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ORIGINAL RESEARCH

Diophantine approximation by Piatetski-Shapiro primes

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Abstract Let $[\,\cdot\,]$ be the floor function. In this paper we show that whenever η is real, the constants λ_i satisfy some necessary conditions, then for any fixed $1 < c < 38/37$ there exist infinitely many prime triples $p_1,\ p_2,\ p_3$ satisfying the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{\frac{37c-38}{26c}} (\log \max p_j)^{10}$$

and such that $p_i = [n_i^c]$, $i = 1,\ 2,\ 3$.

Keywords Diophantine approximation · Piatetski-Shapiro primes

Mathematics Subject Classification 11D75 · 11P32

1 Introduction and statement of the result

Let $\mathbb{P}$ denotes the set of all prime numbers. In 1953 Piatetski-Shapiro [6] showed that for any fixed $\gamma \in (11/12, 1)$ the set

$$\mathbb{P}_\gamma = \{p \in \mathbb{P} \ \mid \ p = [n^{1/\gamma}] \quad \text{for some } n \in \mathbb{N}\}$$

is infinite. The prime numbers of the form $p = [n^{1/\gamma}]$ are called Piatetski-Shapiro primes of type γ. Subsequently the interval for γ was sharpened many times and the best result up to now belongs to Rivat and Wu [7] for $\gamma \in (205/243, 1)$.

Twenty years later Vaughan [9] proved that whenever $\delta > 0$, η is real and constants λ_i satisfy some conditions, there are infinitely many prime triples $p_1,\ p_2,\ p_3$ such that

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi+\delta} \tag{1}$$

for $\xi = 1/10$. Latter the upper bound for ξ was improved several times and the best result up to now is due to K. Matomäki [5] with $\xi = 2/9$. In relation to solvability of inequality (1) with prime numbers of a special form we find papers by the author and Todorova [3] and the author [1]. In order to establish our result we solve the inequality (1) with Piatetski-Shapiro primes. Thus we prove the following theorem.

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Theorem 1 *Suppose that $\lambda_1, \lambda_2, \lambda_3$ are non-zero real numbers, not all of the same sign, η is real, λ_1/λ_2 is irrational and γ be fixed with $37/38 < \gamma < 1$. Then there exist infinitely many ordered triples of Piatetski-Shapiro primes $p_1,\ p_2,\ p_3$ of type γ such that*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{\frac{37-38\gamma}{26}} (\log \max p_j)^{10}\,.$$

2 Notations

The letter p will always denote prime number. As usual $[t]$ and $\{t\}$ denote the integer part of t and the fractional part of t. Moreover $\psi(t) = \{t\} - 1/2$. We write $e(t)$=$\exp(2\pi i t)$. Let γ be a real constant such that $37/38 < \gamma < 1$. Since λ_1/λ_2 is irrational, there are infinitely many different convergents a_0/q_0 to its continued fraction, with

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a_0}{q_0}\right| < \frac{1}{q_0^2}\,, \quad (a_0, q_0) = 1\,, \quad a_0 \neq 0 \tag{2}$$

and q_0 is arbitrary large. Denote

$$X = q_0^{13/6}\,; \tag{3}$$

$$\Delta = X^{-\frac{12}{13}} \log X\,; \tag{4}$$

$$\varepsilon = X^{\frac{37-38\gamma}{26}} \log^{10} X\,; \tag{5}$$

$$H = \frac{\log^2 X}{\varepsilon}\,; \tag{6}$$

$$S(\alpha, X) = \sum_{\substack{\lambda_0 X < p \le X \\ p \in \mathbb{P}_\gamma}} p^{1-\gamma} e(\alpha p) \log p\,, \quad 0 < \lambda_0 < 1\,; \tag{7}$$

$$\Sigma(\alpha, X) = \gamma \sum_{\lambda_0 X < p \le X} e(\alpha p) \log p\,; \tag{8}$$

$$\Omega(\alpha, X) = \sum_{\lambda_0 X < p \le X} p^{1-\gamma} \big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) e(\alpha p) \log p\,; \tag{9}$$

$$I(\alpha, X) = \gamma \int\limits_{\lambda_0 X}^{X} e(\alpha y) dy\,. \tag{10}$$

