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ORIGINAL RESEARCH

Optimization problems for locally convex cone-valued functions

A. Azizi Mayvan · M. R. Motallebi

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Abstract We study the optimal solutions of optimization problems and well-posedness for locally convex cone-valued functions. Using the non-linear scalarization functions, we prove some existence results and obtain characterizations of the optimal solutions. Then, we consider the well-posedness in Tykhonov sense and discuss the scalar optimization problems.

Keywords Locally convex cones · Optimization problems · Well-posedness · Scalarization functions

Mathematics Subject Classification 46N10 · 58E30 · 49K40 · 46A30

1 Introduction

The theory of locally convex cones was developed by K. Keimel and W. Roth in [5]. The basic idea is to extend the notion of a locally convex space to a more general underlying algebraic structure. Afterwards, many topics on locally convex cones have been studied; for example, see [6–14]. The original optimization problems and pointwise well-posedness for locally convex cone-valued functions have been defined and studied in [1]. Our aim in the present work is the study of optimal solutions and well-posedness for the original and weak local (global) optimization problems for locally convex cone-valued functions. In Section 2, considering the original and local (global) preorders, we define three kind of optimization problems for locally convex cone-valued functions and study the optimal solutions. In Section 3, we consider the non-linear scalarization functions for locally convex cones and prove some existence theorems. Then we derive the characterization of optimal solutions and the pointwise well-posed optimization problems by means of Tykhonov well-posed problems.

A *cone* is a set $\mathcal{P}$ endowed with two operations; addition and scalar multiplication for non-negative real numbers $\lambda \geq 0$. The addition is associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, $1.a = a$ and $0.a = 0$ for every $a \in \mathcal{P}$. The cone $\mathcal{P}$ is an *ordered cone* if it carries a preorder, i.e., a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a + c \leq b + c$ and $\lambda a \leq \lambda b$ for all $a, b, c \in \mathcal{P}$ and $\lambda \geq 0$. With usual addition, multiplication and order the extended scalar field $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ of real numbers is a natural example of an ordered cone. In particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\lambda.(+\infty) = +\infty$ for all $\lambda > 0$ and $0.(+\infty) = 0$. A *full locally convex*

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A. Azizi Mayvan · M. R. Motallebi (✉)
Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran
E-mail: motallebi@uma.ac.ir

A. Azizi Mayvan
E-mail: azizimayvan@uma.ac.ir

cone is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, i.e., a subset of $\mathcal{P}$ with the following properties:

(v_1) $0 \leq v$ for all $v \in \mathcal{V}$,
(v_2) for all $u, v \in \mathcal{V}$ there is $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$,
(v_3) $u + v \in \mathcal{V}$ and $\lambda v \in \mathcal{V}$ for all $u, v \in \mathcal{V}$ and $\lambda > 0$.

For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we define

$$v(a) = \{b \in \mathcal{P} \mid b \leq a + v\}, \quad \text{respectively} \quad (a)v = \{b \in \mathcal{P} \mid a \leq b + v\},$$

to be a neighborhood of a in the *upper*, respectively *lower topology* on $\mathcal{P}$. Their common refinement is called *symmetric topology*. We assume all elements of $\mathcal{P}$ to be *bounded below*, i.e., for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. The cone $\overline{\mathbb{R}}$ with abstract neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$ forms a full locally convex cone. Finally, a *locally convex cone* $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex ordered vector space is a locally convex cone in this sense, as it can be embedded into a full locally convex cone (see Example I.2.7 in [5]).

If $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, a linear mapping $T : \mathcal{P} \to \mathcal{Q}$ is called *uniformly continuous* (*u-continuous*) if for every $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. It is immediate from the definition that, u-continuity implies continuity with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$. The set of all u-continuous linear mapping $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ forms a cone which is called the dual cone of $\mathcal{P}$ and denoted by $\mathcal{P}^*$. The *polar* $v^\circ_{\mathcal{P}}$ of a neighborhood $v \in \mathcal{V}$ is defined by $v^\circ_{\mathcal{P}} = \{\mu \in \mathcal{P}^* : a \leq b + v \text{ implies that } \mu(a) \leq \mu(b) + 1\}$.

2 Optimization problems and optimal solutions

In this section, we extend the general classical optimization problems for locally convex cone-valued functions and study the optimal solutions. Also, we define the well-posedness for optimization problems and identify three classes of pointwise well-posed problems.

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and $v \in \mathcal{V}$. According to [5, Ch I, 3.2], the *v-local preorder* $\preceq_v$ on $\mathcal{P}$ is defined by $a \preceq_v b$ for $a, b \in \mathcal{P}$ if and only if $a \leq b + \rho v$ for all $\rho > 0$ and the *global preorder* $\preceq$ on $\mathcal{P}$ is defined as the intersection of all local preorders $\preceq_v$, $v \in \mathcal{V}$, i.e., $a \preceq b$ for $a, b \in \mathcal{P}$ if and only if $a \leq b + v$ for all $v \in \mathcal{V}$. The cone $\mathcal{P}$ is called *separated* if $\overline{a} = \overline{b}$ for $a, b \in \mathcal{P}$ implies that $a = b$, where $\overline{a}$ is the closure of $\{a\}$ with respect to the lover topology. According to [5, Ch I, Proposition 3.9], $(\mathcal{P}, \mathcal{V})$ is separated if and only if the global preorder $\preceq$ is an order, i.e., $\preceq$ is antisymmetric. Note that in $\overline{\mathbb{R}}$ with abstract neighborhood $\mathcal{V} = \{\varepsilon : \varepsilon > 0\}$, the local preorders $\preceq_v$ and the global preorder $\preceq$ coincide with the original order $\leq$. Considering the original preorder $\leq$ and the local preorders $\preceq_v$ $v \in \mathcal{V}$, we define the following relations on $\mathcal{P}$:

$$\begin{aligned} &a \sim b \quad \text{for} \quad a, b \in \mathcal{P} \quad \text{if} \quad a \leq b \quad \text{and} \quad b \leq a. \\ &a < b \quad \text{for} \quad a, b \in \mathcal{P} \quad \text{if} \quad a \leq b \quad \text{and} \quad a \nsim b. \\ &a \approx_v b \quad \text{for} \quad a, b \in \mathcal{P} \quad \text{if} \quad a \preceq_v b \quad \text{and} \quad b \preceq_v a. \end{aligned}$$

Let $G \subseteq \mathcal{P}$. We say that $a \in G$ is a *minimal* element, if $b \leq a$ for some $b \in G$ then $b \sim a$ and *v-local minimal* element, if $b \preceq_v a$ for some $b \in G$ then $b \approx_v a$. Also, $a \in G$ is a *global minimal* element if, a is a v-local minimal element for all $v \in \mathcal{V}$. We will denote by $\mathrm{Min}(G, \leq)$, $\mathrm{Min}(G, \preceq_v)$ and $\mathrm{Min}(G, \preceq)$ the set of all original minimal, v-local minimal and global minimal elements of G, respectively.

Remark 2.1 If $G \subseteq \mathcal{P}$ and for $a \in G$, $\downarrow(G, \leq, a) = \{b \in G : b \leq a\}$, then

(i) $\mathrm{Min}(G, \preceq) = \bigcap_{v \in \mathcal{V}} \mathrm{Min}(G, \preceq_v)$,
(ii) $\downarrow(G, \leq, a) \subseteq \mathrm{Min}(G, \leq)$ for all $a \in \mathrm{Min}(G, \leq)$,
(iii) $\downarrow(G, \preceq_v, a) \subseteq \mathrm{Min}(G, \preceq_v)$ for all $a \in \mathrm{Min}(G, \preceq_v)$,
(iv) $\downarrow(G, \preceq, a) \subseteq \mathrm{Min}(G, \preceq)$ for all $a \in \mathrm{Min}(G, \preceq)$,
(v) If $\mathcal{P}$ is separated then $\mathrm{Min}(G, \preceq) \subseteq \mathrm{Min}(G, \leq)$, for; let $a \in \mathrm{Min}(G, \preceq)$. If $b \leq a$ for some $b \in G$ then $b \preceq a$ which implies $b \preceq_v a$ for all $v \in \mathcal{V}$. Thus, $a \preceq_v b$ for all $v \in \mathcal{V}$, i.e., $a \preceq b$. Since $\mathcal{P}$ is separated we have $a = b$, so $a \in \mathrm{Min}(G, \leq)$.

