Indian J Pure Appl Math (2022) 53:826–829
https://doi.org/10.1007/s13226-021-00176-8

ORIGINAL RESEARCH

A Characterization for $U_3(3)$

M. Reza Salarian

Received: 4 July 2020 / Accepted: 27 August 2021 / Published online: 14 September 2021

Abstract We identify $U_3(3)$ by having a self centralizing element of order 12.

Keywords Finite groups, Finite simple groups

Mathematics Subject Classification 20D08

1 Introduction

Let G be a finite group, P a Sylow p-subgroup of G and N a non-trivial subgroup of P. The normalizer of N in G is called a p-local subgroup of G. In most cases, when the structure of P is known we can determine the structure of p-locals of G. So identifying G with its p-local subgroups is an important subject in group theory and plays a crutial role in the classification of finite simple groups. For this, many of the finite simple groups have been identified with their p-local informations in [1,3,5,7–9,11,12,12–14] and [15].

In this paper we consider the finite simple groups with an extraspecial Sylow 3-subgroups of order 27. In fact we will consider the following question:

Question 1: Is that possible to determine the finite simple groups which have an extraspecial Sylow 3-subgroup P of order 27 such that $C_G(P) = Z(P)$?

As far we know the groups $U_3(3)$, ${}^2F_4(2)'$, $L_3(3)$ and M_{12} are among the groups satisfying the conditions of the question above. But maybe there are also some other examples. The groups M_{12} and $L_3(3)$ are characterized by their 3-local subgroups in [9]. In this paper we will identify the group $U_3(3)$ by its 3-local information and we give an answer to the question above in some cases. In fact by the assumptions of the question above, in many cases the group G has a self- centralizing element of order 12. So at first we have looked at the groups with a self centralizing element of order 12 and have proved the theorem 1. Then in theorem 2 we show that when the group G is $U_3(3)$. To state our theorems we need some assumptions and notation. We note that for a finite groups G and a subgroup H of G we denote by $N_G(H)$ and $C_G(H)$ the normalizer of H in G and the centralizer of H in G, respectively.

Notation: Throughout this paper we keep these notation.

(i) Let G be a finite simple group. Let $k \in G$ be of order 12. Set $M = \langle k \rangle$, $a = k^4$, $t = k^3$, $t^2 = z$ and $T = \langle t \rangle$.

(ii) Let $D \in Syl_2(N_G(M))$ and $D = \langle d \rangle$.

(iii) Let $S \in Syl_2(C_G(t))$ and set $B = O_2(C_G(t))$.

With the notation above to state our theorems we need the following assumptions.

Communicated by B. Sury.

M. R. Salarian (✉)
Department of Mathematics and Computer Science, Kharazmi University, Karaj/Tehran, Iran
E-mail: salarian78@gmail.com; salarian@khu.ac.ir

Assumption *: Let G be a finite simple group.
(i) Assume that $C_G(M) = M$, $N_G(M) \leq C_G(t)$ and $N_G(M)/\langle a\rangle \cong Z_8$.
(ii) We assume that $t \in D$ and then $t = d^2$.

Theorem 1 . *Let G be a finite simple group. Assume that G satisfies the assumption *. Then we have $C_G(t)/N \cong S_3$ where $N = F(C_G(t))$ is the Fitting subgroup of $C_G(t)$. We have further that $B = O_2(N)$ and if B is abelian then B is a characteristic subgroup of S.*

Theorem 2 *Let G be a finite simple group. Assume that G satisfies the assumption * and $C_G(D) = D$. Then $G \cong U_3(3)$.*

Among others as a corollary we can give an answer to the question 1 when the group $U_3(3)$ appears.

