ for all $x \in R$ or equivalently $[[f(x), x], y] \in I$ for all $x, y \in R$.
2. A map $f : R \to R$ is I-commuting if $[f(x), x] \in I$ for all $x \in R$.

We start with the following observations:

Remark 1 1. A map f is (0)-centralizing (resp. (0)-commuting) if and only if f is centralizing (resp. commuting).
2. if a map f is commuting then f is I-commuting.
3. if a map f is centralizing then f is I-centralizing.

The fundamental aim of the following example is to show that these new notions does not coincide with the classical concepts of centralizing and commuting mappings.

Example 1 Let R be a non-commutative prime ring, I be a nonzero ideal of R and $d(x) = ax - xa$ for all $x \in R$ where a is an element of $I \setminus Z(R)$.

1. d is I-commuting but non-commuting.
2. d is I-centralizing but non-centralizing.

Our goal in the following theorem is to confirm that there is a relationship between the structure of the ring R/P and generalized derivations of R that are P-centralizing, where P is a prime ideal of R.

Theorem 1 *Let P be a prime ideal of a ring R and F be a generalized derivation of R associated with a derivation d. If F is P-centralizing then $d(R) \subseteq P$ or R/P is a commutative integral domain.*

Proof We are given that

$$\overline{[F(x), x]} \in Z(R/P) \quad \text{for all } x \in R. \tag{1}$$

If $Z(R/P) \neq \{\overline{0}\}$, then there exists $\overline{z_0} \in Z(R/P)$ and $z_0 \notin P$. A linearization of the above relation leads to

$$\overline{[F(x+z_0), x+z_0]} = [\overline{F(x)}, \overline{z_0}] + [\overline{F(z_0)}, \overline{x}] \in Z(R/P) \quad \text{for all } x \in R \tag{2}$$

thereby obtaining

$$[\overline{F(z_0)}, \overline{x}] \in Z(R/P) \quad \text{for all } x \in R. \tag{3}$$

If we replace x by $xF(z_0)$, the primeness of R/P assures that

$$\overline{F(z_0)} \in Z(R/P).$$

Replacing x by z_0x in (1), we get

$$[\overline{F(z_0)x}, \overline{z_0x}] + \overline{[z_0d(x), z_0x]} \in Z(R/P) \quad \text{for all } x \in R \tag{4}$$

and therefore

$$\overline{[d(x), x]}\ \overline{z_0^2} \in Z(R/P) \quad \text{for all } x \in R. \tag{5}$$

Since $\overline{z_0^2} \neq \overline{0}$, because of $z_0 \notin P$, it follows that $[[d(x), x], y] \in P$ for all $x \in R$. Invoking ([2], Theorem 2.2), then we conclude that $d(R) \subseteq P$ or R/P is commutative.

Now assume that $Z(R/P) = \{\overline{0}\}$, thus our hypothesis reduces to

$$[F(x), x] \in P \quad \text{for all } x \in R. \tag{6}$$

Linearizing equation (6), we obviously get

$$[F(x), y] + [F(y), x] \in P \quad \text{for all } x, y \in R. \tag{7}$$

Replacing y by yx in (7), one can easily verify that

$$y[d(x), x] + [y, x]d(x) \in P \quad \text{for all } x, y \in R. \tag{8}$$

Taking $y = ry$ in (8), we have

$$ry[d(x), x] + r[y, x]d(x) + [r, x]yd(x) \in P \quad \text{for all } x, y, r \in R. \tag{9}$$

Right multiplying (8) by r and comparing with (9) we may write

$$[r, x]yd(x) \in P \quad \text{for all } x, y, r \in R. \tag{10}$$

Since P is prime it follows that for all $x \in R$ either $[r, x] \in P$ or $d(x) \in P$. Consequently, R is a union of two additive subgroups G_1 and G_2, where

$$G_1 = \{x \in R \,/\, d(x) \in P\} \text{ and } G_2 = \{x \in R \,/\, [R, x] \subset P\}.$$

Since a group cannot be a union of two of its proper subgroups, we are forced to conclude that $R = G_1$ or $R = G_2$. Therefore, $d(R) \subset P$ or R/P is a commutative integral domain. □

In ([1], Theorem 3.7) it is proved that if a prime ring R admits a nonzero generalized derivation F associated with a nonzero derivation d such that F is centralizing, then R is commutative. With the further assumption that the ring is 2-torsion free, Oukhtite and Mamouni in [18] obtained the commutativity of the ring by considering a local condition. More precisely, they proved that if the generalized derivation is assumed to be centralizing on a nonzero Jordan ideal J rather that on the wall ring, then R is a commutative integral domain.