3 Preliminary lemmas

Lemma 1 *Let $\varepsilon > 0$ and $k \in \mathbb{N}$. There exists a function $\theta(y)$ which is k times continuously differentiable and such that*

$$\begin{aligned} &\theta(y) = 1 && \textit{for} \quad |y| \le 3\varepsilon/4\,;\\ &0 < \theta(y) < 1 && \textit{for} \quad 3\varepsilon/4 < |y| < \varepsilon\,;\\ &\theta(y) = 0 && \textit{for} \quad |y| \ge \varepsilon\,. \end{aligned}$$

and its Fourier transform

$$\Theta(x) = \int\limits_{-\infty}^{\infty} \theta(y) e(-xy) dy$$

satisfies the inequality

$$|\Theta(x)| \le \min\left(\frac{7\varepsilon}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{k}{2\pi|x|\varepsilon/8}\right)^k\right).$$

Proof See (Lemma 1, [8]). □

Lemma 2 *Let* $|\alpha| \le \Delta$. *Then for the sum denoted by* (8) *and the integral denoted by (10) the asymptotic formula*

$$\Sigma(\alpha, X) = I(\alpha, X) + \mathcal{O}\left(\frac{X}{e^{(\log X)^{1/5}}}\right)$$

holds.

Proof This lemma is very similar to result of Tolev [8]. Inspecting the arguments presented in ([8], Lemma 14), the reader will easily see that the proof of Lemma 2 can be obtained by the same way. □

Lemma 3 *For the sum denoted by* (9) *the upper bound*

$$\Omega(\alpha, X) \ll X^{\frac{37-12\gamma}{26}} \log^5 X\,.$$

holds.

Proof It follows by the same argument used in ([2], (36)). □

Lemma 4 *Suppose that* $\alpha \in \mathbb{R},\ a \in \mathbb{Z},\ q \in \mathbb{N},\ \left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2},\ (a, q) = 1.$

Let

$$\Psi(X) = \sum_{p \le X} e(\alpha p) \log p\,.$$

Then

$$\Psi(X) \ll \left(Xq^{-1/2} + X^{4/5} + X^{1/2}q^{1/2}\right) \log^4 X\,.$$

Proof See ([4], Theorem 13.6). □

4 Outline of the proof

Consider the sum

$$\Gamma(X) = \sum_{\substack{\lambda_0 X < p_1, p_2, p_3 \le X \\ p_i \in \mathbb{P}_\gamma,\, i=1,2,3}} \theta(\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta) p_1^{1-\gamma} p_2^{1-\gamma} p_3^{1-\gamma} \log p_1 \log p_2 \log p_3\,. \tag{11}$$

Using the inverse Fourier transform for the function $\theta(x)$ we get

$$\begin{aligned}\Gamma(X) &= \sum_{\substack{\lambda_0 X < p_1, p_2, p_3 \le X \\ p_i \in \mathbb{P}_\gamma,\, i=1,2,3}} p_1^{1-\gamma} p_2^{1-\gamma} p_3^{1-\gamma} \log p_1 \log p_2 \log p_3 \\ &\quad \times \int\limits_{-\infty}^{\infty} \Theta(t) e\big((\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta)t\big)\, dt \\ &= \int\limits_{-\infty}^{\infty} \Theta(t) S(\lambda_1 t, X) S(\lambda_2 t, X) S(\lambda_3 t, X) e(\eta t)\, dt\,.\end{aligned}$$

We decompose $\Gamma(X)$ as follows

$$\Gamma(X) = \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X)\,, \tag{12}$$

where

$$\Gamma_1(X) = \int\limits_{|t|<\Delta} \Theta(t) S(\lambda_1 t, X) S(\lambda_2 t, X) S(\lambda_3 t, X) e(\eta t)\, dt\,, \tag{13}$$

$$\Gamma_2(X) = \int\limits_{\Delta \le |t| \le H} \Theta(t) S(\lambda_1 t, X) S(\lambda_2 t, X) S(\lambda_3 t, X) e(\eta t)\, dt\,, \tag{14}$$

$$\Gamma_3(X) = \int\limits_{|t| > H} \Theta(t) S(\lambda_1 t, X) S(\lambda_2 t, X) S(\lambda_3 t, X) e(\eta t)\, dt\,. \tag{15}$$

We shall estimate $\Gamma_1(X)$, $\Gamma_2(X)$ and $\Gamma_3(X)$, respectively, in the sections 5, 6 and 7 . In section 8 we shall complete the proof of Theorem 1.

