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ORIGINAL RESEARCH

# Traveling nonsmooth solution and conserved quantities of long nonlinear internal waves

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**Abstract** This work deals with applying a rapidly convergent approximation method to obtain some (weak) nonsmooth solitary or periodic solutions of a generic version of the Korteweg-de Vries and the Benjamin-Bona-Mahony equations, the generalized Gardner equation with dual power-law nonlinearity. A novel approach has been adopted to derive conditions among parameters for which the obtained solution may be bounded. Explicit parameter dependence of few constants of the motion corresponding to the nonsmooth solitary or periodic solutions obtained here has been presented. Results derived here may be helpful in interpretations of large-amplitude internal waves in the ocean and for irregularly large-amplitude waves in other fields of nonlinear physics, e.g., optics and dusty- or magneto-plasmas.

**Keywords** Long nonlinear internal waves · Generalized Gardner equation · Rapidly convergent approximation method · Traveling nonsmooth solution · Constant of the motion

**Mathematics Subject Classification** 34A45 · 35C05 · 35D30

## 1 Introduction

Much of the interest in large internal waves (LIWs) in oceanography began in the 1960s [1–3] and becoming an exciting field of investigations after the confluence of ocean instrumentation, applied mathematics, and remote sensing [4]. Due to their intricate interactions during tides, large internal waves are primarily observed in the coastal oceans and marginal seas or two-layer fluids of stratified density.

There have been several numerical and theoretical modeling studies of internal tides and solitary waves. For example, the Princeton Ocean Model based on the nonlinear primitive equation exhibits a high degree of spatial variability in the phase and amplitude of internal wave currents and vertical displacements [5–7]. On the other

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side, weakly nonlinear and weakly dispersive models based on generalizations of the Korteweg-de-Vries (KdV) equation also exhibit the appearance of intense short-scale solitons from the internal tide [8–11].

The Gardner equation combines the KdV and modified KdV equations, and it behaves with the same properties of the classical KdV. Still, it extends its range to a wider domain of parameters of the internal wave motion in coastal regions [7,12]. The Gardner equation is a completely integrable partial differential equation (PDE). The most important implication of this integrability for the Gardner equation is that its solitary wave solutions are solitons. Accordingly, the properties of the soliton propagation explain why long nonlinear internal solitary-like waves are so commonly observed in coastal areas. Hence, the Gardner equation may be regarded as an appropriate model for highly nonlinear internal solitary waves (ISWs) in shallow water [9,12,13].

The motion of unidirectional, one-dimensional, small-amplitude long waves in nonlinear dispersive media is sometimes well approximated by the BBM-equation [14–16]. Starting in the latter half of the 1960s, the mathematical analysis for such nonlinear, dispersive wave equations came to the fore as an important topic within the nonlinear analysis [17].

The numerical and theoretical modeling studies on ISW phenomena in water waves demonstrate the vital role of nonlinearity, dispersion, the Earth's rotation, the bottom slope, the horizontal variability of density stratification, the background current, etc. These effects lead to a wide variety of nonlinear waves, including forwarding and backward shocks, solitary waves, table-like (or thick) solitons, etc. [18–20].

Over the last five decades, a substantial effort has been dedicated to solving KdV and BBM equations [21–28]. But it is observed that a single (nonlinear) self-interaction term is not always enough to describe subtle aspects of physical phenomena mentioned above. In such cases, self-interactions described by dual power nonlinearity seem to be more realistic [19]. This motivates us to obtain a solution, the nonsmooth or weak solutions in particular, of the GGE with dual power nonlinearity [29,30].

The generalized Gardner equation (GGE) considered here

$$\frac{\partial \varphi}{\partial t}(x,t) + a\frac{\partial \varphi}{\partial x}(x,t) + \{b\,\varphi^n(x,t) + d\,\varphi^{2n}(x,t)\}\frac{\partial \varphi}{\partial x}(x,t) + k\frac{\partial^3 \varphi}{\partial x^3}(x,t) = 0. \tag{1.1}$$

may be regarded as a generic form of the Gardner and Benjamin-Bona-Mahony (BBM) equations often appears in the analysis of surface waves in compressible fluids or acoustic waves in harmonic crystals. Here the unknown function $\varphi(x,t)$ represents the amplitude of long nonlinear internal waves. The first term in Eq. (1.1) is due to the evolution of the water wave, $a$ and $k$ appearing in the linear terms represent dispersion coefficients. The strength of self-interactions is presented by the coefficients of the nonlinear terms (i.e., $b$ and $d$). The independent variables $x$ and $t$ are spatial and temporal coordinates, respectively.

On the other hand, in the 1990s, Camassa and Holm [31] observed the existence of nonsmooth peaked solitary wave solution in the case of the Camassa-Holm (CH) equation. Later, S. Liao [28] showed the existence of solitary peaked waves for other mainstream models like hallow water waves. The existence of a weak traveling wave solution for the nonlinear surface-wave has been studied by Jiang [32], Das et al. [33]. So, the derivation of different nonsmooth solutions of GGE with dual-power self-interactions seems helpful to visualize the lack of smoothness observed in nature and physical phenomena. It may be worthy to mention here that there exist a variety of smooth solutions of the GGE equation in literature, [9,13,29,30,34–40], but the nonsmooth exact solutions (in the sense of generalized function) were not studied yet. Hence, the study of the nonsmooth solutions, boundedness conditions, and conserved quantities of GGE seem exciting and valuable.

Recently, a rapidly convergent approximate method (RCAM) was applied to get smooth and nonsmooth solutions of a variety of nonlinear differential equations [33,41,42]. The success of investigations mentioned above encourages us to use RCAM to get (weak) nonsmooth or periodic exact solution of the GGE (1.1). Availability of such solutions for Eq. (1.1) in terms of exponential or trigonometric functions has been judiciously exploited to derive conditions among parameters involved in the equation and constants present in the solution for which the solution will be bounded. The method adopted here for deriving conditions among parameters is somewhat different as exercised in the dynamical system theory for predicting the qualitative behavior of phase space trajectories. Explicit parameter dependence of some constants of the motion of GGE (1.1) has also been derived here.

Our investigation has been described as follows. The underlying principle of RCAM and its application to GGE is discussed in section 2. The traveling wave solution of the GGE and conditions for a weak solution have also been derived here. The restrictions on bounded nonsmooth solitary and periodic wave solutions and the texture of solitary wave solutions have been presented in section 3. In section 4, parameter dependence of some conserved quantities of Eq. (1.1) has been given. The findings of our investigation have been summarized in section 5.

## 2 The RCAM and Solution of the GGE

Some salient steps of RCAM for finding nonsmooth solitary traveling wave solution of GGE (1.1) have been presented here. The details of the scheme may be found in Refs. [33,42]. Since Eq. (1.1) does not contain independent variables $x$ and $t$ explicitly, it admits Lie group of symmetry transformations generated by the infinitesimal generators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$. So, it is admissible to invoke the similarity transformations

$$\xi = x - vt,\ \varphi(x,t) = u(\xi), \tag{2.1}$$

corresponding to the infinitesimal generators $\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$ in Eq. (1.1) to reduce it to the ODE

$$u''(\xi) + \mu\, u(\xi) - \delta\, u^m(\xi) - \gamma\, u^{2m-1}(\xi) = 0,\ \ \xi \in \mathbb{R} \tag{2.2}$$

with the relation among parameters $a, b, d, k, n$ in the model equation and the $\mu, \delta, \gamma, m$ in the reduced ODE

$$\mu = \frac{a-v}{k},\ \delta = -\frac{b}{k(n+1)},\ \gamma = -\frac{d}{k(2n+1)},\ m = n+1. \tag{2.3}$$

In the process of deriving Eq. (2.2), it is assumed that the traveling solutions are smooth enough in the asymptotic region $|x|$ $(|\xi|) \to \infty$.

In $\mathbb{R}$, Eq. (2.2) can be split into

$$\hat{\mathcal{O}}[u](\xi) = \mathcal{N}[u](\xi), \tag{2.4}$$

where $\hat{\mathcal{O}}[\cdot]$ and $\mathcal{N}[\cdot](\xi)$ are the linear operator and collection of nonlinear terms given by

$$\hat{\mathcal{O}}[\cdot](\xi) = e^{\pm\sqrt{-\mu}\,\xi}\frac{d}{d\xi}\left(e^{\mp 2\sqrt{-\mu}\,\xi}\frac{d}{d\xi}\left(e^{\pm\sqrt{-\mu}\,\xi}[\cdot]\right)\right) \tag{2.5}$$

and

$$\mathcal{N}[u](\xi) = \delta\, u(\xi)^m + \gamma\, u(\xi)^{2m-1}. \tag{2.6}$$

The inverse operator $\hat{\mathcal{O}}^{-1}$ of $\hat{\mathcal{O}}$ may be put in the form of a two-fold integral operator

$$\hat{\mathcal{O}}^{-1}[\cdot](\xi) = e^{\mp\sqrt{-\mu}\xi}\left(\int^{\xi} e^{\pm 2\sqrt{-\mu}s}\left(\int^{s} e^{\mp\sqrt{-\mu}t}[\cdot](t)\,dt\right)ds\right). \tag{2.7}$$

This form helps to get the formula

$$\hat{\mathcal{O}}^{-1}\left(u''(\xi) - \lambda^2 u(\xi)\right) = u(\xi) - c_+\, e^{\sqrt{-\mu}\,\xi} - c_-\, e^{-\sqrt{-\mu}\,\xi} \tag{2.8}$$

where $c_+$ and $c_-$ are two arbitrary integration constants. These results may be used to recast Eq. (2.4) to the nonlinear integral equation of Volterra-type

$$u(\xi) = c_+\, e^{+\sqrt{-\mu}\,\xi} + c_-\, e^{-\sqrt{-\mu}\,\xi} + \hat{\mathcal{O}}^{-1}[\mathcal{N}[u]](\xi) \tag{2.9}$$

by operating $\hat{\mathcal{O}}^{-1}$ on both sides of (2.4).

