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ORIGINAL RESEARCH

On $\mathcal{U}$-weak stability of coarse isometries between L^p spaces

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Abstract In this paper, we study $\mathcal{U}$-stability and weak $\mathcal{U}$-stability of coarse isometries between Banach spaces for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. As a result, for a coarse isometry $f : L_p(\Omega_1, \Sigma_1, \mu_1) \to L_p(\Omega_2, \Sigma_2, \mu_2)\,(1 < p < \infty)$, where $(\Omega_j, \Sigma_j, \mu_j)\,(j = 1, 2)$ are two measure spaces, we prove that the following conditions are equivalent:

(i) The mapping Φ defined by $\Phi(x) = w - \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ for any $x \in L_p(\Omega_1, \Sigma_1, \mu_1)$ is a linear isometry ;
(ii) For any $x \in L_p(\Omega_1, \Sigma_1, \mu_1)$, $\lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ exists and $\lim\limits_{\mathcal{U}} \frac{f(nx)}{n} = \Phi(x)$;
(iii) f is $\mathcal{U}$-weakly stable;
(iv) There exists a $T \in \mathscr{B}(L_p(\Omega_2, \Sigma_2, \mu_2), L_p(\Omega_1, \Sigma_1, \mu_1))$ with $\|T\| = 1$ such that

$$\|Tf(nx) - nx\| = o(n)_{\mathcal{U}}.$$

Keywords L^p space · Coarse isometry · Linear isometry · $\mathcal{U}$-weak stability

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1 Introduction

Let X and Y be two real Banach spaces. For a given $\varepsilon \geq 0$, a mapping $f : X \longrightarrow Y$ is said to be an ε-isometry provided that

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \leq \varepsilon, \text{ for all } x, y \in X.$$

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Especially, we say that f is an isometry if the above $\varepsilon = 0$. A celebrated result in Banach space theory is Mazur-Ulam theorem [13] which states that every surjective isometry f from a real Banach space X to another such space Y with $f(0) = 0$ is a linear map. The simple counterexample $f : \mathbb{R} \longrightarrow \ell_\infty^2$ defined by $f(x) = (x, |x|)$ shows that it does not hold in general without the assumption of surjectivity on f. For non-surjective isometry, Figiel [8] showed the classical result in 1968: For every isometry $f : X \longrightarrow Y$ with $f(0) = 0$, there exists a linear operator $F : L(f) \equiv \overline{span} f(X) \longrightarrow X$ with $\|F\| = 1$ such that $Ff(x) = x$ for all $x \in X$.

In 1945, Hyers and Ulam [11](see also [15]) raised a problem as follows: Whether for every surjective ε-isometry mapping $f : X \longrightarrow Y$ with $f(0) = 0$, there exists a surjective linear isometry $U : X \longrightarrow Y$ and $\gamma > 0$ such that

$$\|f(x) - Ux\| \leq \gamma\varepsilon, \quad \text{for all } x \in X.$$

In 1983, J. Gevirtz [9] given an affirmative answer with $\gamma = 5$ to the question above. After several decades efforts of a number of mathematicians (see for instance [9–11,15]), the following sharp estimate was finally obtained by Omladič and Šemrl [15].

Theorem 1.1 (Omladič and Šemrl) *Suppose that $f : X \to Y$ is a surjective ε-isometry with $f(0) = 0$. Then there is a surjective linear isometry $U : X \to Y$ such that*

$$\|f(x) - Ux\| \leq 2\varepsilon, \quad \textit{for all } x \in X.$$

Upon Figiel's theorem [8] and other remarkable results for non-surjective ε-isometry (see [16,17]), Cheng, Dong and Zhang gave a weak stability theorem [2], which can be regard as a quantitative extension of Figiel's theorem:

Theorem 1.2 (Cheng, Dong and Zhang) *Suppose that $f : X \to Y$ is a standard ε-isometry. Then for every $x^* \in X^*$ there exists $\phi \in Y^*$ with $\|x^*\| = \|\phi\| \equiv r$ so that*

$$\left|\left\langle x^*, x\right\rangle - \langle\phi, f(x)\rangle\right| \leq 4r\varepsilon \quad \textit{for all } x \in X.$$

It has played an profound influence in the study of stability of properties of ε-isometries (see [1–4,7,20]), since we can easily obtain almost all results of former. Recently, Cheng and Dong [6] further showed that the estimate constant "4" can be improved to be "3". Moreover, the constant "3" is accurate.