5 Lower bound of $\Gamma_1(X)$

In order to find the lower bound of $\Gamma_1(X)$ we need to prove the following two lemmas.

Lemma 5 *For the sum denoted by* (7) *and the integral denoted by* (10) *the asymptotic formula*

$$S(\alpha, X) = I(\alpha, X) + \mathcal{O}\left(\frac{X}{e^{(\log X)^{1/5}}}\right) \tag{16}$$

holds.

Proof From (7)–(9) we have

$$\begin{aligned} S(\alpha, X) &= \sum_{\lambda_0 X < p \le X} p^{1-\gamma}\big([-p^\gamma] - [-(p+1)^\gamma]\big) e(\alpha p) \log p \\ &= \sum_{\lambda_0 X < p \le X} p^{1-\gamma}\big((p+1)^\gamma - p^\gamma\big) e(\alpha p) \log p \\ &\quad + \sum_{\lambda_0 X < p \le X} p^{1-\gamma}\big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) e(\alpha p) \log p \\ &= \Sigma(\alpha, X) + \Omega(\alpha, X) + \mathcal{O}\big(\log X\big)\,. \end{aligned} \tag{17}$$

Bearing in mind (17), Lemmas 2 and 3 we obtain the asymptotic formula (16). □

Lemma 6 *Let $\lambda \ne 0$. Then for the sum denoted by (7) and the integral denoted by (10) we have*

$$\text{(i)} \qquad \int\limits_{-\Delta}^{\Delta} |S(\lambda\alpha, X)|^2\, d\alpha \ll X \log^3 X\,,$$

$$\text{(ii)} \qquad \int\limits_{-\Delta}^{\Delta} |I(\lambda\alpha)|^2\, d\alpha \ll X \log X\,,$$

$$\text{(iii)} \qquad \int\limits_{0}^{1} |S(\alpha, X)|^2\, d\alpha \ll X^{2-\gamma} \log^2 X\,.$$

Proof We only prove (i). The inequalities (ii) and (iii) can be proved likewise.

Using (4), (7) and Lagrange's mean value theorem we obtain

$$\begin{aligned} \int\limits_{-\Delta}^{\Delta} |S(\lambda\alpha, X)|^2\, d\alpha &= \sum_{\substack{\lambda_0 X < p_1, p_2 \le X \\ p_i \in \mathbb{P}_\gamma,\, i=1,2}} (p_1 p_2)^{1-\gamma} \log p_1 \log p_2 \int\limits_{-\Delta}^{\Delta} e(\lambda(p_1 - p_2)\alpha) d\alpha \\ &\ll X^{2-2\gamma} (\log X)^2 \sum_{\substack{\lambda_0 X < n_1, n_2 \le X \\ n_i = m_i^{1/\gamma},\, i=1,2}} \min\left(\Delta, \frac{1}{|n_1 - n_2|}\right) \end{aligned}$$

$$\ll \Delta X^{2-\gamma}\log^2 X + X^{2-2\gamma}(\log X)^2 \sum_{\substack{\lambda_0 X<n_1,n_2\le X\\ n_i=m_i^{1/\gamma},\, i=1,2\\ n_1<n_2}} \frac{1}{n_2-n_1}$$

$$\ll \Delta X^{2-\gamma}\log^2 X + X^{2-2\gamma}(\log X)^2 \sum_{\substack{(\lambda_0 X)^\gamma<m_1,m_2\le X^\gamma\\ m_1<m_2}} \frac{1}{m_2^{1/\gamma}-m_1^{1/\gamma}-1}$$