To obtain the non-trivial solution of Eq. (2.4) satisfying physical boundary conditions at $\xi = \pm\infty$, we split the region $\mathbb{R}$ into two disjoint domains $\mathbb{R}^+$ and $\mathbb{R}^-$. One has to choose $c_+ = 0 (c_- = 0)$ to get the bounded solution in $\mathbb{R}^+(\mathbb{R}^-)$ so that

$$u(\xi) = \begin{cases} c_-\, e^{-\sqrt{-\mu}\,\xi} + \hat{\mathcal{O}}^{-1}[\mathcal{N}[u]](\xi) & \text{for } \xi \in \mathbb{R}^+ \\ \\ c_+\, e^{+\sqrt{-\mu}\,\xi} + \hat{\mathcal{O}}^{-1}[\mathcal{N}[u]](\xi) & \text{for } \xi \in \mathbb{R}^-. \end{cases} \tag{2.10}$$

To obtain an exact or approximate solution of the (transformed) Eq. (2.9), we write

$$u(\xi) = \sum_{n=0}^{\infty} u_n(\xi) \tag{2.11}$$

and recast the nonlinear part into the series

$$\mathcal{N}[u](\xi)=\sum_{n=0}^{\infty}\mathcal{A}_n(u_0(\xi),u_1(\xi),\ldots,u_n(\xi))\equiv\sum_{n=0}^{\infty}\mathcal{A}_n(\xi) \tag{2.12}$$

where $\mathcal{A}_n(\xi)(n\geq 0)$ are known as Adomian polynomials [43]. These polynomials can be obtained systematically by using the formula

$$\mathcal{A}_n(\xi)=\frac{1}{n!}\left[\frac{d^n}{d\epsilon^n}\mathcal{N}\left(\sum_{k=0}^{\infty}u_k\epsilon^k\right)\right]_{\epsilon=0}\quad n\geq 0. \tag{2.13}$$

We call the function $u_0(\xi)$ in the definition (2.11) as the leading term and $u_n(\xi),\ n\geq 1$ as the corrections in the subsequent part of this paper. Then corrections $u_n(\xi)(n\geq 1)$ involved in (2.11) can be obtained recursively by using formula

$$u_{n+1}(\xi)=\hat{\mathcal{O}}^{-1}[\mathcal{A}_n](\xi),\quad n\geq 0 \tag{2.14}$$

with

$$u_0(\xi)=\begin{cases}c_-\ e^{-\sqrt{-\mu}\,\xi} & \text{for } \xi\in\mathbb{R}^+,\\ c_+\ e^{+\sqrt{-\mu}\,\xi} & \text{for } \xi\in\mathbb{R}^-.\end{cases} \tag{2.15}$$

For the solutions $u(\xi)$ satisfying the condition $u(-\infty)=0$ in $\mathbb{R}^-$, we use (2.5) and (2.6) into (2.13)–(2.15) and get

$$\begin{aligned}u_0(\xi)=&c_+\ e^{\sqrt{-\mu}\,\xi},\\ u_1(\xi)=&-\frac{\left(c_+\ e^{\sqrt{-\mu}\,\xi}\right)^{m-1}\left\{\gamma\ (m+1)\left(c_+e^{\sqrt{-\mu}\,\xi}\right)^m+4\ \delta\ c_+\ m\ e^{\sqrt{-\mu}\,\xi}\right\}}{4\ \mu\ (m-1)m(m+1)},\end{aligned} \tag{2.16}$$

$$u_2(\xi)=\frac{\left\{\gamma^2(m+1)\left(c_+e^{\sqrt{-\mu}\,\xi}\right)^{2m}+8\gamma\ \delta\ m\left(c_+e^{\sqrt{-\mu}\,\xi}\right)^{m+1}+8\delta^2\ c_+^2\ m\ e^{2\sqrt{-\mu}\,\xi}\right\}}{32\ c_+^{(3-2m)}\mu^2(m-1)^2m(m+1)\ e^{(3-2m)\sqrt{-\mu}\,\xi}}. \tag{2.17}$$

It is important to observe that $u_n$, $n=0,1,2$ mentioned above satisfy the three terms recurrence relation

$$\begin{aligned}u_{n+2}(\xi)=&\\ -&\frac{(mn+m-n)\left\{\gamma(m+1)\left(c_+e^{\sqrt{-\mu}\,\xi}\right)^{2m-2}+4\delta m\left(c_+e^{\sqrt{-\mu}\,\xi}\right)^{m-1}\right\}}{4\mu(m-1)m(m+1)(n+2)}u_{n+1}(\xi)\\ &-\frac{\delta^2\left\{(m-1)n+2\right\}\left(c_+e^{\sqrt{-\mu}\,\xi}\right)^{2m-2}}{4\mu^2(m-1)(m+1)^2(n+2)}u_n(\xi).\end{aligned}$$

This relation has two important aspects. First, one can exploit this algebraic relation to obtain higher-order corrections $u_i(\xi)(i\geq 3)$ in (2.11) recursively and avoid the laborious time-consuming process of multifold integration for their evaluation. The second one is more interesting. The recurrence relation mentioned above is the same for a sequence of coefficients (functions) for the series expansion of a (generating) function (in the notation $\varsigma_\pm=c_\pm e^{\pm\sqrt{-\mu}\,\xi}$)

$$\begin{aligned}&g(\xi,\epsilon)=\\ &\left[\frac{4\ \mu^2\ m\ (m+1)^2\ \varsigma_+^{m+1}}{\epsilon\left\{\gamma\mu(m+1)^2+\delta^2m\epsilon\right\}\varsigma_+^{2m}+4\delta\mu m(m+1)\epsilon\varsigma_+^{m+1}+4\mu^2m(m+1)^2\varsigma_+^2}\right]^{\frac{1}{m-1}}\end{aligned} \tag{2.18}$$

in powers of $\epsilon$. This observation may be exploited to obtain the exact solution $u(\xi)$ (given as infinite sum in (2.11)) of equation (2.2) in a closed-form as

$$\begin{aligned} u(\xi) &= \sum_{n=0}^{\infty} u_n(\xi) = g(\xi, 1) \\ &= \left[ \frac{4\,\mu^2\, m\,(m+1)^2\, \varsigma_+^{m+1}}{\left\{\gamma\,\mu\,(m+1)^2 +\,\delta^2\, m\,\right\} \varsigma_+^{2m} + 4\,\delta\,\mu\, m(m+1)\varsigma_+^{m+1} + 4\,\mu^2\, m\,(m+1)^2\,\varsigma_+^2} \right]^{\frac{1}{m-1}} \end{aligned} \tag{2.19}$$

where $c_+$ is an arbitrary constant.

Similarly in domain $\mathbb{R}^+$, the solution satisfying boundary condition $u(\infty) = 0$ can be found as

$$u(\xi) = \left[ \frac{4\mu^2 m(m+1)^2 \varsigma_-^{m+1}}{\left\{\gamma\mu(m+1)^2 + \delta^2 m\right\} \varsigma_-^{2m} + 4\delta\mu m(m+1)\varsigma_-^{m+1} + 4\mu^2 m(m+1)^2 \varsigma_-^2} \right]^{\frac{1}{m-1}} \tag{2.20}$$

with $c_-$ is another arbitrary constant.

It is instructive to mention here that the solution obtained in (2.19) vanishing at $-\infty$ ( $\infty$ in case of (2.20)) may be infinite at the other end $\infty(-\infty)$. One can combine these two solutions with (2.3) to get non-trivial nonsmooth solution $u(\xi)$ of Eq. (2.2) that vanishes at $\pm\infty$ and is smooth in $\mathbb{R} - \{0\}$ as

$$u(\xi) = \begin{cases} \left( \frac{1}{A_+\, e^{n\sqrt{-\mu}\,\xi} + B + c_+^{-n}\, e^{-n\sqrt{-\mu}\,\xi}} \right)^{\frac{1}{n}} & \text{for } \xi < 0, \\ \left( \frac{1}{A_-\, e^{-n\sqrt{-\mu}\,\xi} + B + c_-^{-n}\, e^{n\sqrt{-\mu}\,\xi}} \right)^{\frac{1}{n}} & \text{for } \xi > 0, \end{cases} \tag{2.21}$$

where

$$A_\pm = \left( \frac{b^2(2n+1) - d(n+1)(n+2)^2(a-v)}{4(n+1)^2(n+2)^2(2n+1)(a-v)^2} \right) c_\pm^n \tag{2.22}$$

and

$$B = \frac{-b}{(n+1)(n+2)(a-v)}. \tag{2.23}$$

It may be observed that this solution is differentiable in $\mathbb{R} - \{0\}$, but its first- and second-order derivatives may not exist at 0 in the classical sense. As a result, one may regard the solution $\varphi(x,t) = u(x - vt)(= u(\xi))$ as a traveling nonsmooth (at $x = vt$) solution of (1.1). This type of solution often appears as compacton, cuspion, etc. in the study of nonlinear waves.

## 2.1 Nonsmooth Traveling Wave Solution

Since the derivatives of nonsmooth functions do not exist in the classical sense, we adopt here the sense of distributional derivative to derive restrictions on parameters for the existence of nonsmooth traveling wave solution of Eq. (1.1). In the recent past, Jiang had studied the existence of a weak traveling wave solution for the nonlinear surface-wave equation [32]. Keeping this in mind, here we present some Lemmas and Theorems.