Let X be a non-empty Hausdorff topological space and $f : X \to (\mathcal{P}, \mathcal{V})$ be a mapping. The original, v-local and global, general optimization problem for f is formalized by $(X, f, \leq)$ $\mathrm{Min}(f(X), \leq)$,

$$(X, f, \preceq_v)\ \ \mathrm{Min}(f(X), \preceq_v) \quad \text{and} \quad (X, f, \preceq)\ \ \mathrm{Min}(f(X), \preceq);$$

respectively, where $f(X) = \{f(x) : x \in X\}$. An element $\hat{x} \in X$ is called a *minimal solution* of problem $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$) if $f(\hat{x})$ is an original (respectively, v-local and global) minimal element of $f(X)$. In general, we say that $\hat{x}$ is an *optimal solution* of $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$) and denote by $\mathrm{Eff}(X, f, \leq)$ (respectively, $\mathrm{Eff}(X, f, \preceq_v)$ and $\mathrm{Eff}(X, f, \preceq)$) the set of all minimal solutions of $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$).

Remark 2.2 If for $x \in X$, $S(f, \leq, f(x)) = f^{-1}(\downarrow (f(X), \leq, f(x))$, then

(i) $S(f, \leq, f(\hat{x})) \subseteq \mathrm{Eff}(X, f, \leq)$ for all $\hat{x} \in \mathrm{Eff}(X, f, \leq)$,
(ii) $S(f, \preceq_v, f(\hat{x})) \subseteq \mathrm{Eff}(X, f, \preceq_v)$ for all $\hat{x} \in \mathrm{Eff}(X, f, \preceq_v)$,
(iii) $S(f, \preceq, f(\hat{x})) \subseteq \mathrm{Eff}(X, f, \preceq)$ for all $\hat{x} \in \mathrm{Eff}(X, f, \preceq)$.

For a full locally convex cone $(\mathcal{P}, \mathcal{V})$, we denote by $Conv(\mathcal{P})$ the set of all non-empty convex subsets of $\mathcal{P}$ which is a cone with its usual addition and multiplication by non-negative scalars. If we identify the elements of $\mathcal{V}$ with singleton sets $\overline{v} = \{v\}$, then $\overline{\mathcal{V}} = \{\overline{v} : v \in \mathcal{V}\}$ is a subset of $Conv(\mathcal{P})$, which can be ordered using the order of $\mathcal{P}$. For $A, B \in Conv(\mathcal{P})$, we define

$$A \leq B \text{ if for each } a \in A \text{ there is some } b \in B \text{ such that } a \leq b,$$

then $(Conv(\mathcal{P}), \overline{\mathcal{V}})$ becomes a full locally convex cone. If E is a locally convex space with 0-neighborhood base $\mathcal{V}$, then $(Conv(E), \overline{\mathcal{V}})$ is a full locally convex cone and by the embedding $x \to \{x\} : E \to Conv(E)$, $(E, \mathcal{V})$ is a locally convex cone. For details see [5].

Example 2.1 (i) If for $X = [0, 1]$, the mapping $f : X \to (Conv(\overline{\mathbb{R}}), \overline{\mathcal{V}})$, $\mathcal{V} = \{\varepsilon : \varepsilon > 0\}$, is defined by

$$f(x) = \begin{cases} (-\infty, 0) & \text{if } x = 0, \\ [0, 1 - x] & \text{if } 0 < x \leq 1, \end{cases}$$

for all $x \in X$, then $\mathrm{Eff}(X, f, \leq) = \{0\}$ and $\mathrm{Eff}(X, f, \preceq_{\overline{\varepsilon}}) = \mathrm{Eff}(X, f, \preceq) = \{0, 1\}$ for all $\varepsilon \in \mathcal{V}$.

(ii) We consider $Conv(\mathbb{R})$ with set inclusion and abstract neighborhood system $\mathcal{V} = \{(-\varepsilon, \varepsilon) : \varepsilon > 0\}$. If for $X = [0, 1]$, the mapping $f : X \to (Conv(\mathbb{R}), \mathcal{V})$ is defined by

$$f(x) = \begin{cases} (0, \frac{3}{2}) & \text{if } x = 0, \\ [0, \frac{3}{2} - x) & \text{if } 0 < x \leq \frac{1}{2}, \\ [0, 2 - x] & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

for all $x \in X$, then $\mathrm{Eff}(X, f, \leq) = \{0, \frac{1}{2}\}$ and $\mathrm{Eff}(X, f, \preceq_\varepsilon) = \mathrm{Eff}(X, f, \preceq) = \{\frac{1}{2}, 1\}$ for all $\varepsilon > 0$.

(iii) For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} : a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider the the singleton neighborhood system $\mathcal{V} = \{0\}$. The elements of $\overline{\mathbb{R}}_+$ are obviously bounded below with respect to the neighborhood $v = 0$, hence $\overline{\mathbb{R}}_+$ is a full locally convex cone [13, Example 2.1]. Evidently, the local preorder $\preceq_0$ coincides with the orginal one on $\overline{\mathbb{R}}_+$. If for $X = [0, 1]$, the mapping $g : X \to \overline{\mathbb{R}}_+$ is defined by

$$g(x) = \begin{cases} +\infty & \text{if } x = 0, \\ x^2 & \text{if } 0 < x \leq \frac{1}{2}, \\ 1 - x & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

then $\mathrm{Eff}(X, g, \leq) = \mathrm{Eff}(X, g, \preceq_0) = \{1\}$.

According to [5], with the v-local (global) preorder $\preceq_v$ ($\preceq$) the neighborhood system $\mathcal{V}$ forms a locally convex cone topology on $\mathcal{P}$ which is coarser than the original one. Suppose $\{a_i\}_{i \in \mathcal{I}}$ is a net in $\mathcal{P}$ and let $a \in \mathcal{P}$. We write $a_i \uparrow a$ ($a_i \downarrow a$) in the $\leq$ (respectively, $\preceq_v$ and $\preceq$)-topology if $\{a_i\}_{i \in \mathcal{I}}$ converges to a with respect to the upper (respectively, lower) topology on $\mathcal{P}$ induced by $\leq$ (respectively, $\preceq_v$ and $\preceq$) (cf. [6,9]). For $v \in \mathcal{V}$, an element $a \in \mathcal{P}$ is called *v-bounded* if $a \leq \lambda v$ for some $\lambda > 0$, and a is called *bounded* if it is v-bounded for all $v \in \mathcal{V}$.

Definition 2.1 If $\hat{x}$ is a minimal solution of $(X, f, \leq)$ ($(X, f, \preceq_v)$ and $(X, f, \preceq)$), then a net $\{x_i\}_{i\in\mathcal{I}} \subseteq X$ is said to be *minimizing net* for $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$) at $\hat{x}$ if $f(x_i) \uparrow f(\hat{x})$ in $\leq$ (respectively, $\preceq_v$ and $\preceq$)-topology. We say that $(X, f, \leq)$ ($(X, f, \preceq_v)$ and $(X, f, \preceq)$) is *well-posed* at $\hat{x}$, if every minimizing net for $(X, f, \leq)$ ($(X, f, \preceq_v)$ and $(X, f, \preceq)$) at $\hat{x}$ converges to $\hat{x}$. Also, $(X, f, \leq)$ ($(X, f, \preceq_v)$ and $(X, f, \preceq)$) is *generalized well-posed* at $\hat{x}$ if every corresponding minimizing net $\{x_i\}_{i\in\mathcal{I}}$ at $\hat{x}$ has a cluster point in $S(f, \leq, f(\hat{x}))$ (respectively, $S(f, \preceq_v, f(\hat{x}))$ and $S(f, \preceq, f(\hat{x}))$).