As an application of Theorem 1, we get the followings results with no further assumption on the characteristic of R.

Corollary 1 *[1, Theorem 3.7] Let R be a prime ring. If R admits a generalized derivation F associated with a nonzero derivation d such that $[F(x), x] \in Z(R)$ for all $x \in R$. Then R is a commutative integral domain.*

Corollary 2 *[19, Theorem 2.] Let R be a prime ring. If R admits a nonzero derivation d such that $[d(x), x] \in Z(R)$ for all $x \in R$. Then R is a commutative integral domain.*

Algebraic identities involving prime ideal

In [3], Bell and Daif investigated commutativity in rings admitting a derivation d strong commutativity preserving (SCP) on a nonzero right ideal. More precisely, they proved that if a semi-prime ring R admits a derivation d satisfying $[d(x), d(y)] - [x, y] = 0$ for all $x, y \in R$, then R is commutative.

Motivated by the above result we investigate a more general context of differential identities involving a prime ideal by omitting the semi-primeness assumption imposed on the ring. This approach allows us to generalize the preceding result of Bell and Daif in two directions. First of all, we will assume that $[d(x), d(y)] - [x, y]$ belong to a prime ideal P rather than $[d(x), d(y)] - [x, y] = 0$. Secondly we will investigate the behaviour of the more general expression $[F(x), d(y)] - [x, y]$ involving a generalized derivation instead of the expression $[d(x), d(y)] - [x, y]$.

Theorem 2 *Let R be a ring and P a prime ideal of R. If R admits a generalized derivation F associated with a derivation d, then the following conditions are equivalent:*

1) $[F(x), d(y)] + [x, y] \in P$ *for all* $x, y \in R$;
2) $[F(x), d(y)] - [x, y] \in P$ *for all* $x, y \in R$;
3) R/P *is a commutative integral domain.*

Proof It's obvious that (3) implies both of (1) and (2).

(1) $\Rightarrow$ (3) Assume that

$$[F(x), d(y)] + [x, y] \in P \quad \text{for all } x, y \in R. \tag{11}$$

If $d(R) \subseteq P$, then R/P is commutative so we may suppose that $d(R) \nsubseteq P$. Substituting ry for y in (11) and using (11), we obtain

$$d(r)[F(x), y] + [F(x), r]d(y) \in P \quad \text{for all } x, y, r \in R. \tag{12}$$

Replacing r by $d(x)r$ in (12), we get

$$d^2(x)r[F(x), y] + d(x)d(r)[F(x), y] + d(x)[F(x), r]d(y) \in P. \tag{13}$$

Left multiplying (12) by $d(x)$ and comparing with equation (13) we find that

$$d^2(x)r[F(x), y] \in P \quad \text{for all } x, y, r \in R.$$

In light of primeness of P, the last relation yields that

$$[F(x), y] \in P \quad \text{or} \quad d^2(x) \in P \quad \text{for all } x, y \in R. \tag{14}$$

Let us set $R_1 = \{x \in R \,/\, [F(x), R] \subseteq P\}$ and $R_2 = \{x \in R \,/\, d^2(x) \in P\}$. Clearly R_1 and R_2 are additive subgroups of R such that $R = R_1 \cup R_2$. According to Brauer's trick, it follows that either $R = R_1$ or $R = R_2$.

Assume that $R = R_1$, then $[F(x), y] \in P$ for all $x, y \in R$ and therefore

$$[F(x), x] \in P \quad \text{for all } x \in R. \tag{15}$$

Since equation (15) is the same as equation (6), reasoning in the same manner as in the proof of Theorem 1, we get the required result.