$$\ll \Delta X^{2-\gamma}\log^2 X + X^{1-\gamma}(\log X)^2 \sum_{\substack{(\lambda_0 X)^\gamma<m_1,m_2\le X^\gamma\\ m_1<m_2}} \frac{1}{m_2-m_1}$$

$$\ll \Delta X^{2-\gamma}\log^2 X + X\log^3 X\,.$$

$$\ll X\log^3 X\,.$$

The lemma is proved. □

Put

$$S_i = S(\lambda_i t, X),$$
$$I_i = I(\lambda_i t, X).$$

We use the identity

$$S_1S_2S_3 = I_1I_2I_3 + (S_1-I_1)I_2I_3 + S_1(S_2-I_2)I_3 + S_1S_2(S_3-I_3)\,. \tag{18}$$

Replace

$$J(X) = \int\limits_{|t|<\Delta} \Theta(t)I(\lambda_1 t, X)I(\lambda_2 t, X)I(\lambda_3 t, X)e(\eta t)\,dt\,. \tag{19}$$

Now from (13), (18), (19), Lemmas 1, 5 and 6 it follows

$$\begin{aligned}
\Gamma_1(X) - J(X) &= \int\limits_{|t|<\Delta} \Theta(t)\Big(S(\lambda_1 t, X) - I(\lambda_1 t, X)\Big)I(\lambda_2 t, X)I(\lambda_3 t, X)e(\eta t)\,dt\\
&\quad + \int\limits_{|t|<\Delta} \Theta(t)S(\lambda_1 t, X)\Big(S(\lambda_2 t, X) - I(\lambda_2 t, X)\Big)I(\lambda_3 t, X)e(\eta t)\,dt\\
&\quad + \int\limits_{|t|<\Delta} \Theta(t)S(\lambda_1 t, X)S(\lambda_2 t, X)\Big(S(\lambda_3 t, X) - I(\lambda_3 t, X)\Big)e(\eta t)\,dt\\
&\ll \varepsilon\frac{X}{e^{(\log X)^{1/5}}}\Bigg(\int\limits_{|t|<\Delta} |I(\lambda_2 t, X)I(\lambda_3 t, X)|\,dt\\
&\quad + \int\limits_{|t|<\Delta} |S(\lambda_1 t, X)I(\lambda_3 t, X)|dt + \int\limits_{|t|<\Delta} |S(\lambda_1 t, X)S(\lambda_2 t, X)|dt\Bigg)\\
&\ll \varepsilon\frac{X}{e^{(\log X)^{1/5}}}\Bigg(\int\limits_{|t|<\Delta} |I(\lambda_2 t, X)|^2dt + \int\limits_{|t|<\Delta} |I(\lambda_3 t, X)|^2dt\\
&\quad + \int\limits_{|t|<\Delta} |S(\lambda_1 t, X)|^2dt + \int\limits_{|t|<\Delta} |S(\lambda_2 t, X)|^2dt\Bigg)\\
&\ll \varepsilon\frac{X^2}{e^{(\log X)^{1/6}}}\,.
\end{aligned} \tag{20}$$

On the other hand for the integral defined by (19) we write

$$J(X) = B(X) + \Phi\,, \tag{21}$$

where

$$B(X) = \gamma^3 \int\limits_{\lambda_0 X}^{X} \int\limits_{\lambda_0 X}^{X} \int\limits_{\lambda_0 X}^{X} \theta(\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 + \eta)\, dy_1\, dy_2\, dy_3$$

and

$$\Phi \ll \int\limits_{\Delta}^{\infty} |\Theta(t)||I(\lambda_1 t, X) I(\lambda_2 t, X) I(\lambda_3 t, X)|\, dt\,. \tag{22}$$