We say that a mapping $f : X \longrightarrow Y$ is a coarse isometry if $\varepsilon_f(t) = o(t)$ as $t \to \infty$, where

$$\varepsilon_f(t) = \sup_{x,y \in X, \|x-y\| \leq t} \{\left|\|f(x) - f(y)\| - \|x - y\|\right|\} \quad \text{for all } t \geq 0.$$

In [14], Lindenstrsuss and Szankowski studied a surjective coarse isometry which yields a linear isometric approximation in some special case. They defined $\varphi_f(t) = \sup\{\left|\|f(x) - f(y)\| - \|x - y\|\right| : \|f(x) - f(y)\| \leq t \text{ or } \|x - y\| \leq t\}$, and proved that when f is a surjective map with $\int_1^{+\infty} \frac{\varphi_f(t)}{t^2} dt < \infty$, then there exits a surjective linear isometry $U : X \longrightarrow Y$ such that

$$\|f(x) - Ux\| = o(\|x\|) \text{ as } \|x\| \to \infty.$$

Furthermore, they gave a counterexample to illustrate that the convergence of the improper integration is necessary even in finite dimensional spaces.

Inspired by Lindenstrauss' counterexample [14] and Cheng etal's results of weak stability of ϵ-isometry [1,2], Sun and Zhang studied the stability of coarse isometry between L^p space by sequence limitation [19]. Cheng [5] proposed the definition of weak stability of a coarse isometry with respect to $\mathcal{U}$ as follows:

Definition 1.3 Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and $f : X \to Y$ be a standard coarse isometry.

i) f is said to be stable with respect to $\mathcal{U}$, or, simply, $\mathcal{U}$-stable if there is a linear isometry $U : X \to Y$ so that

$$\|f(nx)/n - Ux\| = o(1)_{\mathcal{U}}, \text{ for all } x \in X, \text{ as } n \overset{\mathcal{U}}{\to} \infty.$$

ii) We say that f is weak stable with respect to $\mathcal{U}$, or simply, $\mathcal{U}$-weak stable, if for every $x^* \in X^*$ there $\varphi \in Y^*$ with $\|\varphi\| = \|x^*\|$ so that for each $x \subset X$

$$\left|\langle x^*, x\rangle - \langle\varphi, f(nx)/n\rangle\right| = o(1)_{\mathcal{U}}, \quad \text{as } n \overset{\mathcal{U}}{\to} \infty.$$

In this paper, the letters X, Y are used to be real Banach spaces, and X^*, Y^* are their dual spaces respectively. $\mathcal{U}$ denotes a free ultrafilter on $\mathbb{N}$, $\lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ and $w - \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ stand for filter limitations in norm topology and weak topology respectively.

2 On weak $\mathcal{U}$-stability of coarse isometry between L^p spaces

The purpose of this paper is to study properties of $\mathcal{U}$- stable and $\mathcal{U}$-weak stable of coarse isometries between $L^p(1 < p < \infty)$ spaces. Free ultrafilter limits shall play an important role in the paper.

Definition 2.1 i) A filter $\mathcal{F}$ on a set Ω is a collection of subsets of Ω satisfying a. $\emptyset \notin \mathcal{F}$; b. $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$ and c. $A \in \mathcal{F}$ and $A \subset B \subset \Omega$ entail $B \in \mathcal{F}$.

ii) A filter $\mathcal{F}$ is said to be free if $\cap\{F \in \mathcal{F}\} = \emptyset$;

iii) A filter $\mathcal{U}$ is called an ultrafilter if for any $A \subset \Omega$, either $A \in \mathcal{U}$, or, $\Omega \setminus A \in \mathcal{U}$.

iv) Let K be a topological space, and $f : \Omega \to K$ be a function. We say f is convergent to some $k \in K$ with respect to a filter $\mathcal{F}$ if for every neighborhood U of k, we have $f^{-1}(U) \in \mathcal{F}$; in this case, we denote $\lim_{\mathcal{F}} f = k$.