Corollary 1 *Let H be a finite simple group, $P \in Syl_3(H)$ be an extraspecial group of order 27 such that $N_H(P)/P \cong Z_8$, $C_H(P) = Z(P)$ and $C_H(x) \leq N_H(P)$ for each $1 \neq x \in P$. Let $Q \in Syl_2(N_H(P))$ and $C_H(Q) = Q$. Then $H \cong U_3(3)$.*

2 Proofs of Theorems 1 and 2

In this section we will prove Theorems 1 and 2. We keep our notation and assumptions as in the last section. In fact G is a finite simple group which satisfies the assumption *. If $x, y \in G$, we denote $C_G(x) \cap C_G(y)$ by $C_G(x, y)$. First we need the following theorem due to Higman.

Theorem 21 *[3, Theorem 8.2]*
Let H be a finite group with a normal 2-subgroup Q such that H/Q is isomorphic to $PSL(2, 2^n)$, $n \geq 2$, and suppose that an element of H of order 3 acts fixed point free on Q. Then Q is elementary abelian, and is the direct product of minimal normal subgroups of H, each of order 2^{2n}. The Sylow 2-subgroup P of H is of class 2, and if $|Q| > 2^{2n}$, Q is the only abelian subgroup of H of index 2^n.

The next theorem drives directly from the main theorem in [1].

Theorem 22 *We have $C_G(t)$ has a nilpotent normal subgroup N such that one of the following holds;*
(i) We have $C_G(t)/N$ is isomorphic to either S_3 or Z_3;
(ii) We have $C_G(t)/N$ is isomorphic to $L_2(7)$ and $N = \langle t\rangle$;
(iii) $C_G(t)/N \cong A_5$, in which case, $N/\langle t\rangle$ is an elementary abelian 2-group.

Proof We have $C_G(t, a) \cong Z_{12}$, so the theorem follows from [1]. We note that in case iii), by [1] we have $C_G(t)/N \cong A_5$ and N is a 2-group, then 21 will give us that $N/\langle t\rangle$ is an elementary abelian 2-group . □

We recall that $T = \langle t\rangle$ and $t^2 = z$.

Lemma 23 *We have $C_G(t)/N$ is not isomorphic to Z_3 or $L_2(7)$. Further, we have $N \neq T$, $d \notin N$ and $C_G(t) = N_G(T)$.*

Proof We have $D \leq C_G(t)$ and d inverts a, so $d \notin N$ and $C_G(t)/N \ncong Z_3$. Assume that $C_G(t)/N = L_2(7)$. Then by theorem 22 we have $N = T$ and then $C_G(t)$ is a nonsplit extension of Z_4 by $L_2(7)$. But the Schur multiplier of $L_2(7)$ is 2, so $C_G(t)/N \ncong L_2(7)$. Now by theorem 22, $C_G(t)/N \cong S_3$ or A_5. We have $C_G(T)$ is a normal subgroup of index at most 2 in $N_G(T)$. So if $C_G(t) \neq N_G(T)$, then $N_G(T)/N \cong S_5$, $A_5 \times Z_2$ or $S_3 \times Z_2$. This implies that 16 divides the order of $N_G(M)$ which is a contradiction to our assumption *. Assume that $N = T$ and $C_G(t)/N \cong S_3$. Then $D \in Syl_2(C_G(t))$ and since T is a characteristic subgroup of D and $C_G(T) = N_G(T)$ we deduce that $D \in Syl_2(G)$, which is impossible as no non abelian simple group has a cyclic Sylow 2-subgroup. Assume that $C_G(t)/N \cong A_5$. Since the Schur multiplier of A_5 is also 2, we get that $N \neq T$ and the lemma holds. □

We note that by lemma 23 we get that t is not conjugate to t^{-1} in G.