Remark 2.3 Obviously, nets (generalized) well-posedness is formally stronger than (generalized) well-posedness in sequential sense. Note that, if the abstract neighborhood system $\mathcal{V}$ has a bounded element, then nets well-posedness is equivalent to well-posedness in sequential sense. Indeed, nets well-posedness entails well-posedness in the sequential sense. For the converse, let $(X, f, \leq)$ (respectively, $(X, f, \preceq)$ and $(X, f, \preceq_v)$ be sequential well-posed at $\hat{x}$ and it fails to be nets well-posed. Then, there exists some net $\{x_i\}_{i\in\mathcal{I}} \subseteq X$ such that $f(x_i) \uparrow f(\hat{x})$ in the $\leq$ (respectively, $\preceq$ and $\preceq_v$)-topology and $x_i \not\to \hat{x}$. Thus we can find some neighborhood U of $\hat{x}$ such that for every $i \in \mathcal{I}$ there exists $j \geq i$ with $x_j \notin U$. Consider a bounded element $v \in \mathcal{V}$. One can take $i_1 \in \mathcal{I}$ such that $f(x_j) \in v(f(\hat{x}))$ for all $j \geq i_1$ and take $j_1 \geq i_1$ such that $x_{j_1} \notin U$. By proceeding in this way, one can find a sequence $\{x_{j_n}\}_{n\in\mathbb{N}}$ such that $f(x_{j_n}) \in \frac{1}{n}v(f(\hat{x}))$ and $x_{j_n} \notin U$ for all $n \in \mathbb{N}$. Since v is bounded, we have $f(x_{j_n}) \uparrow f(\hat{x})$ in $\leq$ (respectively, $\preceq$ and $\preceq_v$)-topology, when $n \to \infty$. So $x_{j_n} \to \hat{x}$ which contradicts $x_{j_n} \notin U$.

Proposition 2.1 *If $\hat{x}$ is a minimal solution of $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$), then*

(a) *if $(X, f, \leq)$ ($(X, f, \preceq_v)$ and $(X, f, \preceq)$) is well posed at $\hat{x}$, then $S(f, \leq, f(\hat{x}))$ (respectively, $S(f, \preceq_v, f(\hat{x}))$, and $S(f, \preceq, f(\hat{x}))$) is a singleton,*

(b) *if $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$) is generalized well-posed at $\hat{x}$ then $S(f, \leq, f(\hat{x}))$ (respectively, $S(f, \preceq_v, f(\hat{x}))$ and $S(f, \preceq, f(\hat{x}))$) is compact.*

Proof We assume that $\hat{x} \in \mathrm{Eff}(X, f, \leq)$. In other cases the proofs are almost the same. For (a), if $x' \in S(f, \leq, f(\hat{x}))$, we have $f(x') \leq f(\hat{x}) + v$ for all $v \in \mathcal{V}$. This means that, for the constant net $x_i := x'$, $f(x_i) \uparrow f(\hat{x})$ in the $\leq$-topology and so $x_i \to \hat{x}$, i.e., $x' = \hat{x}$. Thus $S(f, \leq, f(\hat{x})) = \{\hat{x}\}$. For (b), consider a net $\{x_i\}_{i\in\mathcal{I}} \subseteq S(f, \leq, f(\hat{x}))$. We have $f(x_i) \leq f(\hat{x}) + v$ for all $i \in \mathcal{I}$ and $v \in \mathcal{V}$, i.e., $f(x_i) \uparrow f(\hat{x})$ in the $\leq$-topology and so $\{x_i\}_{i\in\mathcal{I}}$ has a cluster point in $S(f, \leq, f(\hat{x}))$. □

In the sequel, we identify three classes of pointwise well-posed optimization problems. Consider the set-valued map $F : G \subset \mathcal{P} \to 2^X$. We recall that F is said to be *upper semicontinuous* at $y \in G$ if for every open set U containing $F(y)$, there exists $v \in \mathcal{V}$ such that $F(x) \subseteq U$ for all $x \in v(y) \cap G$.

Theorem 2.1 *Let $\hat{x}$ be a minimal solution of $(X, f, \leq)$. If the set-valued map $f^{-1} : f(X) \to 2^X$ is upper semicontinuous at $f(\hat{x})$ and $f^{-1}(f(\hat{x})) = \{\hat{x}\}$, then $(X, f, \leq)$ is well-posed at $\hat{x}$.*

Proof Let $\{x_i\}_{i\in\mathcal{I}} \subset X$ be a minimizing net for $(X, f, \leq)$ at $\hat{x}$. If $x_i \not\to \hat{x}$, then we can find some neighborhood U of $\hat{x}$ such that for every $i_0 \in \mathcal{I}$ there exists $i \geq i_0$ with $x_i \notin U$. By the upper semicontinuity of f^{-1} at $f(\hat{x})$, there exists $v \in \mathcal{V}$ such that $f^{-1}(v(f(\hat{x})) \cap f(X)) \subset U$. As $f(x_i) \uparrow f(\hat{x})$ in the $\leq$-topology, there is $i_0 \in \mathcal{I}$ such that $f(x_i) \in v(f(\hat{x}))$ for all $i \geq i_0$ which implies $x_i \in f^{-1}(v(f(\hat{x})) \cap f(X)) \subset U$ for all $i \geq i_0$ which is a contradiction. □

Remark 2.4 From the proof of Theorems 2.1, we see that the same assertions hold in the case that $\hat{x}$ is a minimal solution of $(X, f, \preceq_v)$ and $(X, f, \preceq)$.

Theorem 2.2 *If X is compact and f is continuous with respect to the original lower topology of $\mathcal{P}$, then $(X, f, \preceq)$ (respectively, $(X, f, \preceq_v)$) is generalized well-posed at each $\hat{x} \in \mathit{Eff}(X, f, \preceq)$ (respectively, $\hat{x} \in \mathit{Eff}(X, f, \preceq_v)$).*

Proof Let $\hat{x} \in \mathrm{Eff}(X, f, \preceq)$ and $v \in \mathcal{V}$. If $\{x_i\}_{i\in\mathcal{I}} \subseteq X$ such that $f(x_i) \uparrow f(\hat{x})$ in the $\preceq$-topology, then there exists $i_0 \in \mathcal{I}$ such that $f(x_i) \preceq f(\hat{x}) + \frac{v}{4}$ for all $i \geq i_0$, so $f(x_i) \leq f(\hat{x}) + \frac{v}{4} + \frac{v}{4} = f(\hat{x}) + \frac{v}{2}$ for all $i \geq i_0$. By compactness of X, there exist a subnet $\{x_j\}_{j\in\mathcal{J}}$ of $\{x_i\}_{i\in\mathcal{I}}$ and an element $u \in X$ such that $x_j \to u$. Since f is continuous in the original lower topology of $\mathcal{P}$, $f(x_j) \downarrow f(u)$ in the $\leq$-topology, so there exists $j_0 \in \mathcal{J}$ such that $f(u) \leq f(x_j) + \frac{v}{2}$ for all $j \geq j_0$. If we choose $j_1 \in \mathcal{J}$ with $j_1 \geq i_0, j_0$, then for all $j \geq j_1$ we have $f(u) \leq f(x_j) + \frac{v}{2}$ and $f(x_j) \leq f(\hat{x}) + \frac{v}{2}$ which imply that $f(u) \leq f(\hat{x}) + v$. Since v is arbitrary, $f(u) \preceq f(\hat{x})$ which means that $u \in S(f, \preceq, f(\hat{x}))$, i.e., $\{x_i\}_{i\in\mathcal{I}}$ has a cluster point in $S(f, \preceq, f(\hat{x}))$. Now let $\hat{x} \in \mathrm{Eff}(X, f, \preceq_v)$ for a fixed $v \in \mathcal{V}$, and consider a net $\{x_i\}_{i\in\mathcal{I}} \subseteq X$ such that $f(x_i) \uparrow f(\hat{x})$ in the $\preceq_v$-topology. For every $\rho > 0$, there exists $i_0 \in \mathcal{I}$ such that $f(x_i) \preceq_v f(\hat{x}) + \frac{\rho v}{4}$ for all $i \geq i_0$ which yields $f(x_i) \leq f(\hat{x}) + \frac{\rho v}{4} + \frac{\rho v}{4} = f(\hat{x}) + \frac{\rho v}{2}$ for all $i \geq i_0$. Similar to the first part, there exist a subnet $\{x_j\}_{j\in J}$ of $\{x_i\}_{i\in\mathcal{I}}$ and $u \in X$, $j_0 \in \mathcal{J}$ such that $x_j \to u$, $f(u) \leq f(x_j) + \frac{\rho v}{2}$ and $f(x_j) \leq f(\hat{x}) + \frac{\rho v}{2}$ for all $j \geq j_0$. Thus, $f(u) \leq f(\hat{x}) + \rho v$. Since ρ is arbitrary, $f(u) \preceq_v f(\hat{x})$ which means that $u \in S(f, \preceq_v, f(\hat{x}))$ and the proof is complete. □

The function $f : X \to \mathcal{P}$ is called *convex* on a non-empty convex set $S \subseteq X$ if $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ for all $x_1, x_2 \in S$ and $\lambda \in [0, 1]$.