According to ([3], Lemma 4) we have

$$B(X) \gg \varepsilon X^2\,. \tag{23}$$

By (10) we get

$$I(\alpha, X) \ll \frac{1}{|\alpha|}\,. \tag{24}$$

Using (22), (24) and Lemma 1 we deduce

$$\Phi \ll \frac{\varepsilon}{\Delta^2}\,. \tag{25}$$

Bearing in mind (4), (20), (21), (23) and (25) we obtain

$$\Gamma_1(X) \gg \varepsilon X^2\,. \tag{26}$$

6 Upper bound of $\Gamma_2(X)$

Suppose that

$$\left|\alpha - \frac{a}{q}\right| \le \frac{1}{q^2}\,, \quad (a, q) = 1 \tag{27}$$

with

$$q \in \left[X^{\frac{1}{13}}, X^{\frac{12}{13}}\right]\,. \tag{28}$$

Then (8), (27), (28) and Lemma 4 yield

$$\Sigma(\alpha,\ X) \ll X^{\frac{25}{26}} \log^4 X\,. \tag{29}$$

Now (17), (29) and Lemma 3 give us

$$S(\alpha,\ X) \ll X^{\frac{37-12\gamma}{26}} \log^5 X\,. \tag{30}$$

Let

$$\mathfrak{S}(t, X) = \min\left\{|S(\lambda_1 t,\ X)|\,, |S(\lambda_2 t,\ X)|\right\}\,. \tag{31}$$

We shall prove the following lemma.

Lemma 7 *Let* $t,\ X,\ \lambda_1,\ \lambda_2 \in \mathbb{R}$,

$$\Delta \le |t| \le H\,, \tag{32}$$

where Δ *and* H *are denoted by (4) and (6),* $\lambda_1/\lambda_2 \in \mathbb{R}\backslash\mathbb{Q}$ *and* $\mathfrak{S}(t, X)$ *is defined by (31). Then there exists a sequence of real numbers* $X_1,\ X_2, \ldots \to \infty$ *such that*

$$\mathfrak{S}(t, X_j) \ll X_j^{\frac{37-12\gamma}{26}} \log^5 X_j\,, \quad j = 1, 2, \ldots\,.$$

Proof Our aim is to prove that there exists a sequence $X_1,\ X_2,\ \ldots \to \infty$ such that for each $j = 1, 2, \ldots$ at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ with t, subject to (32) can be approximated by rational numbers with denominators, satisfying (28). Then the proof follows from (30) and (31).

Since $\lambda_1, \lambda_2, \lambda_3$ are not all of the same sign one can assume that $\lambda_1 > 0,\ \lambda_2 > 0$ and $\lambda_3 < 0$. Let us notice that there exist $a_1,\ q_1 \in \mathbb{Z}$, such that

$$\left|\lambda_1 t - \frac{a_1}{q_1}\right| < \frac{1}{q_1 q_0^2}\,, \quad (a_1,\ q_1) = 1, \qquad 1 \le q_1 \le q_0^2, \quad a_1 \ne 0\,. \tag{33}$$

From Dirichlet's approximation theorem it follows the existence of integers a_1 and q_1, satisfying the first three conditions. If $a_1 = 0$ then

$$|\lambda_1 t| < \frac{1}{q_1 q_0^2}$$

and (32) gives us

$$\lambda_1 \Delta < \lambda_1 |t| < \frac{1}{q_0^2}\,, \qquad q_0^2 < \frac{1}{\lambda_1 \Delta}\,.$$

The last inequality, (3) and (4) yield

$$X^{\frac{12}{13}} < \frac{X^{\frac{12}{13}}}{\lambda_1 \log X}\,,$$

which is impossible for large X. Therefore $a_1 \ne 0$. By analogy there exist $a_2,\ q_2 \in \mathbb{Z}$, such that

$$\left|\lambda_2 t - \frac{a_2}{q_2}\right| < \frac{1}{q_2 q_0^2}\,, \qquad (a_2,\ q_2) = 1, \qquad 1 \le q_2 \le q_0^2, \qquad a_2 \ne 0\,. \tag{34}$$