Assume now that $d^2(R) \subseteq P$; replacing y by $d(x)$ in (11) we obviously get

$$[d(x), x] \in P \quad \text{for all } x \in R.$$

Since d is a generalized derivation associated with d, hence the last equation is exactly equation (15) for the particular generalized derivation $\mathcal{F} = d$. Arguing as above, we conclude that R/P is a commutative integral domain.

(2) $\Rightarrow$ (3) Follows from the first implication with a slight modification. □

Let R be a prime ring. Letting $P = (0)$ in the previous theorem, we deduce that, if $[F(x), d(y)] + [x, y] = 0$ (resp. $[F(x), d(y)] - [x, y] = 0$) for all $x, y \in R$, then R is commutative. The following corollary shows that the same conclusion remains satisfied for semi-prime rings.

Corollary 3 *Let R be a semiprime ring and F a generalized derivation associated with a derivation d. Then following assertions are equivalent:*

1) $[F(x), d(y)] + [x, y] = 0$ *for all* $x, y \in R$;
2) $[F(x), d(y)] - [x, y] = 0$ *for all* $x, y \in R$;
3) R *is a commutative reduced ring.*

Proof For the non-trivial implication assume that $[F(x), d(y)] + [x, y] = 0$ for all $x, y \in R$. According to semiprimeness, there exists a family Γ of prime ideals such that $\bigcap_{P\in\Gamma} P = (0)$ thereby obtaining $[F(x), d(y)] + [x, y] \in P$ for all $P \in \Gamma$. Invoking Theorem 2, we conclude that R/P is a commutative integral domain which, because of $\bigcap_{P\in\Gamma} P = (0)$, assures that R is commutative. Similarly, if $[F(x), d(y)] - [x, y] = 0$ for all $x, y \in R$, then the same reasoning proves the required result. □

The aim of the next theorem is to study the case when the generalized derivation satisfies some conditions involving commutators and anti-commutators. More specifically, we will treat the special identities

$$\Big(F(x) \circ d(y) - x \circ y \in P \text{ for all } x, y \in R\Big) \text{ and } \Big(F(x) \circ d(y) + x \circ y \in P \text{ for all } x, y \in R\Big).$$

Our results are of a different kind because we will characterize not only the structure of the ring R/P but we will also determine its characteristic. More precisely we will prove the following result.

Theorem 3 *Let R be a ring, P a prime ideal of R and F a generalized derivation of R associated with a derivation d. If F satisfies one of the following conditions:*

1) $F(x) \circ d(y) + x \circ y \in P$ *for all* $x, y \in R$;
2) $F(x) \circ d(y) - x \circ y \in P$ *for all* $x, y \in R$;
3) $[F(x), d(y)] + x \circ y \in P$ *for all* $x, y \in R$;
4) $[F(x), d(y)] - x \circ y \in P$ *for all* $x, y \in R$; *then R/P is a commutative integral domain and* $char(R/P) = 2$.

Proof (1) We are given that

$$F(x) \circ d(y) + x \circ y \in P \quad \text{for all } x, y \in R. \tag{16}$$

Replacing x by $xd(y)$ in (16), we obtain

$$xd^2(y)d(y) + d(y)xd^2(y) + x[d(y), y] \in P \quad \text{for all } x, y \in R. \tag{17}$$

As a special case of (17) when $x = rx$, we may write

$$rxd^2(y)d(y) + d(y)rxd^2(y) + rx[d(y), y] \in P \quad \text{for all } x, y, r \in R. \tag{18}$$

Left multiplying (17) by r and comparing with (18), we conclude that

$$[d(y), r]xd^2(y) \in P \quad \text{for all } x, y, r \in R$$

and by the primeness of P either $[d(y), r] \in P$ for all $y, r \in R$ or $d^2(R) \subseteq P$.

In the first case, it's obvious that $[[d(x), x], y] \in P$ for all $x, y \in R$ proving that d is P-centralizing. Since d is a generalized derivation associated with d itself, by virtue of Theorem 1 either $d(R) \subset P$ or R/P is an integral domain.