Theorem 2.3 *Let $X \subseteq \mathbb{R}^n$ be a closed convex set and f is convex and continuous with respect to the original lower topology of $\mathcal{P}$. For a minimal solution $\hat{x}$ of $(X, f, \preceq)$ ($(X, f, \preceq_v)$) if $S(f, \preceq, f(\hat{x}))$ (respectively, $S(f, \preceq_v, f(\hat{x}))$) is a singleton then $(X, f, \preceq)$ (respectively, $(X, f, \preceq_v)$) is both the well-posed and the generalized well-posed at $\hat{x}$.*

Proof Let $\hat{x} \in \mathrm{Eff}(X, f, \preceq)$ and $S(f, \preceq, f(\hat{x})) = \{\hat{x}\}$. Denote by $\mathbb{B}$ the closed unit ball of $\mathbb{R}^n$. Consider a net $\{x_i\}_{i\in\mathcal{I}} \subseteq X$ such that $f(x_i) \uparrow f(\hat{x})$ in the $\preceq$-topology. By contradiction suppose that $\hat{x}$ is not a cluster point of $\{x_i\}_{i\in\mathcal{I}}$. There exist $\epsilon > 0$ and $i_0 \in \mathcal{I}$ such that $x_i \notin (\hat{x} + \epsilon\mathbb{B}) \cap X$ for all $i \geq i_0$. Since X is convex, we can find $\lambda_i \in (0, 1)$ for all $i \geq i_0$ such that $\lambda_i \hat{x} + (1-\lambda_i)x_i \in (\hat{x} + \epsilon\mathbb{B}) \cap (\hat{x} + \frac{\epsilon}{2}\mathbb{B})^c \cap X$. Since $\hat{x} + \epsilon\mathbb{B}$ is compact, there exists a subnet $\{\lambda_j \hat{x} + (1-\lambda_j)x_j\}_{j\in\mathcal{J}}$ of the $\{\lambda_i \hat{x} + (1-\lambda_i)x_i\}_{i\in\mathcal{I}, i\geq i_0}$ converging to an element $u \in \hat{x} + \epsilon\mathbb{B}$. We have $u \in X$, since X is closed. Since f is continuous with respect to the original lower topology of $\mathcal{P}$, for $v \in \mathcal{V}$ there exists $j_0 \in \mathcal{J}$ such that $f(u) \leq f(\lambda_j \hat{x} + (1-\lambda_j)x_j) + \frac{v}{2}$ for all $j \geq j_0$. Moreover, since $f(x_j) \uparrow f(\hat{x})$ in the $\preceq$-topology there exists $j_1 \in \mathcal{J}$ such that $f(x_j) \leq f(\hat{x}) + \frac{v}{2}$ and so $(1-\lambda_j) f(x_j) \leq (1-\lambda_j) f(\hat{x}) + \frac{v}{2}$ for all $j \geq j_1$. On the other hand, by convexity of f, we have

$$f(\lambda_j \hat{x} + (1-\lambda_j)x_j) \leq \lambda_j f(\hat{x}) + (1-\lambda_j) f(x_j)$$

for all j with $j \geq j_0$ and $j \geq j_1$, which yields

$$\begin{aligned} f(u) &\leq f(\lambda_j \hat{x} + (1-\lambda_j)x_j) + \frac{v}{2} \\ &\leq \lambda_j f(\hat{x}) + (1-\lambda_j) f(x_j) + \frac{v}{2} \\ &\leq \lambda_j f(\hat{x}) + (1-\lambda_j) f(\hat{x}) + \frac{v}{2} + \frac{v}{2} \\ &= f(\hat{x}) + v. \end{aligned}$$

i.e., $f(u) \preceq f(\hat{x})$, since v is arbitrary. By the assumption, $S(f, \preceq, f(\hat{x})) = \{\hat{x}\}$ so $u = \hat{x}$ which contradicts $\lambda_j \hat{x} + (1-\lambda_j)x_j \in (\hat{x} + \frac{\epsilon}{2}\mathbb{B})^c$, $j \in J$. In the similar way, for a fixed $v \in \mathcal{V}$, we prove the assertion, if $\hat{x} \in \mathrm{Eff}(X, f, \preceq_v)$ and $S(f, \preceq_v, f(\hat{x}))$ is a singleton. □

3 Scalarization functions and well-posedness

For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} : a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider $\overline{\mathbb{R}}_+$, with abstract neighborhood system of $\overline{\mathbb{R}}$, as a locally convex cone. According to [1], for every $v \in \mathcal{V}$ and $b \in \mathcal{P}$, the scalarization function $\varphi_{v,b} : \mathcal{P} \to \overline{\mathbb{R}}_+$ for all $a \in \mathcal{P}$ is defined by

$$\varphi_{v,b}(a) = \inf\{t > 0 \ : \ a \leq b + tv\}.$$

In this section, we give more results about the scalarization functions and prove some existence theorems for optimal solutions. We obtain some minimality conditions and characterize the solutions of optimization problems for locally convex cone-valued functions. Finally, we define the Tykhonov well-posed problems for $\overline{\mathbb{R}}_+$-valued functions and characterize the pointwise well-posed optimization problems.

Remark 3.1

(i) If $a \in \mathcal{P}$ is v-bounded then $\varphi_{v,b}(a) < +\infty$. Indeed, there is $\lambda > 0$ such that $a \leq \lambda v$. Since every element of $\mathcal{P}$ is bounded below, there is $\gamma > 0$ such that $0 \leq b + \gamma v$. Thus $a \leq b + (\lambda + \gamma)v$, hence $\{t > 0 : a \leq b + tv\} \neq \emptyset$, i.e., $\varphi_{v,b}(a) < +\infty$.

(ii) If b is v-bounded and $\varphi_{v,b}(a) < +\infty$, then a is v-bounded.

Theorem 3.1 *For every $v \in \mathcal{V}$ and $b \in \mathcal{P}$, the following hold:*

(a) *For every $\epsilon > 0$ and $x, y \in \mathcal{P}$, if $x \leq y + \epsilon v$ then $\varphi_{v,b}(x) \leq \varphi_{v,b}(y) + \epsilon$, i.e., $\varphi_{v,b}$ is u-continuous.*
(b) *$\varphi_{v,b}$ is monotone even with respect to the local preorder $\preceq_v$ of $\mathcal{P}$.*
(c) $\varphi_{v,b}(b) = 0$.
(d) *$x \preceq_v b$ if and only if $\varphi_{v,b}(x) = 0$.*

(e) *For every* $\alpha > 0$, $\varphi_{v,\alpha b}(\alpha x) = \alpha\varphi_{v,b}(x)$.
(f) $\varphi_{v,b}$ *is convex.*
(g) *If* $\mathcal{P}$ *is a full locally convex cone, then* $\varphi_{v,b}(x + rv) \leq \varphi_{v,b}(x) + r$ *for all* $r \geq 0$.
(h) $\varphi_{v,b}(a)$ *is monotonically decreasing with respect to* v, *i.e., if* $u \leq v$ *then* $\varphi_{v,b}(a) \leq \varphi_{u,b}(a)$ *for all* $a \in \mathcal{P}$.