Lemma 24 *Assume that $C_G(t)/N \cong A_5$ and $Z(N) \neq T$. Then $Z(N)$ contains all involutions of $\langle Z(N), a, d\rangle$.*

Proof Assume that $C_G(t)/N \cong A_5$ and $Z(N) \neq T$. Since $Z(N)$ is normal in $C_G(t)$ and $C_G(t)/N \cong A_5$, we have $\langle Z(N), a, d\rangle / Z(N) \cong A_5$. So we may assume that N is abelian. Let $x \in C_G(t) \setminus N$ be an element of order 2. Since $C_G(t)/N$ has just one class of involutions we may assume that $xN = dN$. This gives us that $xd \in N$. We note that by Coprime action theorem (see [6, Theorem 8.2.7]) we have $N = T \times [a, N]$ and $[a, N]$ is an elementary abelian 2-group. We also note that by lemma 23, $C_G(T) = N_G(T)$. Now assume that xd be of order 2, then $d^x = d^{-1}$ which implies that $t^x = t^{-1}$, a contradiction. Assume that xd be of order 4. Since $x \notin D$, we conclude that $xd \in mT$ for some involution $m \in [N, a]$. But this gives us that $t^m = t^{-1}$, a contradiction. Now the lemma is proved.

Lemma 25 *Assume that $C_G(t)/N \cong A_5$. Then $Z(N) = T$. Specially N is nonabelian.*

Proof Assume that $C_G(t)/N \cong A_5$ and we may assume that $Z(N) = N$ and then N is abelian. Let $f \in C_G(t)$ be of order three such that $\langle d, f\rangle N/N \cong A_4$. Set $W =< d, d^f >$. Then W is of order 16 and $W \cap N = T$. By this and lemma 24 we conclude that W has just one involution and $Z(W) = T$. But there is no such a group of order 16 and the lemma is proved.

Lemma 26 *We have $C_G(t)/N \ncong A_5$.*

Proof By lemma 25 we have that N is not abelian. We have $1 \neq N' \leq \langle t\rangle$. We note that $\langle a\rangle$ is a Sylow 3-subgroup of $C_G(t)$ of order 3 and a acts fixed point freely on F/T. Now let $f \in C_G(t)$ be of order three such that $\langle d, f\rangle N/N \cong A_4$. Set $F = O_2(\langle N, d, f\rangle)$. We note that f is conjugate to a in $C_G(t)$. Therefore $C_F(f) = T$ which implies that f acts fixed point freely on F/T. We have also that $< [d, d^f] >$ is of order 2 and is normalized by f. Hence $[d, d^f] \in C_N(f) = T$. We note that by lemma 23 we have $C_G(t) = N_G(T)$. Since G is simple, by Glauberman's Z^*-Theorem (see [6, page 369]) we get that there is $s = t^g \in F$ such that $g \notin C_G(t)$. Now we prove the following claim.
Claim 1: We have $s^2 \notin T$ and $[d, s] \neq 1$.

Proof Assume that $s^2 \in T$ and $s \in N$. Then $s^2 = z$ and $\langle t, s\rangle \cong Z_4 \times Z_2$. This shows that $s = tm$ or $t^{-1}m$ where $m \in N$ is of order 2. For $x \in N$ we have $[x, m] \in T$ which implies that $m^x \in mT$. Since m is of order 2, we get that $m^x = m$ or mz. If $m^x = mz$, then $s^x = s^{-1}$, a contradiction. Therefore $s \in Z(N) = T$ and $t = s$, a contradiction. Therefore $s \notin N$ and we have shown that if $1 \neq x \in N$ is conjugate to t in G, then $x = t$. So we may assume that $s = dm$ for some $m \in N$ and we have $t \notin N^g$. Therefore $[d, s] \neq 1$, $m^2 \in T$. We note that by lemma 23 we get that s is not conjugate to s^{-1} in G. Now we have $s^d = md = s^{-1}tm^2 = s^{-1}t, s, s^{-1}t^{-1}$ or $s^{-1}z$. By this and since $s^2 = (s^d)^2 = z$, we get that $s^d = s$, a contradiction to $[d, s] \neq 1$. Therefore $s^2 \neq z$ and we have shown that if $1 \neq x \in C_G(t)$ is conjugate to t in G and $x^2 \in T$, then $x = t$. Hence $[d, s] \neq 1$ and the claim 1 is proved.