**Definition 1** A function $v(\xi)$ is said to be $i^{th}$ order weak derivative of $u(\xi)$ if it satisfies the relation

$$\int_{-\infty}^{\infty} u(\xi)\, \phi^{(i)}(\xi)\, d\xi = (-1)^i \int_{-\infty}^{\infty} v(\xi)\, \phi(\xi)\, d\xi, \;\; \forall\, \phi \in C_c^\infty \tag{2.24}$$

where $C_c^\infty$ is the space of test functions.

**Lemma 1** *The ODE* $u''(\xi) = f(u)$ *can be recast into* $(u(\xi) - \alpha_0)^2_{\xi\xi} = 2\,(u(\xi) - \alpha_0)\, f(u) + 2(u'(\xi))^2$ *for any algebraic function* $f$ *of* $u$.

**Lemma 2** *A function $u(\xi)$ will be weak solution of the differential equation $u''(\xi) = f(u)$ provided*

$$\int_{-\infty}^{\infty} (u(\xi) - \alpha_0)^2\, \phi''(\xi)\, d\xi = \int_{-\infty}^{\infty} \left\{2\,(u(\xi) - \alpha_0)\, f(u) + 2\,(u'(\xi))^2\right\}\, \phi(\xi)\, d\xi. \tag{2.25}$$

**Theorem 1** *A function $u(\xi)$ will be a weak solution of the differential equation $u''(\xi) = f(u)$, provided any one of the following two conditions holds.*

(i) $u(0^+) = u(0^-)$,
(ii) $\alpha_0 = \frac{u(0^+)+u(0^-)}{2}$ *and* $u'(0^+) = -u'(0^-)$.

*Proof* From Lemma (1), it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} (u(\xi) - \alpha_0)^2\, \phi''(\xi)\, d\xi \\
&= \int_{-\infty}^{0} (u(\xi) - \alpha_0)^2\, \phi''(\xi)\, d\xi + \int_{0}^{\infty} (u(\xi) - \alpha_0)^2\, \phi''(\xi)\, d\xi \\
&= c_1\, \phi'(0) + c_2\, \phi(0) + 2 \int_{-\infty}^{0} \left[(u(\xi) - \alpha_0)\, u''(\xi) + 2\left\{u'(\xi)\right\}^2\right] \phi(\xi)\, d\xi \\
&\quad + 2 \int_{0}^{\infty} \left[(u(\xi) - \alpha_0)\, u''(\xi) + \left\{u'(\xi)\right\}^2\right] \phi(\xi)\, d\xi \\
&= c_1\, \phi'(0) + c_2\, \phi(0) + 2 \int_{-\infty}^{0} \left[(u(\xi) - \alpha_0)\, f(\xi) + \left\{u'(\xi)\right\}^2\right] \phi(\xi)\, d\xi \\
&\quad + 2 \int_{0}^{\infty} \left[(u(\xi) - \alpha_0) f(\xi) + \left\{u'(\xi)\right\}^2\right] \phi(\xi)\, d\xi \\
&= c_1\, \phi'(0) + c_2\, \phi(0) + 2 \int_{-\infty}^{\infty} \left[(u(\xi) - \alpha_0)\, f(\xi) + \left\{u'(\xi)\right\}^2\right] \phi(\xi)\, d\xi
\end{aligned} \tag{2.26}$$

where

$$c_1 = \left\{u(0^-) - \alpha_0\right\}^2 - \left\{u(0^+) - \alpha_0\right\}^2, \tag{2.27}$$

$$c_2 = 2\left[u'(0^-)\, \left\{u(0^-) - \alpha_0\right\} - u'(0^+)\, \left\{u(0^+) - \alpha_0\right\}\right]. \tag{2.28}$$

For weak solution, the choice $c_1 = 0 = c_2$ is desirable simultaneously. The condition $c_1 = 0$ leads to any one of the following two conditions

$$(i)\quad u(0^-) = u(0^+), \tag{2.29}$$

$$(ii)\quad \alpha_0 = \frac{u(0^-) + u(0^+)}{2}. \tag{2.30}$$

From condition (2.29) one gets $c_2 = 0$ if $\alpha_0 = u(0^-) = u(0^+)$. This proves first condition of the theorem.

Now using (2.29), $c_2$ can be expressed as

$$c_2 = \left\{u'(0^-) + u'(0^+)\right\}\left\{u(0^-) - u(0^+)\right\}. \tag{2.31}$$

So, to satisfy $c_2 = 0$, one must have $u'(0^-) = -u'(0^+)$ when $u(0^-) \neq u(0^+)$. This proves the second condition of the theorem.

**Theorem 2** *The nonsmooth solution* (2.21) *of Eq.* (1.1) *will be weak solution, if parameters involved in the equation and integration constants involved in the solution satisfy any one of the following three conditions:*

(i) $c_+ = c_-$,
(ii) $c_+ c_- = \left(\frac{4\,(n+1)^2\,(n+2)^2\,(2n+1)\,(a-v)^2}{b^2\,(2n+1) - d\,(n+1)\,(n+2)^2\,(a-v)}\right)^{\frac{1}{n}}$,
(iii) $\left(\frac{A_+ + B + c_+^{-n}}{A_- + B + c_-^{-n}}\right)^{n+1} = \left(\frac{A_+ - c_+^{-n}}{A_- - c_-^{-n}}\right)^{n}$.

*Proof* The first condition of Theorem (1) can be written as

$$\left(\frac{1}{A_+ + B + c_+^{-n}}\right)^{\frac{1}{n}} = \left(\frac{1}{A_- + B + c_-^{-n}}\right)^{\frac{1}{n}}. \tag{2.32}$$

This gives

$$c_+ = c_- \tag{2.33}$$

or

$$c_+ c_- = \left(\frac{4\,(n+1)^2\,(n+2)^2\,(2n+1)\,(a-v)^2}{b^2\,(2n+1) - d\,(n+1)\,(n+2)^2\,(a-v)}\right)^{\frac{1}{n}}. \tag{2.34}$$

To get the parameter dependence of the second condition, one must have $u'(0^-) = -u'(0^+)$. This implies that

$$\left(\frac{A_+ + B + c_+^{-n}}{A_- + B + c_-^{-n}}\right)^{n+1} = \left(\frac{A_+ - c_+^{-n}}{A_- - c_-^{-n}}\right)^{n}. \tag{2.35}$$

Use of (2.22),(2.33),(2.34) into (2.35) may provide conditions among parameters for which the weak traveling wave solution

$$\varphi(x,t) = \begin{cases} \left(\frac{1}{A_+\, e^{n\sqrt{-\mu}\,(x-v\,t)} + B + c_+^{-n}\, e^{-n\sqrt{-\mu}\,(x-v\,t)}}\right)^{\frac{1}{n}} & \text{for } x < v\,t, \\ \left(\frac{1}{A_-\, e^{-n\sqrt{-\mu}\,(x-v\,t)} + B + c_-^{-n}\, e^{n\sqrt{-\mu}\,(x-v\,t)}}\right)^{\frac{1}{n}} & \text{for } x > v\,t, \end{cases} \tag{2.36}$$

of Eq. (1.1) exists. It may be worthy of mentioning here that the solution $\varphi(x,t)$ given above of the GGE (1.1) is new in the sense that it is a weak solution, nonsmooth along the line $x = vt$ in the $xt$ plane in contrast to the classical solutions available in the literature, e.g. [9,13,29,30,34–40].

## 3 Condition for Solution to be Bounded

The analytical expression for the solution $u(\xi)$ in (2.21) with $\xi = x - v\,t$ has been now used to derive relationships among parameters for which the solution (amplitude of the wave) to be bounded.

### 3.1 Solitary Wave Solution of GGE

Here, the conditions among parameters for bounded solitary wave solution of GGE (1.1) will be derived. Throughout our discussion, we assume that $n > 0$ and the arbitrary constants $c_\pm$ are real and positive. First, we introduce the following notations

$$\zeta_+^P = \frac{-2\,c_+^{-n}\,(v-a)}{\alpha - \beta}, \quad \zeta_+^M = \frac{-2\,c_+^{-n}\,(v-a)}{\alpha + \beta}, \tag{3.1}$$

$$C^{LP} = \left\{\frac{-2\,(v-a)}{\alpha - \beta}\right\}^{\frac{1}{n}}, C^{LM} = \left\{\frac{-2\,(v-a)}{\alpha + \beta}\right\}^{\frac{1}{n}} \tag{3.2}$$

$$b^L = (n+2)\sqrt{\frac{-d\,(v-a)\,(n+1)}{2n+1}}, \quad c^L = \left\{\frac{(n+1)\,(n+2)\,(a-v)}{b}\right\}^{\frac{1}{n}} \tag{3.3}$$

where

$$\alpha = \frac{b}{(n+1)\,(n+2)}, \quad \beta = \sqrt{\frac{-d\,(v-a)}{(n+1)\,(2n+1)}}. \tag{3.4}$$

Next we introduce the following theorem.