Firstly, we extend the uniformly convex spaces' sequence limitation property to filter limitation property.

The following property is classical and is easy to prove.

Proposition 2.2 *Suppose that Ω is a nonempty set , K is a Housdorff compact space, and that $f : \Omega \to K$ is a mapping. Then for every free ultrafilter $\mathcal{U}$ on Ω the limit $\lim_{\mathcal{U}} f$ exists and is unique.*

Lemma 2.3 *Let X be a uniformly convex space, $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, $\{x_n\}_{n=1}^{\infty}$,$\{y_n\}_{n=1}^{\infty}$ be two bounded sequences in X, if $\lim\limits_{\mathcal{U}} \|x_n\| = \lim\limits_{\mathcal{U}} \|y_n\| = r > 0$ and $\lim\limits_{\mathcal{U}} \|x_n + y_n\| = 2r$. Then $\lim\limits_{\mathcal{U}} \|x_n - y_n\| = 0$.*

Proof Since $\{x_n\}, \{y_n\}$ are bounded sequences in X, then $\lim\limits_{\mathcal{U}} \|x_n - y_n\|$ exists. We may assume that $\lim\limits_{\mathcal{U}} \|x_n - y_n\| = a > 0$. By filter convergence definition, for all $n \in \mathcal{N}$, we have

$$A_n = \left\{m \in \mathcal{N} : \big|\|x_m + y_m\| - 2r\big| \le \frac{1}{n}\right\},$$

$$B_n = \left\{m \in \mathcal{N} : \big|\|x_m - y_m\| - a\big| \le \frac{1}{n}\right\},$$

$$C_n = \left\{m \in \mathcal{N} : \big|\|x_m\| - r\big| \le \frac{1}{n}\right\},$$

$$D_n = \left\{m \in \mathcal{N} : \big|\|y_m\| - r\big| \le \frac{1}{n}\right\}.$$

By ultrafilter convergence definition, for all $n \in \mathcal{N}$, we have $A_n, B_n, C_n, D_n \in \mathcal{U}$ and $\emptyset \ne A_n \cap B_n \cap C_n \cap D_n \in \mathcal{U}$. We select sufficient large n such that $\frac{1}{n} \le \frac{2}{a}$ and $k_n \in A_n \cap B_n \cap C_n \cap D_n$, then

$$\big|\|x_{k_n} + y_{k_n}\| - 2r\big| \le \frac{1}{n},$$

$$\big|\|x_{k_n} - y_{k_n}\| - a\big| \le \frac{1}{n},$$

$$\big|\|x_{k_n}\| - r\big| \le \frac{1}{n},$$

$$\big|\|y_{k_n}\| - r\big| \le \frac{1}{n}.$$

So $\{x_n\}$,$\{y_n\}$ has sequences $\{x_{k_n}\}$, $\{y_{k_n}\}$ with $\lim\limits_{n} \|x_{k_n}\| = \lim\limits_{n} \|y_{k_n}\| = r$ and $\lim\limits_{n} \|x_{k_n} + y_{k_n}\| = 2r$. By uniformly convexness of X, it is obvious that $\lim\limits_{n} \|x_{k_n} - y_{k_n}\| = 0$. While, by construction of $\{x_{k_n}\}$, $\{y_{k_n}\}$ above, we have $\lim\limits_{n\to\infty} \|x_{k_n} - y_{k_n}\| = a$. Which is a contraction. □

Theorem 2.4 *Suppose $(\Omega_i, \Sigma_i, \mu_i)$ are two measure spaces, $1 < p < \infty$, $X_i = L^p(\Omega_i, \Sigma_i, \mu_i)(i = 1, 2)$, and let $f : X_1 \longrightarrow X_2$ be a $\mathcal{U}$-weak stable coarse isometry, then $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ exists and defines a linear isometry.*

Proof By Corollary 3.10 in [5], we obtain that $\Phi(x) = w - \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ defines an isometry. It is obvious that $\Phi(0) = 0$. Combined with the strict convexness of $L^p(1 < p < \infty)$, we get that Φ is linear. Hence we just need to show that $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$.