If $q_i \in \left[X^{\frac{1}{13}},\ X^{\frac{12}{13}}\right]$ for $i = 1$ or $i = 2$, then the proof is completed. By (3), (33) and (34) we deduce

$$q_i \le X^{\frac{12}{13}} = q_0^2\,, \quad i = 1, 2\,.$$

It remains to show that the case $q_i < X^{\frac{1}{13}}$, $i = 1, 2$ is impossible. Assume that

$$q_i < X^{\frac{1}{13}}, \quad i = 1, 2\,. \tag{35}$$

From (5), (6), (32)–(35) it follows

$$\begin{aligned} 1 \le |a_i| &< \frac{1}{q_0^2} + q_i \lambda_i |t| < \frac{1}{q_0^2} + q_i \lambda_i H\,, \\ 1 \le |a_i| &< \frac{1}{q_0^2} + \lambda_i X^{\frac{38\gamma-35}{26}} (\log X)^{-8}, \quad i = 1,\ 2\,. \end{aligned} \tag{36}$$

We have

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 t}{\lambda_2 t} = \frac{\frac{a_1}{q_1} + \left(\lambda_1 t - \frac{a_1}{q_1}\right)}{\frac{a_2}{q_2} + \left(\lambda_2 t - \frac{a_2}{q_2}\right)} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathfrak{X}_1}{1 + \mathfrak{X}_2}, \tag{37}$$

where

$$\mathfrak{X}_i = \frac{q_i}{a_i}\left(\lambda_i t - \frac{a_i}{q_i}\right),\ i = 1,\ 2. \tag{38}$$

Bearing in mind (33), (34), (37) and (38) we get

$$|\mathfrak{X}_i| < \frac{q_i}{|a_i|} \cdot \frac{1}{q_i q_0^2} = \frac{1}{|a_i| q_0^2} \le \frac{1}{q_0^2}\,, \quad i = 1, 2\,,$$

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} \cdot \frac{1 + \mathcal{O}\left(\frac{1}{q_0^2}\right)}{1 + \mathcal{O}\left(\frac{1}{q_0^2}\right)} = \frac{a_1 q_2}{a_2 q_1}\left(1 + \mathcal{O}\left(\frac{1}{q_0^2}\right)\right).$$

Thus

$$\frac{a_1 q_2}{a_2 q_1} = \mathcal{O}(1)$$

and

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + \mathcal{O}\left(\frac{1}{q_0^2}\right). \tag{39}$$

Therefore, both fractions $\frac{a_0}{q_0}$ and $\frac{a_1 q_2}{a_2 q_1}$ approximate $\frac{\lambda_1}{\lambda_2}$. Using (3), (33), (35) and inequality (36) with $i = 2$ we obtain

$$|a_2| q_1 < 1 + \lambda_2 X^{\frac{38\gamma - 33}{26}} (\log X)^{-8} < \frac{q_0}{\log X}\,. \tag{40}$$

Consequently $|a_2| q_1 \ne q_0$ and $\frac{a_0}{q_0} \ne \frac{a_1 q_2}{a_2 q_1}$. Now (40) implies

$$\left|\frac{a_0}{q_0} - \frac{a_1 q_2}{a_2 q_1}\right| = \frac{|a_0 a_2 q_1 - a_1 q_2 q_0|}{|a_2| q_1 q_0} \ge \frac{1}{|a_2| q_1 q_0} > \frac{\log X}{q_0^2}\,. \tag{41}$$

On the other hand, from (2) and (39) we deduce

$$\left|\frac{a_0}{q_0} - \frac{a_1 q_2}{a_2 q_1}\right| \le \left|\frac{a_0}{q_0} - \frac{\lambda_1}{\lambda_2}\right| + \left|\frac{\lambda_1}{\lambda_2} - \frac{a_1 q_2}{a_2 q_1}\right| \ll \frac{1}{q_0^2}\,,$$

which contradicts (41). This rejects the assumption (35). Let $q_0^{(1)},\ q_0^{(2)},\ \ldots$ be an infinite sequence of values of q_0, satisfying (2). Then using (3) one gets an infinite sequence $X_1,\ X_2,\ \ldots$ of values of X, such that at least one of the numbers $\lambda_1 t$ and $\lambda_2 t$ can be approximated by rational numbers with denominators, satisfying (28). Hence, the proof is completed. □