**Table 1** Conditions for the parameters mentioned in Theorem 3. Here, (B) is used to represent a sub-case, not the same defined in (2.23)

| Case | $(v-a)$ | Sub case | Limit $k$ | $d$ | $b$ | $c_\pm$ |
|---|---|---|---|---|---|---|
| $A_\pm \neq 0$ | $(v-a)<0$ | $(A)$ | $\in(-\infty,\ 0)$ | $\in(0,\ \infty)$ | $\in(-b^L,\ \infty)\backslash\{b^L\}$ | $< C^{LM}$ |
| $d(v-a)<0$ | | $(B)$ | $\in(-\infty,\ 0)$ | $\in(0,\ \infty)$ | $\in(-\infty,\ -b^L)$ | $<\infty$ |
| $\zeta_\pm^M, \zeta_\pm^P \in \mathbb{R}$ | $(v-a)>0$ | $(C)$ | $\in(0,\ \infty)$ | $\in(-\infty,\ 0)$ | $\in(-\infty,\ b^L)\backslash\{-b^L\}$ | $< C^{LP}$ |
| | | $(D)$ | $\in(0,\ \infty)$ | $\in(-\infty,\ 0)$ | $\in(b^L,\ \infty)$ | $<\infty$ |
| $A_\pm \neq 0$, $d(v-a)>0$ | $(v-a)<0$ | $(E)$ | $\in(-\infty,\ 0)$ | $\in(-\infty,\ 0)$ | $\in(-\infty,\ \infty)$ | $<\infty$ |
| $\zeta_\pm^M, \zeta_\pm^P \in \mathbb{C}$ | $(v-a)>0$ | $(F)$ | $\in(0,\ \infty)$ | $\in(0,\ \infty)$ | $\in(-\infty,\ \infty)$ | $<\infty$ |
| $A_\pm = 0$ | $(v-a)<0$ | $(G)$ | $\in(-\infty,\ 0)$ | $\in(0,\ \infty)$ | $b^L$ | $< c^L$ |
| $b(v-a)<0$ | $(v-a)>0$ | $(H)$ | $\in(0,\ \infty)$ | $\in(-\infty,\ 0)$ | $-b^L$ | $< c^L$ |
| $A_\pm = 0$ | $(v-a)<0$ | $(I)$ | $\in(-\infty,\ 0)$ | $\in(0,\ \infty)$ | $-b^L$ | $<\infty$ |
| $b(v-a)>0$ | $(v-a)>0$ | $(J)$ | $\in(0,\ \infty)$ | $\in(-\infty,\ 0)$ | $b^L$ | $<\infty$ |

**Theorem 3** *The piecewise smooth solution* (2.21) *of GGE* (1.1) *vanishing in the asymptotic region, is bounded for* $(x,t)\in\mathbb{R}\times\mathbb{R}$*, if the parameters* $k,\ d,\ b,\ n,\ a$ *involved in the equation,* $v$ *involved in the transformation* $\xi = x - vt$ *and the arbitrary constant* $c_\pm$ *present in the solution satisfy any one of the conditions stated in the following table along with the condition that* $c_\pm^n \in \mathbb{R}^+$.

*Proof* Use of the symbol $\zeta = e^{n\sqrt{-\mu}\,\xi}$ in the definition (2.21) for $\xi\in(-\infty,0)$ leads to

$$u(\zeta) = \zeta^{\frac{1}{n}}\left(\frac{1}{A_+\,\zeta^2 + B\,\zeta + c_+^{-n}}\right)^{\frac{1}{n}},\ \zeta\in(0,1). \tag{3.5}$$

The solution $u(\zeta)$ will be bounded if the roots $\zeta^M, \zeta^P$ given in (3.1) of the quadratic equation $A_+\,\zeta^2 + B\,\zeta + c_+^{-n} = 0$ lie outside (0, 1) or they are complex. The requirement $u(\xi)\to 0$ as $|\xi|\to\infty$ will be satisfied if $\mu<0$, i.e., $k\,(v-a)>0$. It implies that $k$ and $(v-a)$ are of the same sign. When the roots $\zeta^M, \zeta^P$ are real with either $\zeta^M, \zeta^P>1$ or $\zeta^M, \zeta^P<0$, the solution $u(\zeta)$ in (3.5) will be real for $\zeta\in(0,1)$ if $A_+>0$ (which yields either $b>b^L$ or $b<-b^L$). Again, when $\zeta^M, \zeta^P\in\mathbb{R}$ with one of them being less than 0 and the other being greater than 1, the condition for $u(\zeta)$ to be real for $\zeta\in(0,1)$ is $A_+<0$ (which gives $-b^L<b>b^L$).

The solution $u(\zeta)$ will be real for $\zeta\in(0,1)$ if the polynomial $A_+\,\zeta^2 + B\,\zeta + c_+^{-n}$ is positive for all values of $\zeta\in(0,1)$ which demands that $c_+^{-n}>0$.

Since $\beta>0$, depending upon the sign of $(v-a)$ one can consider the following cases.

**(a)** For $(v-a)<0$,

(i) $\frac{-2\,c_+^{-n}(v-a)}{\alpha-\beta} > \frac{-2\,c_+^{-n}(v-a)}{\alpha+\beta}$, when both $(\alpha+\beta), (\alpha-\beta)>0$,

(ii) $\frac{-2\,c_+^{-n}(v-a)}{\alpha-\beta} < \frac{-2\,c_+^{-n}(v-a)}{\alpha+\beta}$, when $\alpha+\beta>0$ and $\alpha-\beta<0$.

**(b)** For $(v-a)>0$,

(i) $\frac{-2\,c_+^{-n}(v-a)}{\alpha-\beta} < \frac{-2\,c_+^{-n}(v-a)}{\alpha+\beta}$, when both $(\alpha+\beta), (\alpha-\beta)>0$,

(ii) $\frac{-2\,c_+^{-n}(v-a)}{\alpha-\beta} > \frac{-2\,c_+^{-n}(v-a)}{\alpha+\beta}$, when $\alpha+\beta>0$ and $\alpha-\beta<0$.

Now, with the use of the notation introduced in (3.1)–(3.4) the conditions on the real roots $\zeta_+^P,\ \zeta_+^M\in\mathbb{R}\backslash(0,\ 1)$ can be summarized in the following cases (Table 1):

For $d>0,\ (v-a)<0$ one gets from the case **a(i)**
$(1.a)$ $\zeta_+^P>\zeta_+^M>1$ or $(1.b)$ $\zeta_+^M<\zeta_+^P<0$ or $(1.c)$ $\zeta_+^P>1,\ \zeta_+^M<0$.
Again, for the same conditions on $d$ and $(v-a)$ one gets from the case **a(ii)**
$(2.a)$ $\zeta_+^M>\zeta_+^P>1$ or $(2.b)$ $\zeta_+^P<\zeta_+^M<0$ or $(2.c)$ $\zeta_+^M>1,\ \zeta_+^P<0$.
Similarly, for $d<0,\ (v-a)>0$, the conditions from the cases **b(i)** and **b(ii)** can be obtained as
$(3.a)$ $\zeta_+^M>\zeta_+^P>1$ or $(3.b)$ $\zeta_+^P<\zeta_+^M<0$ or $(3.c)$ $\zeta_+^M>1,\ \zeta_+^P<0$.
$(4.a)$ $\zeta_+^P>\zeta_+^M>1$ or $(4.b)$ $\zeta_+^M<\zeta_+^P<0$ or $(4.c)$ $\zeta_+^P>1,\ \zeta_+^M<0$.

The condition $\zeta_+^M > 1$ in $(1.a)$ gives $c_+ < C^{LM}$ provided $\alpha + \beta > 0$ which again leads to $b > -b^L$. Similarly another condition $\zeta_+^P > 1$ in case $(1.a)$ provides $c_+ < C^{LP}$ since $\alpha - \beta > 0$ which yields $b > b^L$. Combining these one can get the boundedness condition for $(1.a)$ as

$$d > 0,\ (v - a) < 0,\ 0 < c_+ < C^{LM},\ b^L < b < \infty. \tag{3.6}$$

Proceeding in similar way the conditions for the case $(1.b)$ can be obtained as

$$d > 0,\ (v - a) < 0,\ -\infty < c_+ < \infty,\ -\infty < b < -b^L. \tag{3.7}$$

In the case of $(1.c)$, the condition $\zeta_+^P > 1$ gives $\alpha - \beta > 0$ and the condition $\zeta_+^M < 0$ gives $\alpha + \beta < 0$ which are contradictory. Hence, the case $(1.c)$ fails to provide any suitable restriction for a bounded solution.

On the other hand, the condition $\zeta_+^P > 1$ in $(2.a)$ leads to $\alpha - \beta > 0$ which contradicts our assumption in $(\alpha - \beta) < 0$ in **a(ii)**. The case $(2.b)$ leads to a similar situation.

In the case of $(2.c)$, the condition $\zeta_+^M > 1$ restricts $c_+$ to $c_+ < C^{LM}$ provided $\alpha + \beta > 0$ i.e. $b > -b^L$ while the condition $\zeta_+^P < 0$ is obvious since $\alpha - \beta < 0$ i.e. $b < b^L$. Thus the boundedness conditions can be obtained from the conditions in $(2.c)$ as

$$d > 0,\ (v - a) < 0,\ 0 < c_+ < C^{LM}, -b^L < b < b^L. \tag{3.8}$$

Following similar arguments as in case $(1.a)$, the conditions for bounded solution can be derived from cases $(3.a)$ and $(3.b)$ as

$$d < 0,\ (v - a) > 0,\ c_+ < C^{LP},\ -\infty < b < -b^L. \tag{3.9}$$

and

$$d < 0,\ (v - a) > 0,\ -\infty < c_+ < \infty,\ b^L < b < \infty. \tag{3.10}$$

Cases $(3.c)$, $(4.a)$, $(4.b)$ leading to contradiction, fail to provide any suitable restriction. The last case $(4.c)$ gives the conditions

$$d < 0,\ (v - a) > 0,\ c_+ < C^{LP},\ -b^L < b < b^L. \tag{3.11}$$

Combination of the conditions (3.6) and (3.8) gives the statement $(A)$ of the theorem while the combination of (3.9) and (3.11) yields the statement $(C)$. The conditions obtained in (3.7), (3.10) are stated in $(B)$ and $(D)$ respectively.