Proof For (a)-(d), see [1, Theorem 3.2]. For (e), obviously, $\varphi_{v,b}(x) = +\infty$ if and only if $\varphi_{v,\alpha b}(\alpha x) = +\infty$. Now, let $t > 0$ such that $\alpha x \leq \alpha b + tv$, then $x \leq b + \frac{t}{\alpha}v$, so $\varphi_{v,b}(x) \leq \frac{1}{\alpha}\varphi_{v,\alpha b}(\alpha x)$ which implies that $\alpha\varphi_{v,b}(x) \leq \varphi_{v,\alpha b}(\alpha x)$. Similarly, we can see that $\varphi_{v,\alpha b}(\alpha x) \leq \alpha\varphi_{v,b}(x)$. Thus, $\varphi_{v,\alpha b}(\alpha x) = \alpha\varphi_{v,b}(x)$. For (f), let $x, y \in \mathcal{P}$ and $\lambda \in [0, 1]$. If $\varphi_{v,b}(x) = +\infty$ or $\varphi_{v,b}(y) = +\infty$ then

$$\varphi_{v,b}(\lambda x + (1 - \lambda)y) \leq \lambda\varphi_{v,b}(x) + (1 - \lambda)\varphi_{v,b}(y).$$

Suppose $\varphi_{v,b}(x) < +\infty$ and $\varphi_{v,b}(y) < +\infty$. For each $\epsilon > 0$ we have $x \leq b + (\varphi_{v,b}(x) + \epsilon)v$ and $y \leq b + (\varphi_{v,b}(y) + \epsilon)v$, hence

$$\lambda x + (1 - \lambda)y \leq b + (\lambda\varphi_{v,b}(x) + (1 - \lambda)\varphi_{v,b}(y) + \epsilon)v,$$

thus

$$\varphi_{v,b}(\lambda x + (1 - \lambda)y) \leq \lambda\varphi_{v,b}(x) + (1 - \lambda)\varphi_{v,b}(y).$$

For (g), if $\varphi_{v,b}(x) < +\infty$ and $r > 0$ then, for each $\epsilon > 0$ we have $x \leq b + (\varphi_{v,b}(x) + \epsilon)v$ which implies that $x + rv \leq b + (\varphi_{v,b}(x) + \epsilon + r)v$, hence $\varphi_{v,b}(x + rv) \leq \varphi_{v,b}(x) + r$. If $\varphi_{v,b}(x) = +\infty$, the inequality is obvious. For (h), if $\varphi_{u,b}(a) = +\infty$, the assertion is obvious. If $\varphi_{u,b}(a) < +\infty$ then, for every$\epsilon > 0$, $a \leq (\varphi_{u,b}(a) + \epsilon)u + b \leq (\varphi_{u,b}(a) + \epsilon)v + b$, so $\varphi_{u,b}(a) \leq \varphi_{v,b}(a) + \epsilon$. Thus $\varphi_{u,b}(a) \leq \varphi_{v,b}(a)$. □

For a non-empty set $A \subseteq \mathcal{P}$, $\overline{A}$ denotes the *closure* of A with respect to the original lower topology of $\mathcal{P}$. In particular, $\overline{a}$ is closure of the singleton set $\{a\}$ and $\overline{a} = \bigcap_{v \in \mathcal{V}} v(a) =\downarrow (\mathcal{P}, \preceq, a)$ [5, Ch I, 3.3].

Remark 3.2 For every $b \in \mathcal{P}$ and $v \in \mathcal{V}$, the set $\downarrow (\mathcal{P}, \preceq_v, b)$ is closed with respect to the original lower topology of $\mathcal{P}$. Indeed, the singleton $\{0\}$ is closed respect to the lower topology of $(\overline{\mathbb{R}}_+, \mathcal{V})$, $\mathcal{V} = \{\varepsilon : \varepsilon > 0\}$ and $\downarrow (\mathcal{P}, \preceq_v, b) = {\varphi_{v,b}}^{-1}(0)$. By Theorem 3.1 (a), $\varphi_{v,b} : \mathcal{P} \to \overline{\mathbb{R}}_+$ is u-continuous, so $\downarrow (\mathcal{P}, \preceq_v, b)$ is closed in original lower topology. Moreover, we have $\overline{b} =\downarrow (\mathcal{P}, \preceq, b) = \bigcap_{v \in \mathcal{V}} \downarrow (\mathcal{P}, \preceq_v, b)$.

Corollary 3.1 *If* X *is compact and* f *is continuous with respect to the original lower topology of* $\mathcal{P}$, *then* *Eff*$(X, f, \preceq_v) \neq \emptyset$ *for all* $v \in \mathcal{V}$.

Proof From Remark 3.2, for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$, $\downarrow (\mathcal{P}, \preceq_v, a)$ is closed with respect to original lower topology of $\mathcal{P}$, so continuity of f implies that the set $S(f, \preceq_v, f(x)) = f^{-1}(\downarrow (\mathcal{P}, \preceq_v, f(x)))$ is closed for all $x \in X$. Now, the assertion holds by [4, Theorem 5.1]. □

For $v \in \mathcal{V}$, if $\xi_v : \mathcal{P} \to \overline{\mathbb{R}}_+$ is defined by $\xi_v(x) = \inf\{t > 0 : x \leq tv\}$, then $\xi_v = \varphi_{v,0}$ and we have:

Proposition 3.1 *For every* $v \in \mathcal{V}$ *and* $x \in \mathcal{P}$, $\xi_v(x) = max\{\mu(x) : \mu \in v^\circ_{\mathcal{P}}\}$.

Proof If $\xi_v(x) < +\infty$ then, for each $\epsilon > 0$, we have $x \leq (\xi_v(x) + \epsilon)v$ so $\mu(x) \leq \xi_v(x) + \epsilon$ for all $\mu \in v^\circ_{\mathcal{P}}$ and $\epsilon > 0$. Thus

$$\xi_v(x) \geq \sup\{\mu(x) : \mu \in v^\circ_{\mathcal{P}}\}.$$

Obviously, the above inequality satisfies in the case that $\xi_v(x) = +\infty$. By Theorem 3.1 (e), $\xi_v(\alpha x) = \alpha\xi_v(x)$ for all $\alpha > 0$. For $x, y \in \mathcal{P}$, if $A = \{t > 0 : x \leq tv\}$ and $B = \{t > 0 : y \leq tv\}$, then for $t_1 \in A$ and $t_2 \in B$ we have $x + y \leq (t_1 + t_2)v$, so $\xi_v(x + y) \leq \inf(A + B) = \inf A + \inf B = \xi_v(x) + \xi_v(y)$, i.e., ξ_v is a sublinear functional on $\mathcal{P}$. Thus, by Theorem 3.1 (a) and [12, Corollary 3.3], the assertion holds. □

Note that if $(E, \mathcal{V})$ is indeed locally convex space with set inclusion as order, then for $V \in \mathcal{V}$, ξ_V is the well known Minkowski functional, i.e., $\xi_V(x) = \inf\{t > 0 : x \in tV\}$.

Definition 3.1 A net $\{x_i\}_{i\in\mathcal{I}} \subset \mathcal{P}$ is called *lower* (*upper*) *Cauchy* if for every $v \in \mathcal{V}$ there is $i_v \in \mathcal{I}$ such that $x_i \leq x_j + v$ (respectively, $x_j \leq x_i + v$) for all $i, j \in \mathcal{I}$ with $i \geq j \geq i_v$. Also $\{x_i\}_{i\in\mathcal{I}}$ is called *symmetric Cauchy* if it is lower and upper Cauchy, i.e., for each $v \in \mathcal{V}$ there is $i_v \in \mathcal{I}$ such that $x_i \leq x_j + v$ and $x_j \leq x_i + v$ for all $i, j \in \mathcal{I}$ with $i, j \geq i_v$. The locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *lower* (*upper* and *symmetric*) *complete* if every lower (respectively, upper and symmetric) Cauchy net converges in lower (respectively, upper and symmetric) topology. In general, a set $A \subset \mathcal{P}$ is called *lower* (*upper* and *symmetric*) *complete* if every lower (respectively, upper and symmetric) Cauchy net is convergent to an element of A in corresponding topology. For details see [6,9].