By calim 1 we get that $s^2 \neq z$ and therefore $s \notin N$. Since $d \notin N$ and $C_G(t)/N$ has just one class of involutions, we may assume that $s = dm$ for some $m \in N$. Then $m^2 \in T$. Set $l = [d, m]$, then $[l, d] \in T$. Since $[d, s] \neq 1$, we have $l \neq 1$. As $m^2 \in T$, we conclude that $m^2 = (m^d)^2 = mlml$. This implies that $m = lml$. As $m^d = ml$, we have $s^d = dml = sl$ and since $(d^l)^2 = t$ and $[l, d] \in T$, we get that $d^l = dz$ or d. Assume that $d^l = dz$ and then $l^d = lz$. Now we have $s = s^t = s^{d^2} = (sl)^d = sllz$ which gives us that $l^2 = z$. From this and as $m = lml$, we imply that $[l, m] = z$ and then we have $s = s^s = s^{dm} = s^m l^m = dlmlz = dmz = sz$, which tells us that $z = 1$, a contradiction. Therefore $[d, l] = 1$ and then $s = s^t = s^{d^2} = (dml)^d = dml^2$. This gives us that $l^2 = 1$ and since $m = lml$, we get that $[m, l] = 1$. Hence $[s, l] = 1$. Now we have $(s^2)^d = (s^d)^2 = (sl)^2 = s^2l^2 = s^2$ which implies that $[s^2, d] = 1$. We note that $s^d = sl$, therefore $[s, s^d] = 1$. We have shown that $s \neq s^d \in C_G(s)$ and $(s^d)^2 = s^2$. But this is a contradiction to s being conjugate to t and the lemma holds. □

Now we have the following theorem.

Theorem 27 *We have $C_G(t)/N \cong S_3$ and if B is abelian, then B is a characteristic subgroup of S.*

Proof By lemmas 22, 23 and 25, we get that $C_G(t)/N \cong S_3$ and N is nilpotent. So $B = O_2(N)$. We have also that $d \notin N$ and d inverts a. We note that since $B \neq T$ and a acts fixed point free on B/T, we get that S is not abelian. Assume that B is abelian and $g \in Aut(S)$, then $B^g \leq S$ and by Coprime action theorem (see [6, Theorem 8.2.7]) we have $B = T \oplus [B, a]$. Since B and B^g are of index 2 in S and $S/[B, a]$ is abelian, we get that $S' \leq B \cap B^g \cap [B, a]$. Since a is fixed point free on $[B, a]$ and d inverts a, we have $C_S(d) \leq DS'$. This shows that if $d \in B^g$, then we have $B^g = DS'$. This gives us that TS' is of index 2 in B and $S' \cap (S')^a \leq C_S(d, d^a) = 1$. But this implies that $|S'| = 2$ and then $B \cong Z_4 \times Z_2 \times Z_2$ which shows that B has no element of order 8, a contradiction. Therefore $D \cap B^g = T$. Assume that $B \neq B^g$ and $x \in B^g \setminus B$. Then $x^{-1}d \in B \cap B^g = Z(S)$. This gives us that $d \in B^g$, a contradiction. This gives us that $B = B^g$ and the lemma holds. □

Theorem 27 proves theorem 1. So in the remaining of this paper we prove theorem 2. By lemmas 22, 23 and 25, we get that $C_G(t)/N \cong S_3$. By our assumption in theorem 2 we have $C_G(D) = D$. Now we have the following lemma.

Lemma 28 *If $C_G(D) = D$, then $N_S(D)/D \cong Z_2$.*

Proof We have $C_S(D) = D$ and since G is simple, we get that $D \neq S$ and $N_S(D)/D$ is a non trivial subgroup of $Z_2 \times Z_2$. Since $N_S(D) \leq C_G(t)$, we deduce that $N_S(D)/D \cong Z_2$ and the lemma holds.

In the next lemma we show that S is of order 32.