Now, if the roots $\zeta_+^M$, $\zeta_+^P$ are complex, the solution will be bounded. This provides us the condition $d(v-a) > 0$ which also ensures the requirement $A_+ > 0$ for solution $u(\zeta)$ to be real. This can be split into statements $(E)$, $(F)$ of the theorem.

Next, we will find condition for bounded solution in case $A_+ = 0$ i.e., $b^2\,(2n+1) - d\,(n+1)(n+2)(a-v) = 0$. For the requirement of bounded solution, the root of the equation $B\,\zeta_+ + c_+^{-n} = 0$ i.e., $\zeta_+ = \frac{-c_+^{-n}}{B}$ must lie outside $(0, 1)$.

The condition $\zeta_+ > 1$ gives $c_+ < c^L$ when $b\,(a - v) > 0$. For $b\,(a - v) < 0$, $\zeta_+$ is always negative. Depending on the sign of $(v - a)$, $b$ takes the values $b^L$ or $-b^L$. These conditions are stated in $(G) - (I)$ in the theorem.

## 3.2 Texture of Solitary Wave Solution

If the solution is bounded, it may have optima (in case of $A_\pm \neq 0$) and point of inflection (for $A_\pm = 0$) in $\mathbb{R}^+$ and $\mathbb{R}^-$ separately. The location of optima ($\eta_{opt}^+$ and $\eta_{opt}^-$ in $\mathbb{R}^+$ or $\mathbb{R}^-$) and point of inflection ($\eta_{inf}^+$ and $\eta_{inf}^-$ in $\mathbb{R}^+$ or $\mathbb{R}^-$) are found by solving $u'(\xi)|_{\eta_{opt}} = 0$ and $u''(\xi)|_{\eta_{inf}} = 0$ and their dependence on parameter values are given by

$$\eta_{opt}^- = \sqrt{\frac{-k}{a-v}}\left[\frac{1}{2\,n}\ln\left\{\frac{4(1+n)^2(2+n)^2(1+2n)(a-v)^2}{b^2(1+2n)^2 - d(1+n)(2+n)^2(a-v)}\right\} - \ln c_+\right], \tag{3.12}$$

$$\eta_{opt}^+ = \sqrt{\frac{-k}{a-v}}\left[\frac{1}{2\,n}\ln\left\{\frac{b^2(1+2n)^2 - d(1+n)(2+n)^2(a-v)}{4(1+n)^2(2+n)^2(1+2n)(a-v)^2}\right\} + \ln c_-\right] \tag{3.13}$$

and

$$\eta_{inf}^{-} = \sqrt{\frac{-k}{a-v}}\left[-\frac{1}{n}\ln\left\{\frac{-bn}{(1+n)(2+n)(a-v)}\right\} - \ln c_{+}\right], \tag{3.14}$$

$$\eta_{inf}^{+} = \sqrt{\frac{-k}{a-v}}\left[\frac{1}{n}\ln\left\{\frac{-bn}{(1+n)(2+n)(a-v)}\right\} + \ln c_{-}\right]. \tag{3.15}$$

If $\eta_{inf}^{-}$ $(\eta_{inf}^{+}) \in \mathbb{R}^{-}(\mathbb{R}^{+})$ for a specific parameter value, the solution has optima on both halves $\mathbb{R}^{-}$ and $\mathbb{R}^{+}$ and it represents solitary waves with two crests (M/W shape). In that case, the distance between points of optima may be found as

$$\begin{aligned} d_{opt} &= \eta_{opt}^{+} - \eta_{opt}^{-} \\ &= \sqrt{\frac{-k}{a-v}}\ln\left[4^{-\frac{1}{n}}c_{+}c_{-}\left\{\frac{\frac{b^2}{(2+n)^2} - \frac{d(1+n)(a-v)}{(1+2n)}}{(1+n)^2(a-v)^2}\right\}^{\frac{1}{n}}\right]. \end{aligned} \tag{3.16}$$

The solution behaves differently for $A_{\pm} = 0$. It looks like table top solution. The width of the crest depends on the distance between two points of inflection given by

$$\begin{aligned} d_{inf} &= \eta_{inf}^{+} - \eta_{inf}^{-} \\ &= \sqrt{\frac{-k}{a-v}}\ln\left[c_{+}c_{-}\left\{\frac{-bn}{(n+2)(n+1)(a-v)}\right\}^{\frac{2}{n}}\right]. \end{aligned} \tag{3.17}$$

To test the validity of predictions on the restriction of parameters stated in Theorem 3, some parameter values in conformity with the conditions in each case have been prescribed in Table 2, and the corresponding solutions have been depicted in Figure 1$(A) - (J)$. It is observed that the solitary wave solution has a single sharp crest whenever $c_{\pm}$ are close to $C^{LM}/C^{LP}$. If $c_{\pm}$ has no upper bound i.e., for the cases $(B),\ (D),\ (E), (F)$, the solution has two crests for large $c_{\pm}$. In the case of $A_{\pm} = 0$, the behavior of the solution remains the same except for a higher value of $c_{\pm}$. For large $c_{\pm}$, the solution has tabletop soliton (e.g. $(I),\ (J)$). The geometric nature of the solution for each set of parameters presented in Table 2 justifies the restriction on parameters stated in Theorem 3.

### 3.3 Periodic Solution

The condition on parameters involved in the equation and arbitrary constants present in the solution for the existence of bounded periodic solution of Eq. (1.1) may be found as follows. For simplicity, we introduce here the symbols

$$\begin{aligned} d^{LP} &= \frac{b^2(2n+1)}{(n+1)(n+2)^2(a-v)}, \\ C^{L} &= \left(\frac{b^2(2n+1) - d(n+1)(n+2)^2(a-v)}{4(n+1)^2(n+2)^2(2n+1)(a-v)^2}\right)^{-\frac{1}{2n}}. \end{aligned} \tag{3.18}$$

**Theorem 4** *For $k\ (a-v) > 0$, the piecewise smooth solution* (2.21) *of Eq.* (1.1) *is periodic and bounded provided any one the following conditions holds:*
$(a)$ $(a-v) > 0,\ \ b < 0,\ \ 0 < d < d^{LP},\ \ k > 0,\ \ c_{-} = c_{+} = C^{L}$.
$(b)$ $(a-v) < 0,\ \ b > 0,\ \ d^{LP} < d < 0,\ \ k < 0,\ \ c_{-} = c_{+} = C^{L}$.

*Proof* To reduce the solution given in (2.21) in terms of trigonometric functions, we use the substitution

$$A_{\pm} = \frac{D_{\pm}}{2}e^{\pm i\Delta},\ c_{\pm}^{-n} = \frac{D_{\pm}}{2}e^{\mp i\Delta} \tag{3.19}$$

and get

$$u(\xi)=\begin{cases}\left\{\frac{1}{D_+\,cos(n\sqrt{\mu}\xi+\Delta)+B}\right\}^{\frac{1}{n}} & for\ \xi<0,\\ \left\{\frac{1}{D_-\,cos(n\sqrt{\mu}\xi+\Delta)+B}\right\}^{\frac{1}{n}} & for\ \xi>0,\end{cases} \tag{3.20}$$

with

$$\Delta=cos^{-1}\left(\frac{A_\pm+c_\pm^{-n}}{D_\pm}\right),\ D_\pm=2\sqrt{A_\pm\,c_\pm^{-n}}. \tag{3.21}$$

Now, $\Delta$ will be well defined if

$$-1\le\frac{A_\pm+c_\pm^{-n}}{D_\pm}\le 1. \tag{3.22}$$

Using the value of $D_\pm$ from (3.21) the above condition can be reduced to

$$|A_\pm+c_\pm^{-n}|\le 2\sqrt{A_\pm c_\pm^{-n}}$$

which amounts to

$$A_\pm=c_\pm^{-n}. \tag{3.23}$$

Use of (2.22) in the above relation gives the value of arbitrary constants $c_+,\ c_-$ as

$$c_+=c_-=\left(\frac{b^2(2n+1)-d(n+1)(n+2)^2(a-v)}{4(n+1)^2(n+2)^2(2n+1)(a-v)^2}\right)^{-\frac{1}{2n}}\left(\cong C^L\right). \tag{3.24}$$

Hence from (3.21) and (3.23) one gets

$$A_\pm=c_\pm^{-n}=\sqrt{\frac{b^2(2n+1)-d(n+1)(n+2)^2(a-v)}{4(n+1)^2(n+2)^2(2n+1)(a-v)^2}}, \tag{3.25}$$

$$D_\pm=2\sqrt{\frac{b^2(2n+1)-d(n+1)(n+2)^2(a-v)}{4(n+1)^2(n+2)^2(2n+1)(a-v)^2}}. \tag{3.26}$$

Now $A_\pm,\ D_\pm,\ c_\pm^{-n}\in\mathbb{R}$, if

$$b^2(2n+1)-d(n+1)(n+2)^2(a-v)>0. \tag{3.27}$$

Since $k\ (a-v)>0\ (\mu>0)$, the last expression provides the following two cases

$$\begin{gathered}(a-v)>0,\ k>0,\ d<d^{LP};\\ (a-v)<0,\ k<0,\ d^{LP}<d.\end{gathered} \tag{3.28}$$

The solution in (3.20) will be bounded provided $D_+\cos(n\sqrt{\mu}\,\xi+\Delta)+B$ is always positive. This requirement leads to the condition $B>D_\pm$, which provides the restrictions $b\ (a-v)<0$ and $d\ (a-v)>0$. These two conditions can be split into the following cases:

$$\begin{gathered}(a-v)>0,\ b<0,\ d>0;\\ (a-v)<0,\ b>0,\ d<0.\end{gathered} \tag{3.29}$$

Combination of (3.28) and (3.29) yields

$$\begin{gathered}(a-v)>0,\ \ b<0,\ \ 0<d<d^{LP},\ \ k>0;\\ (a-v)<0,\ \ b>0,\ \ d^{LP}<d<0,\ \ k<0.\end{gathered} \tag{3.30}$$

To examine our predictions on the restriction of parameters for the appearance of bounded periodic solution of Eq. (1.1), the solutions for parameter values $n=4,\ v=2,\ a=3,\ k=10,\ b=-15,\ d=5$ and $n=7,\ v=3,\ a=2,\ k=-10,\ b=15,\ d=-1$ in conformity with the conditions for the cases $(a-v)>0$ and $(a-v)<0$ respectively have been found to be bounded, as is evident from Figure 1($K$) and ($L$). The bounded periodic natures of the solutions ascertain the correctness of the restrictions of domain of parameters stated in Theorem 4.