Now assume that $d^2(R) \subseteq P$; substituting $d(y)$ for y in (16), we easily obtain

$$x \circ d(y) \in P \quad \text{for all } x, y \in R. \tag{19}$$

Writing zrx instead of x in the last relation, one can obviously verify that $zr[x, d(y)] + ((zr) \circ d(y))x \in P$ which, in light of (19), yields that $zR[x, d(y)] \subset P$. In particular, we may write

$$[x, d(y)]R[x, d(y)] \in P \quad \text{for all } x, y \in R.$$

Since P is prime, it follows that $[x, d(y)] \in P$ for all $x, y \in R$. Arguing as above, we have either $d(R) \subseteq P$ or R/P is a commutative integral domain.

Suppose $d(R) \subseteq P$; then Eq.(16) reduces to $x \circ y \in P$ for all $x, y \in R$. Replacing y by yr we find that $[x, y]r \in P$ and using the fact that a prime ideal is proper, we conclude that $[R, R] \subset P$ so that R/P is commutative. Consequently, in both cases R/P is a commutative integral domain.

Finally, we claim that R/P is with characteristic 2. Indeed, assume that R/P is 2-torsion free. As we have already seen that R/P is commutative then Eq.(16) assures that $2\big(F(x)d(y) + xy\big) \in P$ for all $x, y \in R$. If we replace y by ry, we get $F(x)rd(y) \in P$ for all $x, y, r \in R$. Accordingly, $d(R) \subseteq P$. Once again using 2-torsion freeness together with equation (16), we find that $xy \in P$ for all $x, y \in R$ and thus $P = R$, a contradiction. Consequently, R/P is an integral domain with characteristic 2.

(2) If $F(x) \circ d(y) - x \circ y \in P$ for all $x, y \in R$, then using similar argument as in the first case with a suitable slight modification, we get the required result.

(3) Assume that

$$[F(x), d(y)] + x \circ y \in P \quad \text{for all } x, y \in R. \tag{20}$$

Substituting $xd(y)$ for x in (20), we obtain

$$xd^2(y)d(y) + x[d(y), y] - d(y)xd^2(y) \in P \quad \text{for all } x, y \in R. \tag{21}$$

In particular, setting $x = rx$ in (21) we have

$$rxd^2(y)d(y) + rx[d(y), y] - d(y)rxd^2(y) \in P \quad \text{for all } x, y, r \in R. \tag{22}$$

Right multiplying (21) by r and combining with (22), we obviously get

$$[d(y), r]xd^2(y) \in P \quad \text{for all } r, x, y \in R. \tag{23}$$

Invoking the primeness of P, the last relation assures that either $[d(y), r] \in P$ for all $y, r \in R$ or $d^2(R) \subseteq P$. By an argument similar to that in the proof of (1), we again conclude that either $d^2(R) \subseteq P$ or R/P is commutative.

Assume that $d^2(R) \subseteq P$; replacing y by $d(y)$ in (20) we obtain

$$x \circ d(y) \in P \quad \text{for all } x, y \in R$$

this relation is exactly (19), then as above R/P is commutative or $d(R) \subseteq P$. In the later case, our hypothesis reduces to $x \circ y \in P$ for all $x, y \in R$ therefore in both cases R/P is a commutative integral domain.

Suppose that R/P is a 2-torsion free prime ring; as we have already seen that R/P is commutative then our hypothesis becomes $x \circ y \in P$ for all $x, y \in R$ and therefore $P = R$, contradicting the fact that a prime ideal is proper. Consequently, R/P is necessarily with characteristic 2.

(4) Assume that $[F(x), d(y)] - x \circ y \in P$ for all $x, y \in R$, then using the same techniques as used above with necessary variations we get the required result. □

In ([11], Proposition 1.) M. Hongan proved that if R is a 2-torsion free ring with identity 1, then there is no derivation d of R such that $d(x \circ y) + x \circ y = 0$ for all $x, y \in R$ or $d(x \circ y) - x \circ y = 0$ for all $x, y \in R$. Applying Theorem 3, we get a similar result as follows:

Corollary 4 *Let R be a 2-torsion free ring which is either a semiprime or a unitary ring. There is no generalized derivation F associated with a derivation d satisfying one of the following conditions:*

1) $F(x) \circ d(y) + x \circ y = 0$ *for all* $x, y \in R$;
2) $F(x) \circ d(y) - x \circ y = 0$ *for all* $x, y \in R$;
3) $[F(x), d(y)] + x \circ y = 0$ *for all* $x, y \in R$;
4) $[F(x), d(y)] - x \circ y = 0$ *for all* $x, y \in R$.