Firstly, by definition of coarse isometry, we have

$$\|\Phi(x)\| = \|x\| = \lim_{\mathcal{U}} \|\frac{f(nx)}{n}\|.$$

On one hand, by weak lower semi-continuousness of norm, we have

$$2\|\Phi(x)\| = \|w - \lim_{\mathcal{U}} \frac{f(nx)}{n} + \Phi(x)\| \le \lim_{\mathcal{U}} \|\frac{f(nx)}{n} + \Phi(x)\|.$$

On the other hand, by norm's triangle inequality, we have

$$\lim_{\mathcal{U}} \|\frac{f(nx)}{n} + \Phi(x)\| \le \lim_{\mathcal{U}} \|\frac{f(nx)}{n}\| + \|\Phi(x)\| = 2\|\Phi(x)\|.$$

Hence, $\lim\limits_{\mathcal{U}} \|\frac{f(nx)}{n} + \Phi(x)\| = 2\|\Phi(x)\|$. By Lemma 2.3, we get that $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$, then we complete the proof. □

Theorem 2.5 *Let $X_1, X_2, \mathcal{U}, f$ be as above, then the following propositions are equivalent:*

(i) $\Phi(x) = w - \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ *is a linear isometry;*

(ii) $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ *exists and defines a linear isometry;*

(iii) There exists $T \in \mathscr{B}(X_2, X_1)$ with $\|T\| = 1$ such that

$$\|Tf(nx) - nx\| = o(n)_{\mathcal{U}};$$

(iv) f is $\mathcal{U}$-weak stable.

Proof (i)⇒(ii) Repeat the above process of the existence of $\lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$, then (ii) holds immediately.

(ii)⇒(iii) Since $\Phi : X_1 \longrightarrow X_2$ is a linear isometry, by [12] theorem 3 (p.p. 162), $\Phi(X_1)$ is one-complemented in X_2. Then there exists a closed subspace N of X_2 such that $X_2 = \Phi(X_1) \oplus N$ with P projects X_2 onto $\Phi(X_1)$ along N is norm one. Notice the reflexivity of X_2, we get that Φ is a conjugate operator and $X_2^* = \Phi(X_1)^* \oplus N^*$.

Let $T = \Phi^{-1}P$. It is obvious that $\|P\| = 1$ and

$$\begin{aligned}\|T\frac{f(nx)}{n} - x\| &= \|\Phi^{-1}P\frac{f(nx)}{n} - x\|\\ &= \|P\frac{f(nx)}{n} - \Phi(x)\|\\ &\to 0(n \overset{\mathcal{U}}{\to} \infty).\end{aligned}$$

(iii)⇒(iv) Since $\|T\| = 1$, then $\|T\| = \|T^*\| = 1$, and for any $x^* \in X_1^*$, $\|T^*x^*\| \le \|T^*\|\|x^*\|$.
On the other hand, for any $\theta \ne x^* \in X_1^*$, select $x \in \partial\|x^*\|$, by (iii), we have

$$\begin{aligned}\|x^*\| = \langle x^*, x\rangle &= \lim_{\mathcal{U}}\langle x^*, T\frac{f(nx)}{n}\rangle = \lim_{\mathcal{U}}\langle T^*x^*, \frac{f(nx)}{n}\rangle\\ &\le \|T^*x^*\| \lim_{\mathcal{U}} \|\frac{f(nx)}{n}\| \le \|x^*\|\|x\| = \|x^*\|.\end{aligned}$$

So $\|T^*x^*\| = \|x^*\|$ and $T^* : X_2^* \longrightarrow X_1^*$ is a linear isometry.