**Table 2** Particular values of parameters $k$, $d$, $b$ and arbitrary constant $c_{\pm}$ and their limits satisfying the conditions in Theorem 3 have been presented here. For case $v < a$, we have taken $v = 2,\ a = 3$ and for $v > a,\ v = 3, a = 2$. In all the cases $n = 4$

| $v,\ a$ | Case | Parameter k Limit | Value | d Limit | Value | b Limit | Value |
|---|---|---|---|---|---|---|---|
| $v < a$ | $(A)$ | $(-\infty, 0)$ | $-10$ | $(0, \infty)$ | 15 | $(0, \infty)$ | 10 |
| | $(B)$ | $(-\infty, 0)$ | $-10$ | $(0, \infty)$ | 15 | $(-\infty, -17.3)$ | $-17.4$ |
| $v > a$ | $(C)$ | $(0, \infty)$ | 10 | $(-\infty, 0)$ | $-15$ | $(-\infty, 0)$ | $-10$ |
| | $(D)$ | $(0, \infty)$ | 10 | $(-\infty, 0)$ | $-15$ | $(17.3, \infty)$ | 17.4 |
| $v < a$ | $(E)$ | $(-\infty, 0)$ | $-10$ | $(-\infty, 0)$ | $-7$ | $(0, \infty)$ | 10 |
| $v > a$ | $(F)$ | $(0, \infty)$ | 10 | $(0, \infty)$ | 7 | $(0, \infty)$ | 100 |
| $v < a$ | $(G)$ | $(-\infty, 0)$ | $-10$ | $(0, \infty)$ | 10 | $10\sqrt{2}$ | $10\sqrt{2}$ |
| $v > a$ | $(H)$ | $(0, \infty)$ | 10 | $(-\infty, 0)$ | $-10$ | $-10\sqrt{2}$ | $-10\sqrt{2}$ |
| $v < a$ | $(I)$ | $(-\infty, 0)$ | $-10$ | $(0, \infty)$ | 10 | $-10\sqrt{2}$ | $-10\sqrt{2}$ |
| $v > a$ | $(J)$ | $(0, \infty)$ | 10 | $(-\infty, 0)$ | $-10$ | $10\sqrt{2}$ | $10\sqrt{2}$ |
| $v,\ a$ | Case | Arbitrary Const. $c_+$ Limit | Value | $c_-$ Limit | Value | | |
| $v < a$ | $(A)$ | $< 1.14$ | 1.1 | $< 1.14$ | 1.1 | | |
| | $(B)$ | $< \infty$ | 70 | $< \infty$ | 70 | | |
| $v > a$ | $(C)$ | $< 1.2$ | 1.1 | $< 1.2$ | 1.1 | | |
| | $(D)$ | $< \infty$ | 300 | $< \infty$ | 300 | | |
| $v < a$ | $(E)$ | $< \infty$ | 100 | $< \infty$ | 196.8 | | |
| $v > a$ | $(F)$ | $< \infty$ | 5 | $< \infty$ | 5 | | |
| $v < a$ | $(G)$ | $< 1.2$ | 1.1 | $< 1.2$ | 1.1 | | |
| $v > a$ | $(H)$ | $< 1.2$ | 1.15 | $< 1.2$ | 1.15 | | |
| $v < a$ | $(I)$ | $< \infty$ | 40 | $< \infty$ | 40 | | |
| $v > a$ | $(J)$ | $< \infty$ | 30 | $< \infty$ | 30 | | |

## 4 Conserved Quantities

Constants of the motion/conserved quantities of an evolution (ordinary or partial differential) equation play an important role in the assessment of the appropriateness of the model for the analysis of the concerned physical process. So, the availability of expressions for the conserved quantities in terms of parameters appearing in the model is useful for investigators of the process. Since we have considered here group invariant solution corresponding to the space-time translations, we present here explicit parameter dependence of the invariants, momentum $(p_x)$ and energy $(\mathcal{E})$ corresponding to the translations in space and time variables, respectively. For their derivations, it is convenient to introduce a few symbols given below.

Using the notations introduced in (2.3), (2.23), the relation $C_{\pm} = c_{\pm}^{-n}$ and $u(\xi)$ in (2.19) the product of distinct powers of $u(\xi)^p$ and its derivative $(u'(\xi))^q$ can be recast into the form

$$u(\xi)^p(u'(\xi))^q = \begin{cases} e^{(qm+p)\sqrt{-\mu}\,\xi}\,\dfrac{(-\mu)^{\frac{q}{2}}\left(C_+e^{-(m-1)\sqrt{-\mu}\,\xi}-A_+e^{(m-1)\sqrt{-\mu}\xi}\right)^q}{\left(A_+e^{2(m-1)\sqrt{-\mu}\xi}+Be^{(m-1)\sqrt{-\mu}\,\xi}+C_+\right)^{\frac{qm+p}{m-1}}} & \xi\in\mathbb{R}^-, \\ e^{-(qm+p)\sqrt{-\mu}\,\xi}\,\dfrac{(-\mu)^{\frac{q}{2}}\left(C_-e^{(m-1)\sqrt{-\mu}\xi}-A_-e^{-(m-1)\sqrt{-\mu}\,\xi}\right)^q}{\left(A_-e^{-2(m-1)\sqrt{-\mu}\,\xi}+Be^{-(m-1)\sqrt{-\mu}\,\xi}+C_-\right)^{\frac{qm+p}{m-1}}} & \xi\in\mathbb{R}^+. \end{cases} \tag{4.1}$$

We denote $I^{p,\,q}_{m,\,\mu,\,\delta,\,\gamma,\,c_{\pm}} = \int_{-\infty}^{\infty} u(\xi)^p\left(u'(\xi)\right)^q d\xi$ and get

$$\begin{aligned} I^{p,\,q}_{m,\,\mu,\,\delta,\,\gamma,\,c_{\pm}} &= (-\mu)^{\frac{q}{2}}\int_{-\infty}^{0} e^{(qm+p)\sqrt{-\mu}\xi}\,\frac{\left(C_+e^{-(m-1)\sqrt{-\mu}\xi}-A_+e^{(m-1)\sqrt{-\mu}\xi}\right)^q}{\left(A_+e^{2(m-1)\sqrt{-\mu}\xi}+B\,e^{(m-1)\sqrt{-\mu}\xi}+C_+\right)^{\frac{qm+p}{m-1}}}d\xi \\ &+(-\mu)^{\frac{q}{2}}\int_{0}^{\infty} e^{-(qm+p)\sqrt{-\mu}\xi}\,\frac{\left(C_-e^{(m-1)\sqrt{-\mu}\xi}-A_-e^{-(m-1)\sqrt{-\mu}\xi}\right)^q}{\left(A_-e^{-2(m-1)\sqrt{-\mu}\xi}+B\,e^{-(m-1)\sqrt{-\mu}\xi}+C_-\right)^{\frac{qm+p}{m-1}}}d\xi. \end{aligned} \tag{4.2}$$