Remark 3.3 A net $\{x_i\}_{i\in\mathcal{I}} \subset \mathcal{P}$ is lower (upper) Cauchy if and only if for every $v \in \mathcal{V}$ there is some $i_v \in \mathcal{I}$ such that $\varphi_{v,x_i}(x_j) \leq 1$ (respectively, $\varphi_{v,x_j}(x_i) \leq 1$) for all i, j with $j \geq i \geq i_v$. Also $\{x_i\}_{i\in\mathcal{I}}$ is symmetric Cauchy if and only if for every $v \in \mathcal{V}$ there is some $i_v \in \mathcal{I}$ such that $\varphi^s_v(x_i, x_j) = max\{\varphi_{v,x_i}(x_j), \varphi_{v,x_j}(x_i)\} \leq 1$ for all i, j with $i, j \geq i_v$.

A subset $\mathcal{U}$ of $\mathcal{V}$ is called a base of $\mathcal{V}$ if for every $v \in \mathcal{V}$ there is $u \in \mathcal{U}$ and $\lambda > 0$ such that $\lambda u \leq v$. Clearly, for $a, b \in \mathcal{P}$, if $a \preceq_u b$ for all $u \in \mathcal{U}$ then $a \preceq b$.

Remark 3.4 If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone and $\mathcal{U}$ is a base for $\mathcal{V}$, then $\{x_i\}_{i\in\mathcal{I}} \subset \mathcal{P}$ is lower (upper and symmetric) Cauchy if for every $u \in \mathcal{U}$ and $\epsilon > 0$ there is some $i_{u,\epsilon} \in \mathcal{I}$ such that $\varphi_{u,x_i}(x_j) \leq \epsilon$ (respectively, $\varphi_{u,x_j}(x_i) \leq \epsilon$ and $\varphi_u(x_i, x_j) \leq \epsilon$) for all i, j with $j \geq i \geq i_{u,\epsilon}$; indeed, for $v \in \mathcal{V}$ there is $u \in \mathcal{U}$ and $\lambda > 0$ such that $\lambda u \leq v$. By the assumption we can choose $i_{u,\frac{1}{\lambda}} \in \mathcal{I}$ such that $\varphi_{u,x_i}(x_j) \leq \frac{1}{\lambda}$ for all i, j with $i, j \geq i_{u,\frac{1}{\lambda}}$. By Theorem 3.1 (h), we have $\varphi_{v,x_i}(x_j) \leq \varphi_{\lambda u,x_i}(x_j) = \lambda\varphi_{u,x_i}(x_j)$, so $\varphi_{v,x_i}(x_j) \leq 1$. Thus $\{x_i\}_{i\in\mathcal{I}}$ is lower Cauchy by Remark 3.3.

Theorem 3.2 *Let $(\mathcal{Q}, \mathcal{W})$ be separated with a countable base $\mathcal{U} = \{u_n : n \in \mathbb{N}\}$ for $\mathcal{W}$ and $X \subset \mathcal{Q}$ be symmetric complete. Let $f : X \to (\mathcal{P}, \mathcal{V})$ such that $f(x_0)$ is bounded for some $x_0 \in X$ and let for every $n \in \mathbb{N}$ there exists $v_n \in \mathcal{V}$ such that*

$$\varphi^s_{u_n}(x, y) \leq \xi_{v_n}(f(y)) - \xi_{v_n}(f(x))$$

for all $x, y \in S(f, \leq, f(x_0))$ with $f(x) \leq f(y)$. If for every $x \in S(f, \leq, f(x_0))$ the set $S(f, \leq, f(x))$ is closed with respect to the original lower topology of $\mathcal{Q}$, then $(X, f, \leq)$ has at least a minimal solution in $S(f, \leq, f(x_0))$.

Proof According to the property (v_2) of $\mathcal{V}$, we may assume that $v_{n+1} \leq v_n$ for all $n \in \mathbb{N}$. Since $f(x_0)$ is bounded, we have $0 \leq \xi_{v_n}(f(x_0)) < \infty$ for all $n \in \mathbb{N}$. We construct a sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq S(f, \leq, f(x_0))$ with initial point x_0 in the following way: Choose $x_n \in S(f, \leq, f(x_{n-1}))$ for all $n \in \mathbb{N}$ such that

$$\xi_{v_n}(f(x_n)) \leq \inf\{\xi_{v_n}(f(x)) : x \in S(f, \leq, f(x_{n-1}))\} + \frac{1}{2^n}.$$

We note that $f(x_n) \leq f(x_{n-1})$, so by the transitive property of $\leq$, we have $S(f, \leq, f(x_n)) \subseteq S(f, \leq, f(x_{n-1}))$ for all $n \in \mathbb{N}$. Thus for a fixed $n_0 \in \mathbb{N}$, $\{\xi_{v_{n_0}}(f(x_n))\}_{n\in\mathbb{N}}$ is a positive decreasing sequence by Theorem 3.1 (b), so it has a limit. On the other hand, $\varphi^s_{u_{n_0}}(x_n, x_m) \leq \xi_{v_{n_0}}(f(x_m)) - \xi_{v_{n_0}}(f(x_n))$ for all n, m with $n \geq m \geq n_0$, whence $\varphi^s_{u_{n_0}}(x_n, x_m) \to 0$ when $n > m \to \infty$, so $\{x_n\}_{n\in\mathbb{N}}$ is symmetric Cauchy by Remark 3.4 hence, by the symmetric completeness of X, there exists an element $\hat{x} \in X$ such that $x_n \to \hat{x}$ and so $x_n \downarrow \hat{x}$, since the symmetric topology is finer than the lower topology. By the assumption, $S(f, \leq, f(x_n))$ is closed in the lower topology, so $f(\hat{x}) \in S(f, \leq, f(x_n))$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, for every $x \in S(f, \leq, f(\hat{x}))$ and $n_0 \in \mathbb{N}$, we have

$$\begin{aligned}
&\xi_{v_{n_0}}(f(\hat{x})) - \xi_{v_{n_0}}(f(x)) \\
&\quad \leq \xi_{v_{n_0}}(f(x_n)) - \xi_{v_{n_0}}(f(x)) \\
&\quad \leq \xi_{v_{n_0}}(f(x_n)) - \inf\{\xi_{v_{n_0}}(f(x)) : x \in S(f, \leq, f(x_{n-1}))\} \\
&\quad \leq \frac{1}{2^n} \to 0,
\end{aligned}$$

when $n \to \infty$, which implies that $\xi_{v_{n_0}}(f(x)) = \xi_{v_{n_0}}(f(\hat{x}))$, hence $\varphi^s_{u_{n_0}}(x, \hat{x}) = 0$. Thus Theorem 3.1 (d) yields $x \preceq \hat{x}$ and $\hat{x} \preceq x$. Since $\mathcal{Q}$ is separated, we have $x = \hat{x}$. Thus $\hat{x}$ is a minimal solution of $(X, f, \leq)$. □

Theorem 3.3 *Suppose $(\mathcal{Q}, \mathcal{W})$ is a locally convex cone with the countable base $\mathcal{U} = \{u_n : n \in \mathbb{N}\}$ for $\mathcal{W}$, $X \subset \mathcal{Q}$ is symmetric complete and let $(\mathcal{P}, \mathcal{V})$ be separated and $f : X \to (\mathcal{P}, \mathcal{V})$ be a continuous map with respect to the lower topologies. Let $f(x_0)$ be bounded for some $x_0 \in X$ and let for every $n \in \mathbb{N}$ there exists $v_n \in \mathcal{V}$ such that*

$$\varphi^s_{u_n}(x, y) \leq \xi_{v_n}(f(y)) - \xi_{v_n}(f(x))$$

for all $x, y \in S(f, \preceq, f(x_0))$ with $f(x) \preceq f(y)$. Then $(X, f, \leq)$ has at least a minimal solution in $S(f, \preceq, f(x_0))$.