Let $\varphi = T^*x^*$, then $\|\varphi\| = \|x^*\|$ and

$$\begin{aligned}&\frac{\left|\langle x^*, nx\rangle - \langle \varphi, f(nx)\rangle\right|}{n}\\ &= \frac{\left|\langle x^*, nx\rangle - \langle T^*x^*, f(nx)\rangle\right|}{n}\\ &= \frac{\left|\langle x^*, nx\rangle - \langle x^*, Tf(nx)\rangle\right|}{n}\\ &\le \frac{\|nx - Tf(nx)\|}{n}\|x^*\| \overset{n\overset{\mathcal{U}}{\to}\infty}{\longrightarrow} 0.\end{aligned}$$

(iv)⇒(i) By corollary 3.10 in [5] and the proceeding of the existence of $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ in theorem 2.4, it is easy to get this conclusion. □

Corollary 2.6 *Suppose that X is a Banach space, that Y is an L^p space for some $1 < p < \infty$, and let $f : X \longrightarrow Y$ be a coarse isometry. Then f is $\mathcal{U}$-weak stable if and only if $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ exists.*

Proof Necessity. By corollary 3.10 in [5], $\Phi(x) = w-\lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ is an isometry. Repeat the proof of the existence of filter limitation in theorem 2.4, $\lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ exists, then $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$.

Sufficiency. Since $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ exists, it is obvious that $\Phi(x) = w - \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ is a linear isometry. We obtain that f is $\mathcal{U}$-weak stable immediately by corollary 3.10 in [5].

When Y is a Hilbert space, notice that every closed subspace of Hilbert space is one complemented, similar to the process of theorem 2.5, we have the following proposition. □

Proposition 2.7 *Suppose that X is a Banach space, that Y is a Hilbert space, and let $f : X \longrightarrow Y$ be a coarse isometry. Then the following propositions are equivalent:*

(i) $\Phi(x) = w - \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ *is a liner isometry;*
(ii) $\Phi(x) = \lim\limits_{\mathcal{U}} \frac{f(nx)}{n}$ *exists and defines a linear isometry;*

$$\Phi : X \longrightarrow Y;$$

(iii) There exists a $T \in \mathcal{B}(Y, X)$ with $\|T\| = 1$ such that

$$\|Tf(nx) - nx\| = o(n)_{\mathcal{U}};$$

(iv) f is $\mathcal{U}$-weak stable.

Notice that the sequence limitation is a special filter limitation, as applications, we can easily get Sun and Zhang's result in [19] as follows:

Proposition 2.8 *([19] Theorem 2.12) Let X be a Banach space and $Y = L_p(\Omega, \Sigma, \mu)$ $(1 < p < \infty)$ for some σ-finite measure space (Ω, Σ, μ). Suppose that $f : X \to Y$ is a standard coarse isometry. If for every $x^* \in S_{X^*}$, there exists $\phi \in S_{Y^*}$ such that*

$$|\langle x^*, x\rangle - \langle \phi, f(x)\rangle| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

Then there is a linear isometry $U : X \to Y$ satisfying

$$\|f(x) - Ux\| = o(\|x\|) \quad \text{as } \|x\| \to \infty.$$

Proposition 2.9 *([19] Theorem 2.14) Let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be two σ-finite measure spaces and $1 < p < \infty$. Suppose that $X = L_p(\Omega_1, \Sigma_1, \mu_1)$, $Y = L_p(\Omega_2, \Sigma_2, \mu_2)$ and that $f : X \to Y$ is a standard coarse isometry. If $U : X \to Y$ is a linear isometry and $P : Y \to U(X)$ is a projection with $\|P\| = 1$, then the following statements are equivalent:*

(i) $\|f(x) - Ux\| = o(\|x\|)$ *as* $\|x\| \to \infty$;
(ii) $\|Pf(x) - Ux\| = o(\|x\|)$ *as* $\|x\| \to \infty$.

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