Proof Assume that R is a 2-torsion free semiprime ring. Then there exists a family Γ of prime ideals such that $\bigcap_{P \in \Gamma} P = (0)$. Since the proof is similar and does not depend on the chosen identity we may assume existence of a generalized derivation (F, d) satisfying 1) that is $F(x) \circ d(y) + x \circ y = 0$ for all $x, y \in R$. Accordingly, $F(x) \circ d(y) + x \circ y \in P$ for all $P \in \Gamma$ and Theorem 3 yields that R/P is commutative. Therefore, for all $x, y \in R$ we have $[x, y] \in P$ (for all $P \in \Gamma$) so that $[x, y] = 0$ proving that R is commutative. Hence our hypothesis reduces to $F(x)d(y) + xy = 0$ for all $x, y \in R$. Replacing x by xy in the above relation, we find that $d(y)xd(y) = 0$ for all $x, y \in R$ and the semiprimeness forces $d = 0$. Consequently, $xy = 0$ for all $x, y \in R$ so that $R = \{0\}$, a contradiction. In the case R is an arbitrary 2-torsion free unitary ring, setting $y = 1$ our hypothesis reduces to $x = 0$ for all $x \in R$, a contradiction. □

Theorem 4 *Let R be a ring, P a prime ideal and F a generalized derivation associated with a derivation d. If F satisfies one of the following conditions:*

1) $F(x) \circ d(y) + [x, y] \in P$ *for all* $x, y \in R$;
2) $F(x) \circ d(y) - [x, y] \in P$ *for all* $x, y \in R$; *then R/P is a commutative integral domain. Moreover, if R/P is 2-torsion free then $d(R) \subseteq P$.*

Proof (1) We are given that

$$F(x) \circ d(y) + [x, y] \in P \quad \text{for all } x, y \in R. \tag{24}$$

Substituting $xd(y)$ for x in (24), we obtain

$$xd^2(y)d(y) + d(y)xd^2(y) + x[d(y), y] \in P \quad \text{for all } x, y \in R \tag{25}$$

in particular, we may write

$$rxd^2(y)d(y) + d(y)rxd^2(y) + rx[d(y), y] \in P \quad \text{for all } x, y, r \in R. \tag{26}$$

Right multiplying (25) by r and comparing with (26), one can easily see that

$$[d(y), r]xd^2(y) \in P \quad \text{for all } r, x, y \in R. \tag{27}$$

Invoking arguments used in the proof of Theorem 3 it follows that either R/P is commutative or $d^2(R) \subseteq P$. In the latter case, writing $d(y)$ instead of y in (24), we get

$$[d(y), x] \in P \quad \text{for all } x, y \in R. \tag{28}$$

Therefore d is P-centralizing and Theorem 1 yields that either R/P is commutative or $d(R) \subset P$ in which case our hypothesis reduces to $[x, y] \in P$ for all $x, y \in R$ proving again that R/P is a commutative integral domain. (2) Proceeding on the same lines with necessary variations, we can prove that $F(x) \circ d(y) - [x, y] \in P$ for all $x, y \in R$ assures the commutativity of R/P.

Now assume that R/P is 2-torsion free, then $F(x)d(y) \in P$ for all $x, y \in R$. Replacing x by $xd(r)$ we get $xd^2(r)d(y) \in P$ so that $d^2(R) \subset P$. Since $d^2(xy) = d^2(x)y + xd^2(y) + 2d(x)d(y)$, it follows that $d(x)d(y) \in P$ for all $x, y \in R$ proving that $d(R) \subset P$. □

Following is an immediate corollary of the above theorem.