Proof Since f is continuous, $S(f, \preceq, f(x))$ is closed with respect to the lower topology for all $x \in S(f, \preceq, f(x_0))$. Thus the global preorder $\preceq$ satisfies in all assumptions of Theorem 3.2, hence we can find an element $\hat{x} \in S(f, \preceq, f(x_0))$ such that $x \preceq \hat{x}$ and $\hat{x} \preceq x$ for all $x \in S(f, \preceq, f(\hat{x}))$. On the other hand, continuity of f implies the monotonicity of f [5, Ch I, 3.11]. If $x \in S(f, \leq, f(\hat{x}))$ then $x \in S(f, \preceq, f(\hat{x}))$ which implies $\hat{x} \preceq x$. Thus, we have $f(\hat{x}) \preceq f(x)$. Since $(\mathcal{P}, \mathcal{V})$ is separated, $f(\hat{x}) = f(x)$ which means that $\hat{x}$ is a minimal solution of $(X, f, \leq)$. □

We say that $b \in G$ is *lower closed* on G if the set $\downarrow (G, \leq, b)$ is closed in G with respect to the original lower topology of $\mathcal{P}$. In this case, for $a \in G$, if $a \preceq b$ then for the constant net $x_i := b$ we have $x_i \downarrow a$, so $a \in \downarrow (G, \leq, b)$, i.e., $a \leq b$. This means $\downarrow (G, \preceq, b) = \downarrow (G, \leq, b)$. Here, we provide some minimality conditions for $G \subseteq \mathcal{P}$ via the scalarization function $\varphi_{v,b}$.

Theorem 3.4 *For $b \in G$ the following hold:*

(a) *$b \in Min(G, \leq)$ is lower closed on G if and only if for every $a \in G$ with $a \nsim b$ there is $v \in \mathcal{V}$ such that $\varphi_{v,b}(a) > 0$.*
(b) $b \in Min(G, \preceq_v)$ if and only if for every $a \in G$ with $a \not\approx_v b$, $\varphi_{v,b}(a) > 0$.
(c) *$b \in Min(G, \preceq)$ if and only if for every $a \in G$ and $v \in \mathcal{V}$, $a \not\approx_v b$ implies $\varphi_{v,b}(a) > 0$.*

Proof (a) Suppose $b \in \mathrm{Min}(G, \leq)$ is lower closed on G and for $a \in G$ let $a \nsim b$. If $\varphi_{v,b}(a) = 0$ for all $v \in \mathcal{V}$, by Theorem 3.1 (d), we deduce that $a \preccurlyeq b$ which implies $a \leq b$. Since $b \in \mathrm{Min}(G, \leq)$, we have $a \sim b$ which is a contradiction. Conversely, suppose that for every $a \in G$ with $a \nsim b$ there is $v \in \mathcal{V}$ such that $\varphi_{v,b}(a) > 0$. If $a \in G$ such that $a \preceq b$ and $a \nsim b$, then there is $v \in \mathcal{V}$ such that $\varphi_{v,b}(a) > 0$. By Theorem 3.1 (d), we have $\varphi_{v,b}(a) = 0$ which is a contradiction. Thus $a \sim b$, so $b \in \mathrm{Min}(G, \leq)$ and b is lower closed on G. For (b), let $b \in \mathrm{Min}(G, \preceq_v)$ and $a \in G$ such that $a \not\approx_v b$. If $\varphi_{v,b}(a) = 0$, by Theorem 3.1 (d), we deduce that $a \preceq_v b$ which implies that $b \preceq_v a$, so $a \approx_v b$ which is a contradiction. Conversely, for each $a \in G$ with $a \not\approx_v b$, suppose $\varphi_{v,b}(a) > 0$ and let $b \notin \mathrm{Min}(G \preceq_v)$. There exists $a \in G$ such that $a \preceq_v b$ and $a \not\approx_v b$, so by Theorem 3.1 (d), we have $\varphi_{v,b}(a) = 0$, which is a contradiction. Part (c) holds by Remark 2.1 and (b). □

Theorem 3.5 *An element $\hat{x} \in X$ is a minimal solution of $(X, f, \leq)$ and $f(\hat{x})$ is lower closed on $f(X)$ if and only if for every $x \in X$ with $f(x) \nsim f(\hat{x})$ there exists a $\preceq$-monotone function $T : f(X) \to \overline{\mathbb{R}}_+$ such that*

$$f(\hat{x}) \in \mathit{Eff}(f(X), T, \leq) \quad \text{and} \quad f(x) \notin \mathit{Eff}(f(X), T, \leq).$$

Proof Suppose $\hat{x} \in \mathrm{Eff}(X, f, \leq)$, $f(\hat{x})$ is lower closed on $f(X)$ and let $f(x) \nsim f(\hat{x})$ for some $x \in X$. By Theorem 3.4 (a), there is $v \in \mathcal{V}$ such that $\varphi_{v, f(\hat{x})}(f(x)) > 0$. If for every $x \in X$, we set $T(f(x)) = \varphi_{v, f(\hat{x})}(f(x))$ then, by Theorem 3.1 (b) and (c), T is $\preceq$-monotone and $\varphi_{v, f(\hat{x})}(f(\hat{x})) = 0$. Then $f(\hat{x})$ is a minimal solution of $(f(X), T, \leq)$ and $f(x) \notin \mathrm{Eff}(f(X), T, \leq)$, since $\varphi_{v, f(\hat{x})}(f(x)) > 0$. Conversely, let $x \in X$ such that $f(x) \preceq f(\hat{x})$. If $f(x) \nsim f(\hat{x})$, then there is a monotone function $T : f(X) \to \overline{\mathbb{R}}_+$ with $f(\hat{x}) \in \mathrm{Eff}(f(X), T, \leq)$ and $f(x) \notin \mathrm{Eff}(f(X), T, \leq)$. But $T(f(x)) \leq T(f(\hat{x}))$ which contradicts $f(x) \notin \mathrm{Eff}(f(X), T, \leq)$. Thus $f(x) \sim f(\hat{x})$, so $\hat{x}$ is a minimal solution of $(X, f, \leq)$ and $f(\hat{x})$ is lower closed on $f(X)$. □

Theorem 3.6 *An element $\hat{x} \in X$ is a minimal solution of $(X, f, \preceq_v)$ if and only if there exists $\preceq_v$-monotone function $T : f(X) \to \overline{\mathbb{R}}_+$ such that*

$$\mathit{Eff}(f(X), T, \leq) = \{f(x) : x \in X, f(x) \approx_v f(\hat{x})\}.$$

Proof Suppose $\hat{x} \in \mathrm{Eff}(X, f, \preceq_v)$ and set $T(x) = \varphi_{v, f(\hat{x})}(f(x))$ for all $x \in X$. By Theorem 3.1 (b) and (c), T is $\preceq_v$-monotone and $\varphi_{v, f(\hat{x})}(f(\hat{x})) = 0$. Thus, by Theorem 3.4 (b), $\mathrm{Eff}(f(X), T, \leq) = \{f(x) : x \in X, f(x) \approx_v f(\hat{x})\}$. For the converse, if $x \in X$ such that $f(x) \preceq_v f(\hat{x})$, then $T(f(x)) \leq T(f(\hat{x}))$. Since $f(\hat{x}) \in \mathrm{Eff}(f(X), T, \leq)$, we have $f(x) \in \mathrm{Eff}(f(X), T, \leq)$ which implies $f(x) \approx_v f(\hat{x})$. Therefore, $\hat{x}$ is a minimal solution of $(X, f, \preceq_v)$. □

Corollary 3.2 *An element $\hat{x} \in X$ is a minimal solution of $(X, f, \preceq)$ if and only if for every $x \in X$ that $f(x) \not\preceq_v f(\hat{x})$ for some $v \in \mathcal{V}$, there exists a $\preceq_v$-monotone function $T : f(X) \to \overline{\mathbb{R}}_+$ such that $f(\hat{x}) \in Eff(f(X), T, \leq)$ and $f(x) \notin Eff(f(X), T, \leq)$.*

Definition 3.2 Let $h : X \to (\overline{\mathbb{R}}_+, \mathcal{V})$ be a function, where $\mathcal{V} = \{\varepsilon \in \mathbb{R} : \varepsilon > 0\}$. Since the local and global preorders coinside to the original order on $\overline{\mathbb{R}}_+$, the *general Tykhonov optimization problem* for h is formalized by (X, h) Min $h(X)$, where $h(X) = \{h(x) : x \in X\}$. We note that, if $\inf_X h$ denotes the infimum of h on X and $\mathrm{Eff}(X, h) \neq \emptyset$, then $\mathrm{Min}(X, h) = \{\inf_X h\}$. We say that the minimization problem (X, h) is *Tykhonov well-posed* if $\mathrm{Eff}(X, h)$ is a singleton and every minimizing sequence (i.e., $\{x_n\}_{n\in\mathbb{N}} \subseteq X, h(x_n) \to \inf_X h$) converges to $\mathrm{Eff}(X, h)$. Also, it is called *generalized Tykhonov well-posed* if $\mathrm{Eff}(X, h) \neq \emptyset$ and for every minimizing sequence in X there exists a subsequence that converges to an element $\mathrm{Eff}(X, h)$ (cf. [3]).