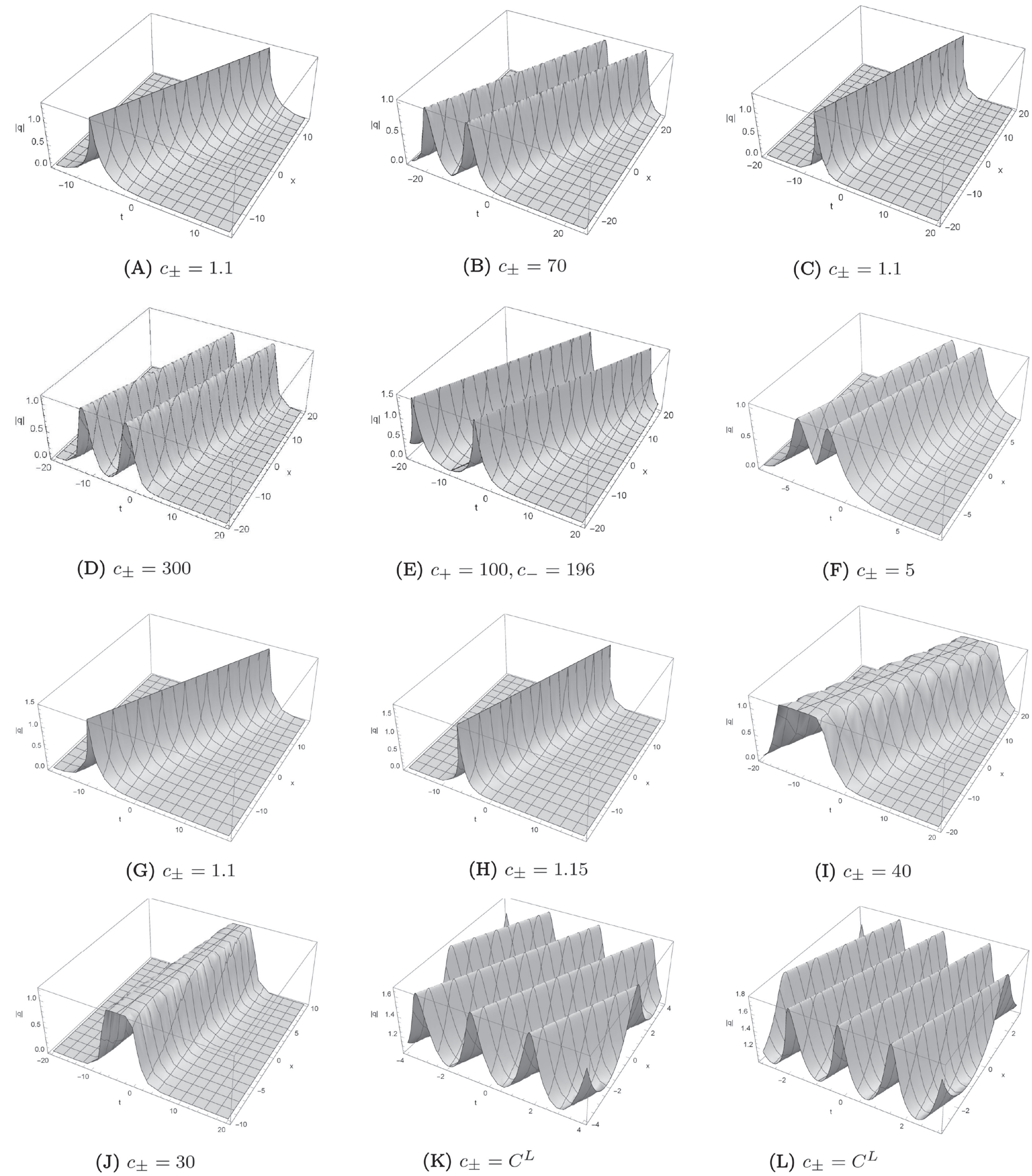


**Fig. 1** Plot of $|\phi(x,\ t)|(= u(x - vt))$ of solitary wave solution (2.21) of GGE Eq. (1.1) for $n = 4$ and values of other parameters given in Table 2(A-J). Periodic solution is presented in $(K)$ (for $n = 4,\ v = 2,\ a = 3,\ k = 10,\ b = -15,\ d = 5$) and in $(L)$ $(n = 7,\ v = 3,\ a = 2,\ k = -10,\ b = 15,\ d = -1)$

Introducing new variables $e^{(m-1)\sqrt{-\mu}\,\xi} = \zeta$ and $e^{-(m-1)\sqrt{-\mu}\,\xi} = \zeta'$ the integrals in the right-hand side can be put into the form

$$I^{p,\,q}_{m,\,\mu,\,\delta,\,\gamma,\,c_\pm} = \frac{(-\mu)^{\frac{q}{2}-\frac{1}{2}}}{(m-1)\sqrt{-\mu}} \left[ (-1)^q \int_0^1 \frac{\zeta^{\frac{qm+p}{m-1}-1}(A_+\zeta - \frac{C_+}{\zeta})^q}{\left(A_+\zeta^2 + B\,\zeta + C_+\right)^{\frac{2km+p}{m-1}}}\,d\zeta \right.$$
$$\left. + \int_0^1 \frac{\zeta'^{\frac{qm+p}{m-1}-1}(A_-\zeta' - \frac{C_-}{\zeta'})^q}{\left(A_-\zeta'^2 + B\,\zeta' + C_-\right)^{\frac{2km+p}{m-1}}}\,d\zeta' \right]. \tag{4.3}$$

Whenever $A_{\pm} \neq 0$, the integral in the right-hand side of (4.3) can be written as

$$I^{p,\,q}_{m,\,\mu,\,\delta,\,\gamma,\,c_{\pm}} =$$
$$\frac{(-\mu)^{\frac{q-1}{2}}}{(m-1)} \sum_{r=0}^{q} (-1)^{q-r} \binom{q}{r} \left[ A_{+}^{q-r} C_{+}^{r-\frac{qm+p}{m-1}} \int_0^1 \frac{\zeta^{\frac{qm+p}{m-1}+q-2r-1}}{(1-\frac{\zeta}{\zeta_{+}^{M}})^{\frac{qm+p}{m-1}} (1-\frac{\zeta}{\zeta_{+}^{P}})^{\frac{qm+p}{m-1}}} \, d\zeta \right.$$
$$\left. + A_{-}^{q-r} C_{-}^{r-\frac{qm+p}{m-1}} \int_0^1 \frac{\zeta'^{\frac{qm+p}{m-1}+q-2r-1}}{(1-\frac{\zeta'}{\zeta_{-}'^{M}})^{\frac{qm+p}{m-1}} (1-\frac{\zeta'}{\zeta_{-}'^{P}})^{\frac{qm+p}{m-1}}} \, d\zeta' \right]. \tag{4.4}$$

Here,

$$\zeta_{+}^{M} = \frac{-B-\sqrt{B^2-4A_{+}C_{+}}}{2A_{+}}, \ \ \zeta_{+}^{P} = \frac{-B+\sqrt{B^2-4A_{+}C_{+}}}{2A_{+}}, \tag{4.5}$$
$$\zeta_{-}'^{M} = \frac{-B-\sqrt{B^2-4A_{-}C_{-}}}{2A_{-}}, \ \ \zeta_{-}'^{P} = \frac{-B+\sqrt{B^2-4A_{-}C_{-}}}{2A_{-}}. \tag{4.6}$$

We now use the integral representation [44] for the Appell function $F_1$ involving the Beta function $B(\alpha, \beta)$ given by

$$F_1(\alpha_1, \beta_1, \beta'; \gamma_1, x, y) = \frac{1}{B(\alpha_1, \gamma_1-\alpha_1)} \int_0^1 \frac{t^{\alpha_1-1}(1-t)^{\gamma_1-\alpha_1-1}}{(1-xt)^{\beta_1}(1-yt)^{\beta'}} \, dt, \tag{4.7}$$

to obtain the parameter dependence of the integral as

$$I^{p,\,q}_{m,\,\mu,\,\delta,\,\gamma,\,c_{\pm}} = \frac{(-\mu)^{\frac{q-1}{2}}}{(m-1)} \sum_{r=0}^{q} (-1)^{r} \binom{q}{r} B\left(\frac{qm+p}{m-1}+q-2r, 1\right) \times$$
$$\left[ A_{+}^{q-r} C_{+}^{r-\frac{qm+p}{m-1}} F_1\left(\frac{qm+p}{m-1}+q-2r, \frac{qm+p}{m-1}, \frac{qm+p}{m-1}; \frac{qm+p}{m-1}+q-2r+1; \right.\right.$$
$$\left. \frac{1}{\zeta_{+}^{M}}, \frac{1}{\zeta_{-}^{P}} \right) + A_{-}^{q-r} C_{-}^{r-\frac{qm+p}{m-1}} F_1\left(\frac{qm+p}{m-1}+q-2r, \frac{qm+p}{m-1}, \frac{qm+p}{m-1}; \right.$$
$$\left.\left. \frac{qm+p}{m-1}+q-2r+1; \frac{1}{\zeta_{-}'^{M}}, \frac{1}{\zeta_{+}'^{P}} \right) \right]. \tag{4.8}$$

For $A_{\pm} = 0$,

$$\bar{I}^{p,\,q}_{m,\,\mu,\,\delta,\,\gamma,\,c_{\pm}} = \int_{-\infty}^{\infty} u(\xi)^{p} \left(u'(\xi)\right)^{q} d\xi$$
$$= \frac{(-\mu)^{\frac{q-1}{2}}}{(m-1)} \left[ C_{+}^{-\frac{q+p}{m-1}} \int_0^1 \frac{\zeta^{\frac{p+q}{m-1}-1}}{\left(1+\frac{B}{C_{+}}\zeta\right)^{\frac{qm+p}{m-1}}} \, d\zeta + C_{-}^{-\frac{q+p}{m-1}} \int_0^1 \frac{\zeta'^{\frac{p+q}{m-1}-1}}{\left(1+\frac{B}{C_{-}}\zeta'\right)^{\frac{qm+p}{m-1}}} \, d\zeta' \right]. \tag{4.9}$$

With the help of the integral representation [44] for the Gauss hypergeometric function

$${}_2F_1(\alpha_1, \beta_1; \gamma_1; x) = \frac{1}{B(\beta_1, \gamma_1-\beta_1)} \int_0^1 \frac{t^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-1}}{(1-xt)^{\alpha_1}} \, dt, \tag{4.10}$$

the integral in (4.9) can be recast into

$$\bar{I}^{p,\,q}_{m,\,\mu,\,\delta,\,\gamma,\,c_{\pm}} = \frac{(-\mu)^{\frac{q-1}{2}}}{(m-1)} B\left(\frac{q+p}{m-1}, 1\right) \times$$

$$\left[C_+^{-\frac{q+p}{m-1}}\,{}_2F_1\left(\frac{qm+p}{m-1},\frac{q+p}{m-1};\frac{q+p}{m-1}+1;-\frac{B}{C_+}\right)\right.$$
$$\left.+C_-^{-\frac{q+p}{m-1}}\,{}_2F_1\left(\frac{qm+p}{m-1},\frac{q+p}{m-1};\frac{q+p}{m-1}+1;-\frac{B}{C_-}\right)\right]. \tag{4.11}$$