Corollary 5 *Let R be a semiprime ring and F a generalized derivation of R associated with a derivation d. If F satisfies one of the following conditions:*

1) $F(x) \circ d(y) + [x, y] = 0$ *for all* $x, y \in R$;
2) $F(x) \circ d(y) - [x, y] = 0$ *for all* $x, y \in R$; *then R is a commutative reduced ring. Moreover, if R is 2-torsion free then F is a left multiplier.*

Invariance of minimal prime ideals under derivations

Given a derivation d of R, an ideal I of R is said to be invariant under d or d-invariant for short if $d(I) \subseteq I$. If I is such an ideal of R then d induces a derivation on R/I which preserves many properties of d. In particular, if we have a family of d-invariant ideals J_i with trivial intersection $\bigcap_i J_i = (0)$, then the ring R can be injected in the direct product of R/J_i and properties of d can arise from properties of induced derivations on R/J_i. In this way, we expect to realize the fundamental goal consisting to restrict questions regarding derivations in semiprime rings to questions about prime rings. Thus the real concern about a given semiprime ring is existence of prime d-ideals with trivial intersection. This leads us to the famous problem

Problem 1. Given a semiprime ring R, does there exist a collection of prime ideals P_i, such that $\bigcap_i P_i = (0)$ and such that each P_i is invariant under all derivations of R?

An affirmative answer to Problem 1 has been presented in ([7], Theorem 2.1) for a countable ring R. Furthermore, it is well known that every minimal prime ideal of a torsion-free semiprime ring is invariant under all derivations [10]. In this direction Herstein raised the following problem:

Problem 2. Given a semiprime ring R, does $d(P) \subseteq P$ hold for any minimal prime ideal P of R and for any derivation d of R?

Therefore, the fundamental preoccupation regarding a semi-prime ring is whether the intersection of the invariant minimal prime ideals under a given derivation is 0 or not. This problem was also mentioned in ([5] and [17]). The best result of the conjecture is due to Beidar and Mikhalv ([4], Theorem 8.16): Let R be a ring of bounded index m (i.e. m is a positive integer such that $x^m = 0$ for all nilpotent elements $x \in R$) such that the additive order of every nonzero torsion element of R, if any, is strictly larger than m. Then all minimal prime ideals of R are invariant under all derivations of R. In particular, every minimal prime ideal of a reduced ring is invariant under derivations of the ring ([8], p. 614). However, Problem 2 turns out to be false in general. In fact, Chuang and Lee [6] constructed a semiprime ring R which possesses a minimal prime ideal not invariant under a derivation of the ring.

Our fundamental aim in this section is to study the problem of invariance of minimal prime ideal of a semiprime ring under derivations. More precisely, we will apply our previous theorems to give some sufficient conditions under which a derivation leaves invariant minimal prime ideals of a semiprime ring.

Corollary 6 *Let R be a semiprime ring and F a generalized derivation of R associated with a derivation d. If F is centralizing generalized derivation of R, then*

1) *there exists a minimal prime ideal P invariant under d.*
2) *d has its range in $Z(R)$.*

Proof 1) Assume on the contrary that $d(P) \nsubseteq P$ for any minimal prime ideal P of R. According to Theorem 1, R/P is commutative and combining this fact with $\bigcap_{P minimal} P = (0)$ it follows that R is commutative. Hence, R is a reduced ring and [5, p. 614] assures that every minimal prime ideal of R is d-invariant, a contradiction.

2) Let x be an element in R and P be a minimal prime ideal of R. Using Theorem 1, we have $d(R) \subseteq P$ or R/P is commutative so that $[d(x), y] \in P$ for all $y \in R$ and therefore $d(x) \in Z(R)$. □

Remark 2 In particular, if a semiprime ring R admits a nonzero centralizing derivation d, then there exists a minimal prime ideal P invariant under d and d has its range in $Z(R)$.

Now we are in a position to prove the second result of this section which answers (at least partially) Problem 2. Indeed, for a semiprime ring provided with a generalized derivation (F, d) we establish some conditions under which every minimal prime ideal of R is invariant under the derivation d.