Taking into account (14), (31), Lemmas 1 and 7 we deduce

$$\begin{aligned}
\Gamma_2(X_j) &\ll \varepsilon \int\limits_{\Delta \le |t| \le H} \mathfrak{S}(t, X_j)\Big(\big|S(\lambda_1 t, X_j) S(\lambda_3 t, X_j)\big| + \big|S(\lambda_2 t, X_j) S(\lambda_3 t, X_j)\big|\Big)\, dt \\
&\ll \varepsilon \int\limits_{\Delta \le |t| \le H} \mathfrak{S}(t, X_j)\Big(\big|S(\lambda_1 t, X_j)\big|^2 + \big|S(\lambda_2 t, X_j)\big|^2 + \big|S(\lambda_3 t, X_j)\big|^2\Big)\, dt \\
&\ll \varepsilon X_j^{\frac{37 - 12\gamma}{26}} (\log X_j)^5 T_k\,,
\end{aligned} \tag{42}$$

where

$$T_k = \int\limits_{\Delta}^{H} \left|S(\lambda_k t, X_j)\right|^2 dt.$$

Using Lemma 6 (iii) and working as in ([3], pp. 17–18) we obtain

$$T_k \ll H X_j^{2-\gamma} \log^2 X_j \,. \tag{43}$$

From (5), (6), (42), (43) we get

$$\Gamma_2(X_j) \ll X_j^{\frac{37-12\gamma}{26}} X_j^{2-\gamma} \log^9 X_j \ll X_j^{\frac{89-38\gamma}{26}} \log^9 X_j \ll \frac{\varepsilon X_j^2}{\log X_j} \,. \tag{44}$$

7 Upper bound of $\Gamma_3(X)$

By (7), (15) and Lemma 1 it follows

$$\Gamma_3(X) \ll X^{3-3\gamma} \int\limits_{H}^{\infty} \frac{1}{t}\left(\frac{k}{2\pi t\varepsilon/8}\right)^k dt = \frac{X^{3-3\gamma}}{k}\left(\frac{4k}{\pi\varepsilon H}\right)^k . \tag{45}$$

Choosing $k = [\log X]$ from (6) and (45) we obtain

$$\Gamma_3(X) \ll 1 \,. \tag{46}$$

8 Proof of the Theorem

Summarizing (5), (12), (26), (44) and (46) we deduce

$$\Gamma(X_j) \gg \varepsilon X_j^2 = X_j^{\frac{89-38\gamma}{26}} \log^{10} X_j \,.$$

The last estimation implies

$$\Gamma(X_j) \to \infty \quad \text{as} \quad X_j \to \infty \,. \tag{47}$$

Bearing in mind (11) and (47) we establish Theorem 1.

References

1. S. I. Dimitrov, *Diophantine approximation by special primes*, Appl. Math. in Eng. and Econ. – 44th. Int. Conf., AIP Conf. Proc., **2048**, 050005, (2018).
2. S. I. Dimitrov, *On the distribution of αp modulo one over Piatetski-Shapiro primes*, arXiv:2005.05008v1 [math.NT] 11 May 2020.
3. S. Dimitrov, T. Todorova, *Diophantine approximation by prime numbers of a special form*, Annuaire Univ. Sofia, Fac. Math. Inform., **102**, (2015), 71–90.
4. H. Iwaniec, E. Kowalski, *Analytic number theory*, Colloquium Publications, **53**, Amer. Math. Soc., (2004).
5. K. Matomäki, *Diophantine approximation by primes*, Glasgow Math. J., **52**, (2010), 87–106.
6. I. I. Piatetski-Shapiro, *On the distribution of prime numbers in sequences of the form $[f(n)]$*, Mat. Sb., **33**, (1953), 559–566.
7. J. Rivat, J. Wu, *Prime numbers of the form $[n^c]$*, Glasg. Math. J, **43**, 2, (2001), 237–254.
8. D. Tolev, *On a diophantine inequality involving prime numbers*, Acta Arith., **61**, (1992), 289–306.
9. R. Vaughan, *Diophantine approximation by prime numbers I*, Proc. Lond. Math.Soc., **28**(3), (1974), 373–384.