With the use of (2.1) and (2.3), we get two conserved quantities [27] of the GGE as

$$p_x=\int_{-\infty}^{\infty}\varphi(x-vt)dx=\int_{-\infty}^{\infty}u(\xi)d\xi=\begin{cases}I^{1,0}_{n+1,\frac{a-v}{k},\frac{-b}{k(n+1)},c_\pm} & \text{for } A_\pm\neq 0,\\ \bar{I}^{1,0}_{n+1,\frac{a-v}{k},\frac{-b}{k(n+1)},c_\pm} & \text{for } A_\pm=0,\end{cases} \tag{4.12}$$

and

$$\mathcal{E}=\int_{-\infty}^{\infty}\varphi^2(x-vt)dx=\int_{-\infty}^{\infty}u^2(\xi)d\xi=\begin{cases}I^{2,0}_{n+1,\frac{a-v}{k},\frac{-b}{k(n+1)},c_\pm} & \text{for } A_\pm\neq 0,\\ \bar{I}^{2,0}_{n+1,\frac{a-v}{k},\frac{-b}{k(n+1)},c_\pm} & \text{for } A_\pm=0.\end{cases} \tag{4.13}$$

## 5 Conclusions

Some exact nonsmooth and periodic traveling wave solutions of GGE (1.1) have been derived here. To attain the goal, similarity transformation $\xi = x - v\ t$ corresponding to the infinitesimal generator $\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$ of Lie group of symmetry transformations admitted by Eq. (1.1) has been used to reduce the PDE to an ODE, Eq. (2.2). Subsequently, the systematic steps of RCAM have been exercised judiciously to obtain nonsmooth or periodic solutions involving exponential functions to the reduced ODE. The solution of the GGE (1.1) is then obtained with the aid of inverse transformations of the similarity variables $\xi$, $u$ and variables $x$ , $t$ , $\varphi$ of Eq. (1.1). Analytical expressions for such solutions have been used to derive conditions among parameters and integration constants for which nonsmooth or periodic solutions are bounded. Appropriate values of the parameters and arbitrary constants satisfying the relations in each case presented in the theorems have been chosen to examine the validity of the constraints among parameters. The corresponding solutions have been found bounded and have various solitary textures, as shown in Figure 1. Finally, some integrals involving exact solutions and their derivatives have been evaluated to obtain the expressions of some conserved quantities in terms of the parameters present in the equation and integration constants appearing in the solution. There are lots of smooth solutions of the Eq. (1.1), but nonsmooth solutions derived here are new to the best of our knowledge. All results derived here seem to be useful in finding nonsmooth solutions of the KdV equation, modified KdV equation, and Gardner equation, as these equations are particular cases of GGE (1.1).

Hence, one can conclude that derived results are important in studying various long nonlinear internal waves in the coastal oceans appearing during tides, nonsmooth solitary waves in plasma, and solid-state physics. Moreover, an extension of the methodology of RCAM exercised here for investigating the existence of nonsmooth solutions of mathematical models involving a system of partial differential equations appearing in the ocean dynamics [45–47], magneto-plasma dynamics [48], fiber optics [49], in the process of Bose-Einstein condensation [50], scalar PDE of higher-order appearing in the waves in random and complex media [51,52], and so on, seem open, will be reported elsewhere.

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## References

1. T.B. Benjamin, Journal of Fluid Mechanics **25**(2), 241 (1966)
2. T.B. Benjamin, Journal of Fluid Mechanics **29**(3), 559 (1967)
3. D.J. Benney, Journal of Mathematics and Physics **45**(1-4), 52 (1966)
4. K.R. Helfrich, W.K. Melville, Annu. Rev. Fluid Mech. **38**, 395 (2006)
5. P.D. Craig, Journal of Marine Research **46**(1), 59 (1988)
6. P.E. Holloway, B. Barnes, Continental Shelf Research **18**(1), 31 (1998)
7. R. Grimshaw, E. Pelinovsky, Y. Stepanyants, T. Talipova, Marine and Freshwater Research **57**(3), 265 (2006)
8. N. Smyth, P. Holloway, Journal of Physical Oceanography **18**(7), 947 (1988)

9. P.E. Holloway, E. Pelinovsky, T. Talipova, Journal of Geophysical Research: Oceans **104**(C8), 18333 (1999)
10. M.A. Merrifield, P.E. Holloway, T.S. Johnston, Geophysical Research Letters **28**(4), 559 (2001)
11. R. Grimshaw, E. Pelinovsky, T. Talipova, A. Kurkin, Journal of Physical Oceanography **34**(12), 2774 (2004)
12. R. Grimshaw, D. Pelinovsky, E. Pelinovsky, T. Talipova, Physica D: Nonlinear Phenomena **159**(1-2), 35 (2001)
13. R. Grimshaw, in *Nonlinear waves in fluids: recent advances and modern applications* (Springer, 2005), pp. 1–28
14. J. Bona, W. Pritchard, L. Scott, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences **302**(1471), 457 (1981)
15. J.L. Hammack, H. Segur, Journal of Fluid Mechanics **65**(2), 289 (1974)
16. N. Zabusky, C. Galvin, Journal of Fluid Mechanics **47**(4), 811 (1971)
17. J. Bona, N. Tzvetkov, Discrete & Continuous Dynamical Systems **23**(4), 1241 (2009)
18. R. Grimshaw, D. Pelinovsky, E. Pelinovsky, A. Slunyaev, Chaos: An Interdisciplinary Journal of Nonlinear Science **12**(4), 1070 (2002)
19. L. Ostrovsky, E. Pelinovsky, V. Shrira, Y. Stepanyants, Chaos: An Interdisciplinary Journal of Nonlinear Science **25**(9), 097620 (2015)
20. M. Obregon, N. Raj, Y. Stepanyants, Chaos: An Interdisciplinary Journal of Nonlinear Science **28**(3), 033106 (2018)
21. J.L. Bona, R. Smith, Phil. Trans. R. Soc. Lond. A **278**(1287), 555 (1975)
22. J. Avrin, Nonlinear Analysis: Theory, Methods & Applications **11**(1), 139 (1987)
23. A.M. Wazwaz, M.A. Helal, Chaos, Solitons & Fractals **21**(3), 579 (2004)
24. A.M. Wazwaz, Physics Letters A **355**(4-5), 358 (2006)
25. A. Biswas, S. Konar, Communications in Nonlinear Science and Numerical Simulation **13**(4), 703 (2008)
26. S. Abbasbandy, A. Shirzadi, Communications in Nonlinear Science and Numerical Simulation **15**(7), 1759 (2010)
27. A. Biswas, Communications in Nonlinear Science and Numerical Simulation **15**(10), 2744 (2010)
28. S. Liao, Science China Physics, Mechanics and Astronomy **55**(12), 2469 (2012)
29. X. Li, M. Wang, Physics Letters A **361**(1-2), 115 (2007)
30. D. Lu, C. Liu, Applied Mathematics and Computation **217**(4), 1404 (2010)
31. R. Camassa, D.D. Holm, Physical Review Letters **71**(11), 1661 (1993)
32. B. Jiang, Mathematical Methods in the Applied Sciences **40**(15), 5419 (2017)
33. P.K. Das, S. Mandal, M.M. Panja, Mathematical Methods in the Applied Sciences **41**(17), 7869 (2018)
34. M. Bokaeeyan, A. Ankiewicz, N. Akhmediev, Physical Review E **99**(6), 062224 (2019)
35. M. Bruzón, M. Gandarias, R. de la Rosa, A Mathematical Modeling Approach from Nonlinear Dynamics to Complex Systems pp. 7–26 (2019)
36. S. Hamdi, B. Morse, B. Halphen, W. Schiesser, Natural Hazards **57**(3), 597 (2011)
37. S. Hamdi, B. Morse, B. Halphen, W. Schiesser, Natural hazards **57**(3), 609 (2011)
38. O.E. Hepson, A. Korkmaz, I. Dag, International Journal of Computer Mathematics **97**(4), 837 (2020)
39. S. Mohapatra, R.B. Fonseca, C. Guedes Soares, Journal of Coastal Research **34**(4), 928 (2018)
40. H. Rezazadeh, D. Kumar, T.A. Sulaiman, H. Bulut, Modern Physics Letters B **33**(17), 1950196 (2019)
41. P.K. Das, M.M. Panja, in *Applied Mathematics* (Springer, 2015), pp. 193–201
42. P.K. Das, M.M. Panja, IJSEAS **2**, 334 (2016)
43. J.S. Duan, R. Rach, Applied Mathematics and Computation **218**(8), 4090 (2011)
44. F.W. Olver, D.W. Lozier, R.F. Boisvert, C.W. Clark. NIST handbook of mathematical functions, US Department of Commerce, National Institute of Standards and Technology (2010)
45. X.Y. Gao, Y.J. Guo, W.R. Shan, Chaos, Solitons & Fractals **138**, 109950 (2020)
46. X.Y. Gao, Y.J. Guo, W.R. Shan, Applied Mathematics Letters **104**, 106170 (2020)
47. X.Y. Gao, Y.J. Guo, W.R. Shan, Physics Letters A **384**(31), 126788 (2020)
48. X.X. Du, B. Tian, Q.X. Qu, Y.Q. Yuan, X.H. Zhao, Chaos, Solitons & Fractals **134**, 109709 (2020)
49. C.R. Zhang, B. Tian, Q.X. Qu, L. Liu, H.Y. Tian, Zeitschrift für Angewandte Mathematik und Physik **71**(18), 1 (2020)
50. X. Zhao, B. Tian, Q.X. Qu, Y.Q. Yuan, X.X. Du, M.X. Chu, Modern Physics Letters B **34**(26), 2050282 (2020)
51. X.Y. Gao, Y.J. Guo, W.R. Shan, Waves in Random and Complex Media pp. 1–21 (2021)
52. M. Wang, B. Tian, Y. Sun, Z. Zhang, Computers & Mathematics with Applications **79**(3), 576 (2020)