Corollary 7 *Let R be a semiprime and F a generalized derivation associated with a derivation d. If F satisfies one of the following conditions:*

1) $F(x) \circ d(y) + x \circ y = 0$ *for all* $x, y \in R$;
2) $F(x) \circ d(y) - x \circ y = 0$ *for all* $x, y \in R$;
3) $[F(x), d(y)] + x \circ y = 0$ *for all* $x, y \in R$;
4) $[F(x), d(y)] - x \circ y = 0$ *for all* $x, y \in R$; *then* $d(P) \subseteq P$ *for any minimal prime ideal of* R. *Moreover, there exists a minimal prime ideal* P *such that* $char(R/P) = 2$.

Proof According to Theorem 3, R/P is commutative for any minimal prime ideal and then R is commutative. Therefore R is a reduced ring and [[5], p. 614] assures that $d(P) \subseteq P$ for any minimal prime ideal of R.

Now assume that R/P is 2-torsion free for any minimal prime ideal P of R. Once again using Theorem 3, we have R/P is commutative and $d(R) \subseteq P$. Therefore, our hypothesis reduces to $xy \in P$ for all $x, y \in R$ and combining this fact together with $\bigcap_{P minimal} P = (0)$ we arrive at $xy = 0$ for all $x, y \in R$ and thus $R = \{0\}$, a contradiction. □

References

1. E. Albaş and N. Argaç, *Generalized derivations of prime rings*, Algebra Colloquium, 11 (3) (2004), 399-410.
2. F. A. A. Almahdi , A. Mamouni and M. Tamekkante, *A generalization of Posner's theorem on derivations in rings*, Indian J. Pure Appl. Math., 51 (1) (2020), 187-194.
3. H. E. Bell, M. N. Daif, *On commutativity and strong commutativity preserving maps,* Canad. Math. Bull., 37 (4) (1994), 443-447.
4. K. I. Beidar and A. V. Mikhalv, *Orthogonal completeness and algebraic systems,* Uspekhi Mat. Nauk., 40 (6) (246) (1985), 79-115.

5. J. Bergen, *Automorphic-differential identities in rings,* Proc. Amer. Math. Soc., 106 (1989), 297-305.
6. J. Bergen, I. N. Herstein, and C. Lanski, *Derivations with invertible values,* Canad. J. Math., 35 (2) (1983), 300-310.
7. C. L. Chuang and T. K. Lee, *Semiprime rings with prime ideals invariant under derivations,* J. Algebra, 302 (1) (2006), 305-312.
8. C. L. Chuang, T. K. Lee, *Invariance of minimal prime ideals under derivations,* Proc. Amer. Math. Soc., 113 (1991), 613-616.
9. V. De Filippis, N. Rehman and A. Ansari, *Lie ideals and generalized derivations in semiprime rings*, Iran. J. Math. Sci. Inform., 10 (2) (2015), 45-54.
10. K. R. Goodearl and R. B. Warfield Jr., *Primitivity in differential operator rings,* Math. Z., 180 (1982), 503-523.
11. M. Hongan, *A note on semiprime rings with derivation,* Internat. J. Math. and Math. Sci., 20 (2) (1997), 413-415.
12. B. Hvala, *Generalized derivations in rings*, Comm. Algebra, 26 (4) (1998), 1147-1166.
13. I. N. Herstein, *Topics in ring theory*, University of Chicago Press, Chicago-London, (1969).
14. I. N. Herstein, *A note on derivations*, Canad. Math. Bull., 21 (3) (1978), 369-370.
15. C. Lanski, *Differential identities, Lie ideals and Posner's theorems*, Pacific J. Math., 134 (2) (1988), 275-297.
16. T. K. Lee, *Generalized derivations of left faithful rings*, Comm. Algebra, 27 (8) (1999), 4057-4073.
17. G. Letzter, *Derivations and nil ideals,* Rend. Circ. Mat. Palermo, 37 (2) (1988), 174-176.
18. L. Oukhtite and A. Mamouni, *Generalized derivations centralizing on Jordan ideals of rings with involution*, Turkish J. Math., 38 (2) (2014), 225-232.
19. E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1093-1100.
20. N. Rehman, *On Commutativity of rings with generalized derivations*, Math. J. Okayama Univ., 44 (2002), 43-49.