Remark 3.5 If the Tykhonove optimization problems (X, h) is well-posed at $\bar{x}$, then $\mathrm{Eff}(X, h) = \{\hat{x}\}$ by Proposition 2.1 (a). Thus $f(x_n) \to \inf_X f$ in $\overline{\mathbb{R}}_+$, since $f(x_n) \uparrow \inf_X f$ in $\overline{\mathbb{R}}_+$. Similarly, the generalized well-posedness at $\hat{x} \in \mathrm{Eff}(X, h)$ reduces to the generalized Tykhonov well-posedness.

In the following we consider the sequential (generalized) well-posedness for problems $(X, f, .)$ and by the Tykhonov well-posedness, we characterize the pointwise well-posedness of the optimization problems.

Lemma 3.1 *If $x_0 \in X$ and $v \in \mathcal{V}$, then*

$$Eff(X, \varphi_{v, f(x_0)} \circ f) = S(f, \preceq_v, f(x_0)).$$

Proof Since $\varphi_{v, f(x_0)}(f(x_0)) = 0$, we have $\inf_X \varphi_{v, f(x_0)} \circ f = 0$ so Theorem 3.1 (d) implies $\mathrm{Eff}(X, \varphi_{v, f(x_0)} \circ f) = S(f, \preceq_v, f(x_0))$. □

Theorem 3.7 *If $\hat{x}$ is an optimal solution of $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$) and $(X, \varphi_{v, f(\hat{x})} o f)$ is Tykhonov well-posed for some $v \in \mathcal{V}$, then $(X, f, \leq)$ (respectively, $(X, f, \preceq_v)$ and $(X, f, \preceq)$) is well-posed at $\hat{x}$.*

Proof Since $(X, \varphi_{v, f(\hat{x})} o f)$ is Tykhonov well-posed, by Remark 3.3, we have $\mathrm{Eff}(X, \varphi_{v, f(\hat{x})} o f) = \{\hat{x}\}$. Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a minimizing sequence for $(X, f, \leq)$ at $\hat{x}$, i.e., $f(x_n) \uparrow f(\hat{x})$ in the $\leq$-topology. By Theorem 3.1, $\varphi_{v, f(\hat{x})}$ is u-continuous and thus $\varphi_{v, f(\hat{x})}(f(x_n)) \uparrow 0$ in $\overline{\mathbb{R}}_+$ which implies that $\varphi_{v, f(\hat{x})}(f(x_n)) \to 0$ in $\overline{\mathbb{R}}_+$. Since $(X, \varphi_{v, f(\hat{x})} o f)$ is Tykhonov well-posed, $\{x_n\}_{n\in\mathbb{N}}$ converges to $\hat{x}$. This means that $(X, f, \leq)$ is well-posed at $\hat{x}$. □

Theorem 3.8 *For $\hat{x} \in Eff(X, f, \preceq)$, if the scalar problems $(X, \varphi_{v, f(\hat{x})} o f)$ are generalized Tykhonov well-posed for all $v \in \mathcal{V}$, then $(X, f, \preceq)$ is generalized well-posed at $\hat{x}$.*

Proof Suppose $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $f(x_n) \uparrow f(\hat{x})$ in the $\preceq$-topology and let $v \in \mathcal{V}$. Since $\varphi_{v, f(\hat{x})}(f(\hat{x})) = 0$ and $\varphi_{v, f(\hat{x})}$ is u-continuous then $\varphi_{v, f(\hat{x})}(f(x_n)) \to 0$. By the assumption there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converging to an element $u \in S(f, \preceq_v, f(\hat{x}))$. Since $f(x_{n_k}) \uparrow f(\hat{x})$ and $(X, \varphi_{v, f(\hat{x})} o f)$ is generalized Tykhonov well-posed for all $v \in \mathcal{V}$, by the same argument used above, $u \in S(f, \preceq_v, f(\hat{x}))$ for all $v \in \mathcal{V}$ and so $u \in S(f, \preceq, f(\hat{x}))$. This means that $(X, f, \preceq)$ is generalized well-posed at $\hat{x}$. □

Theorem 3.9 *For $\hat{x} \in Eff(X, f, \preceq_v)$, if the scalar problem $(X, \varphi_{v, f(\hat{x})} \circ f)$ is generalized Tykhonov well-posed, then $(X, f, \preceq_v)$ is generalized well-posed at $\hat{x}$.*

Proof The proof is similar to that of previous theorem. □

The function $f : X \to \mathcal{P}$ is called *quasi-convex* on a non-empty convex set $S \subseteq X$, if for every $a \in \mathcal{P}$ and $v \in \mathcal{V} \cup \{0\}$, the set $\{x \in S : f(x) \leq a + v\}$ is convex.

Proposition 3.2 *If f is quasi-convex on the convex set X, then $\varphi_{v,b} \circ f$ is quasi-convex for all $b \in \mathcal{P}$.*

Proof Let $b \in \mathcal{P}$. We show that $S(\varphi_{v,b} \circ f, \leq, \alpha)$ is convex for all $\alpha \geq 0$. Since f is quasi-convex, for every $\epsilon > 0$, $S(f, \leq, b + (\alpha + \epsilon)v)$ is convex, so $S(f, \preceq_v, b + \alpha v) = \bigcap_{\epsilon>0} S(f, \leq, b + (\alpha + \epsilon)v)$ is convex. On the other hand, by Theorem 3.1 (d), we have $S(\varphi_{v,b} \circ f, \leq, \alpha) = S(\varphi_{v,b+\alpha v} \circ f, \leq, 0) = S(f, \preceq_v, b + \alpha v)$. Therefore $S(\varphi_{v,b} \circ f, \leq, \alpha)$ is convex and the proof is complete. □

Theorem 3.10 *Suppose (Y, d) is a locally compact metric space and let $X \subseteq Y$ be closed and convex. Assume that f is quasi-convex on X and is continuous with respect to the original lower topology of $\mathcal{P}$. Then $(X, f, \preccurlyeq_v)$ and $(X, f, \preceq)$ are generalized well-posed at each of their local and global solutions.*

Proof Suppose $\hat{x} \in \text{Eff}(X, f, \preceq)$ and let $v \in \mathcal{V}$. The function $\varphi_{v, f(\hat{x})} o f$ is continuous with respect to the original lower topology of $\mathcal{P}$, so $S(\varphi_{v, f(\hat{x})} o f, \leq, \alpha)$ is closed for all $\alpha \geq 0$. Thus $\varphi_{v, f(\hat{x})} o f$ is lower semicontinuous in usual sense. Also, by Proposition 3.2, $\varphi_{v, f(\hat{x})} o f$ is quasi-convex for all $v \in \mathcal{V}$; so, by [2, Theorem 2.1], $(X, \varphi_{v, f(\hat{x})} o f)$ is generalized Tykhonov well-posed for all $v \in \mathcal{V}$. Now, applying Theorem 3.8 we obtain the result. In the case $\hat{x} \in \text{Eff}(X, f, \preceq_v)$, we go on in a similar way and apply Theorem 3.9. $\square$

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