Lemma 29 *B/T is elementary abelian of order 4 and S is of order 32. Further $Z(S) = T$.*

Proof Let $x \in B \notin T$. Then $[x, x^a] = 1$. Set $W = \langle x, x^a \rangle$. Since d inverts a and W is $\langle a \rangle$-invariant, we get that D acts on W. Then $C_W(D) \neq 1$ and hence $W \cap T \neq 1$. Since $N_B(D)$ is of index 2 in B and by lemma 28 $N_S(D)/D \cong Z_2$, we get that $|B/T| = 4$ and then $|S| = 32$. Now the lemma holds.

Proof of theorem 2: We assume that G is a finite simple group which satisfies the assumption * and $C_G(D) = D$. By lemmas 27, 23 and 29, we have $C_G(t)/N \cong S_3$, $C_G(T) = N_G(T)$ and $|S| = 32$. This implies that $S \in Syl_2(G)$. Now by [2] we get that $G \cong U_3(3)$ and theorem 2 is proved.

Proof of Corollary 1: Let H be a finite simple group, $P \in Syl_3(H)$ be an extraspecial group of order 27 such that $N_H(P)/P \cong Z_8$, $C_H(P) = Z(P)$ and $C_H(x) \leq N_H(P)$ for each $1 \neq x \in P$. Let $Q \in Syl_2(N_H(P))$ and $C_H(Q) = Q$. Let $Q =< q >$ and $Z(P) =< p >$, then $q^2 \in C_H(p)$ and pq^2 is a selt centralizing element of order 12 in H. So H satisfies the assumptions of theorem 2 and hence $H \cong U_3(3)$. Now the corollary is proved.

Acknowledgements This project is supported by grant number 96008646 from INSF. The author is also grateful to the referee for his/her report.

References

1. W. Feit and J. G. Thompson; Finite groups which contain a self-centralizing subgroup of order 3; Nagoya Math. J. 21 (1962),185-197.
2. P. Fong; Some Sylow Subgroups of Order 32 and a Characterization of U(3, 3); J. Algebra, 6, 65-76 (1967).
3. G. Higman; Odd characterizations of finite simple groups; Lecture notes, University of Michigan, 1968.
4. K. Harada; On finite groups having self-centralizing 2-subgroups of small order; J. Algebra. Volume 33, Issue 1, 1975, Pages 144-160.
5. I. Korchagina , Ch. Parker and P. J. Rowley, A 3-local characterization of Co_3. European Journal of Combinatorics, Volume 28, Issue 2 , February 2007, 559-566.
6. H. Kurzweil and B. Stellmacher; The Theory of Finite Groups, An Introduction; Springer, 2004.
7. M. E. O'Nan; Some characterizations by centralizers of elements of order 3; J. Algebra. 48, Issue 1, 1977, 113-141.
8. Ch. Parker; A 3-local characterization of $U_6(2)$ and Fi_{22}; J. Algebra 300 (2006), no. 2, 707–728.
9. Ch. Parker and S. Astil; A 3-local characterization of M_{12} and $SL_3(3)$; Arch. Math. (Basel) 92 (2009), no. 2, 99-110.
10. Ch. Parker and P. J. Rowley; A characteristic 5 identification of the Lyons group. J. London Math. Soc. (2) 69 (2004), no. 1, 128-140.
11. Ch. Parker and C. Wiedorn; A 5-local identification of the Monster. Archiv Math. Volume 83, Number 5, 2004, 404 - 415.
12. A. R. Prince; A characterization of the simple groups $PSp(4, 3)$ and $PSp(6, 2)$; J. Algebra, 45 (2) (1977), pp. 306-320.
13. M. R. Salarian; An identification of Co_1; J. Algebra 320 (2008) 1409-1448.
14. M. R. Salarian; A 3-local characterization of Fi_{24}'; J. Algebra, 324, 2804-2813, (2010).
15. M. R. Salarian and G. Stroth; An Identification for the Monster group; J. Algebra 323 (2010) 1186-1195.